Lecture notes on Electrodynamics–118120

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[1]Comments and typos welcome. Send to avronj@gmail.com.

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Chapter 1

Tensor calculus

Tensor calculus allows us to write equation without committing ourselves to a specific coordinate systems.

1.1 Geometry

Minkowski space-time is the stage on which electrodynamics takes place. Before describing the geometry of space-time lets us collect the tools we need from (Riemanian) geometry.

1.1.1 Euclidean geometry

In constructing physical theories we need to know some facts about nature. For example, we need to know various physical constants such as e, $\hbar$, c mass of particles etc. They are all god given scalars. Besides knowledge about scalars, we also need to know something about the space we live in. For our purposes, physical space is to a good approximation, Euclidean, and we take this to be another God given fact. We are still free to choose any (curvilinear in general) coordinate systems to describe the space.

Remark 1.1 (Euclid, Gauss and Einstein)**.** *Euclid took it for granted that physical space is Euclidean. The first to seriously entertain the possibility that the physical space need not be Euclidean was Gauss. In a Euclidean world the angles of all triangles sum up to* π*. So, when one says that the world is to a good approximation Euclidean one means that the deviations from* π *are small. Gauss who had experience in lad surveying made an experiment which was did not show deviation from Euclidean geometry. Later Einstein taught us that space-time is actually curved and there are many physical tests of this. However, this is a another story.*

1.1.2 The metric tensor

In the Euclidean plane consider the three coordinate systems: (x^1, x^2) a Cartesian coordinate system; $\big((x')^1, (x')^2\big)$ a rotated (blue) Cartesian coordinate system and a polar coordinate systems $\big((x")^1, (x")^2\big) = (r, \theta)$. $d\ell$ is the dis-

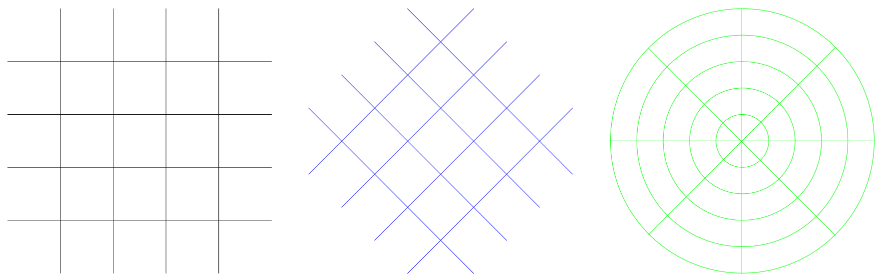

Figure 1.1: Cartesian, rotated Cartesian and polar coordinates for the Euclidean plane

tance measured between two neighborhood points using a standard meter. It does not depend on the choice of coordinates

$$(d\ell)^2 = \sum_{ij} g_{ij} dx^i dx^j$$

g is called the metric tensor, also known as the Riemann metric tensor who in his Thesis founded Riemanian geometry. It is a *second rank tensor* which means it has two indices. It is also symmetric $g_{ij} = g_{ji}$. The components are written downstairs. Downstairs components are called covariant components.

If the two Cartesian coordinate systems use the same yardstick then we may choose the length scale of x^j so that

$$g_{ij} = \delta_{ij}$$

With this choice g is dimensionless and dx^j have dimension of length.

In polar coordinates, $x^1 = r\cos\theta, \quad x^2 = r\sin\theta$

$$(d\ell)^2 = (dr)^2 + r^2\,(d\theta)^2$$

Exercise 1.2. *Verify.*

The moral of this is that there are many distinct metrics for a given space: As many as coordinate transformations. The surface of the sphere is geometrically distinct from the plane. It has the metric

$$(d\ell)^2 = (d\theta)^2 + \sin^2\theta\,(d\phi)^2$$

The geometric distinction means that it *can not* accommodate Cartesian coordinates. This is why maps drawn on a sheet of paper never accurately represent regions earth.

Exercise 1.3. *Compute* $\det g$ *in Cartesian and polar coordinates.*

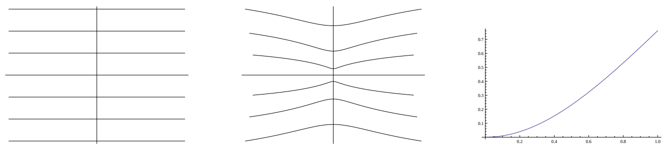

Figure 1.2: The plot on the left shows a Cartesian y-mesh. The plot in the middle shows a curvilinear y-mesh in the plane. The deformation is due to a deformation of the radial polar coordinate $r \to r(\tanh r)$ shown on the right.

1.1.3 Einstein summation convention

This is a short hand which says: Sum over pairs of up-down indices. For example

$$\sum g_{ij} dx^i dx^j = g_{ij} dx^i dx^j$$

It is also called *contraction* of indices.

Remark 1.4 (Dummy indices)**.** *Summation indices are sometimes called running and sometimes dummy. They can be relabeled freely*

$$g_{ja} v^a = g_{jb} v^b$$

Remark 1.5 (Warning)**.** *If you get an equation where the indices are not nicely paired, such as*

$$v_a u_a, \quad v_a u^a w_a$$

it is a good idea to search for a typo.

1.1.4 Coordinate transformations

If g is the metric tensor in the coordinate x and x' is different coordinate system of the same space, then g' is

$$\begin{aligned}(d\ell)^2 &= g_{ij} dx^i dx^j \\ &= g_{ij} \left(\frac{\partial x^i}{\partial (x')^a}\right) d(x')^a \left(\frac{\partial x^j}{\partial (x')^b}\right) d(x')^b \\ &= (g')_{ab} d(x')^a d(x')^b\end{aligned}$$

This says

$$(g')_{ab} = \Lambda^i{}_a \, \Lambda^j{}_b \, g_{ij}, = \Lambda^j{}_b \, g_{ji} \Lambda^i{}_a, \quad \Lambda^i{}_a = \left(\frac{\partial x^i}{\partial (x')^a}\right) \tag{1.1}$$

If one thinks of Λ and g as matrices [1] the relation above can be written as

$$g' = \Lambda^t g \Lambda \tag{1.2}$$

[1]First index is row second is column.

Remark 1.6. *Since g is a symmetric matrix, it can be diagonalized by an orthogonal transformation. So, at any given, fixed point x there is a Λ that diagonalizes g. You can also rescale the coordinates so as to make $g_{ij}(x) = \delta_{ij}$ at one point x). But, in general, you can not make g the identity everywhere. If you can, the space is the Euclidean space.*

This is an expression of the fact that any (Riemannian) manifold is locally Euclidean. Columbus needed to go far to convince everybody that Earth is a sphere.

Exercise 1.7. *By counting the number of free parameters and the number of constraints in the Taylor expansions, show that you can choose coordinate transformations that make g the identity and makes all its first derivative vanish. However, you can not make all the second derivatives vanish by coordinate transformations.*

Exercise 1.8. *Write the metric of the unit sphere using the coordinates z of the spherical projection on the plane.*

1.2 Vectors

Consider a vector $\delta\mathbf{x}$ (say, in the Euclidean plane) associated with a small change of the coordinates δx^j. For the sake of masochism, we allow non-orthogonal coordinate system.

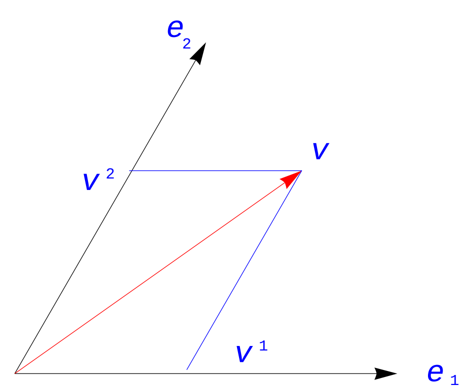

Figure 1.3: Contravariant components and the basis vectors

$$\delta\mathbf{x} = \delta x^1 \mathbf{e}_1 + \delta x^2 \mathbf{e}_2 \tag{1.3}$$

$\mathbf{e}_j$ are vectors pointing along the coordinate lines and δx^j are the coordinates increments. In general $\mathbf{e}_j$ may depend on position as is the case for of polar coordinates.

$(\delta x^1, \delta x^2)$ are the components of the vector. They are called the *contravariant* components. Like the coordinate, have their indices upstairs. The length of the vector is a scalar, and this aloows us to relate the basis vectors $\mathbf{e}_j$ with the metric

$$\delta\mathbf{x}\cdot\delta\mathbf{x} = \sum g_{ij}\,\delta x^i \delta x^j, \quad g_{ij} = \mathbf{e}_i\cdot\mathbf{e}_j \tag{1.4}$$

We recover that fact that g is symmetric and positive

$$g_{ij} = g_{ji}, \quad g \geq 0$$

You also see form this formula that g is diagonal in orthogonal coordinate systems and that the covariant basis vectors $\mathbf{e}_j$ are, in general, not unit vectors.

Exercise 1.9. *Show that in a polar coordinate system the covariant basis vectors and the normalized unit vectors are related by*

$$\mathbf{e}_r = \hat{\mathbf{r}}, \quad \mathbf{e}_\theta = r\hat{\theta}$$

Remark 1.10. *Normalized unit vectors are defined* only *for orthogonal coordinate systems.*

In two coordinate systems we have

$$\delta\mathbf{x} = \delta x^1\mathbf{e}_1 + \delta x^2\mathbf{e}_2 = \delta(x')^1\mathbf{e}'_1 + \delta(x')^2\mathbf{e}'_2 \tag{1.5}$$

so the transformation law for the contravariant components is, by the chain rule

$$\delta x^j = \left(\frac{\partial x^j}{\partial (x')^a}\right)\delta(x')^a \iff \delta x^j = \Lambda^j{}_a\delta(x')^a \iff \underbrace{\delta x = \Lambda\delta(x')}_{Contravariant} \tag{1.6}$$

In the third expression think of the contravariant components δx as column vector and Λ as a matrix. If you compare this with the rule for the metric tensor you see that the prime is on the other side. Covariant and contravariant indices have different transformation laws.

Example 1.11 (Mechanical model)**.** *A particle of unit mass moves on a ring of fixed radius, r. Its orbit in polar coordinates is $(r,\theta(t))$. Its velocity vector is:*

$$\mathbf{v} = (r\dot{\theta})\,\hat{\theta} = \dot{\theta}\,\mathbf{e}_\theta \tag{1.7}$$

The "normalized" component is the tangential velocity*, the contravariant component is the* angular velocity *and its dimension is frequency, not velocity. This is one disadvantage of contravariant components: They may screw dimensions. They do have compensating advantages, however.*

Exercise 1.12. *Show that the matrices*

$$\Lambda^j{}_a = \left(\frac{\partial x^j}{\partial (x')^a}\right), \quad (\Lambda')^b{}_k = \left(\frac{\partial (x')^b}{\partial x^k}\right)$$

are inverses $\Lambda\Lambda' = \mathbb{1}$

1.2.1 Covariants components

The basis vectors $\mathbf{e}_j$ are neither orthogonal nor normalized. In such cases one defines the *dual* basis vectors $\mathbf{e}^j$ to the basis $\mathbf{e}_j$ by

$$\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i{}^j \tag{1.8}$$

and δ is the Kronecker symbol. We could then write a vector can be represented in two different ways

$$\mathbf{v} = v^a \mathbf{e}_a = v_a \mathbf{e}^a \tag{1.9}$$

v_a are the covariant components and v^j the contravariant. Clearly

$$v_j = \mathbf{v} \cdot \mathbf{e_j} \tag{1.10}$$

The covariant components are geometrically interpreted as dropping perpendiculars on the coordinate mesh.

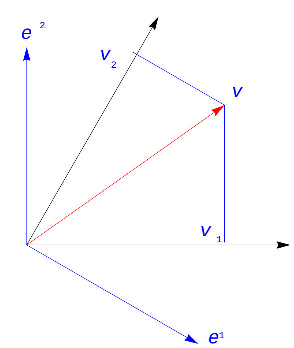

Figure 1.4: Covariant components and the dual basis

From the definitions of the metric tensor, and the notion of duality we get

$$v_k = v_a \, \mathbf{e}^a \cdot \mathbf{e}_k = v^a \, \mathbf{e}_a \cdot \mathbf{e}_k = g_{ka} v^a \tag{1.11}$$

We learn from this that *the metric tensor allows us to push indexes down.*

Exercise 1.13. *Show that*

$$\mathbf{e}^j = (\mathbf{e}^j \cdot \mathbf{e}^a) \, \mathbf{e}_a$$

Everything one can do with contravariant components has an analog in the covariant components. In particular, the length of a vector is evidently given by

$$\mathbf{v} \cdot \mathbf{v} = v_a v_b g^{ab}, \quad g^{jk} = \mathbf{e}^j \cdot \mathbf{e}^k \tag{1.12}$$

Taking the scalar product of exercise 1.13 with $\mathbf{e}_k$ we conclude that the two metric tensors are inversely related

$$\delta^j_k \underbrace{=}_{duality} \mathbf{e}^j \cdot \mathbf{e}_k \underbrace{=}_{ex.10} (\mathbf{e}^j \cdot \mathbf{e}^a)(\mathbf{e}_a \cdot \mathbf{e}_k) = g^{ja} g_{ak} \underbrace{=}_{index\ gym} g^j{}_k$$

The metric tensor with indexes up is the inverse matrix of the metric tensor with indexes down. It raises indexes since

$$g^{ja}v_a = g^{ja}g_{ab}v^b = \delta^j_b v^b = v^j \tag{1.13}$$

From this one can figure the transformation rules for the covariant components

$$\begin{aligned}
(v')_a &= (g')_{ab}(v')^b \\
&= (g')_{ab}(\Lambda^{-1}v)^b \\
&= \Lambda^i{}_a\Lambda^j{}_b g_{ij}(\Lambda^{-1})^b{}_k v^k \\
&= \Lambda^i{}_a\Lambda^j{}_b(\Lambda^{-1})^b{}_k g_{ij}\, v^k \\
&= \Lambda^i{}_a(\Lambda\Lambda^{-1})^j{}_k g_{ij}\, v^k \\
&= \Lambda^i{}_a g_{ij}\, v^j \\
&= \Lambda^i{}_a v_i
\end{aligned}$$

Remark 1.14. *If you think of the covariant components as* a row *vector then the transfromation rule is*

$$\underbrace{v' = v\Lambda}_{covarinat}$$

Note the similarities and differences with the rule of transfromation of the contravariant components.

1.2.2 Orthogonal coordinates

Many of the standard curvilinear coordinate systems one encounters in practice are orthogonal. In this case, the metric g is a diagonal matrix. Orthogonal coordinates admit three types of components: The usual covariant and contravariant components and the "normalized" components. All three are given by

$$\mathbf{V} = v^j\mathbf{e}_j = v_j\mathbf{e}^j = v_{\hat{j}}\mathbf{n}_j$$

with $\mathbf{e}_i \cdot \mathbf{e}_j = g_{ij} = g_i\delta_{ij}$, $\mathbf{e}^i \cdot \mathbf{e}^j = g^{ij} = (g_i)^{-1}\delta_{ij}$ and $\mathbf{n}_j \cdot \mathbf{n}_j = \delta_{ij}$.

Example 1.15 (Polar coordinates)**.** *In polar coordinates*

$$\mathbf{v} = \underbrace{v^r\mathbf{e}_r + v^\theta\mathbf{e}_\theta}_{contravariant\ components} = \underbrace{v_r\mathbf{e}^r + v_\theta\mathbf{e}^\theta}_{covariant\ components} = \underbrace{v_{\hat{r}}\hat{\mathbf{r}} + v_{\hat{\theta}}\hat{\theta}}_{normalized} \tag{1.14}$$

The local frames have basis vectors:

$$\begin{aligned}
\mathbf{e}_r \cdot \mathbf{e}_r = 1, \quad \mathbf{e}_\theta \cdot \mathbf{e}_\theta = r^2, \quad \mathbf{e}_r \cdot \mathbf{e}_\theta = 0 \\
\mathbf{e}^r \cdot \mathbf{e}^r = 1, \quad \mathbf{e}^\theta \cdot \mathbf{e}^\theta = r^{-2}, \quad \mathbf{e}^r \cdot \mathbf{e}^\theta = 0 \\
\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1, \quad \hat{\theta} \cdot \hat{\theta} = 1, \quad \hat{\mathbf{r}} \cdot \hat{\theta} = 0
\end{aligned}$$

1.2.3 Contraction makes scalars

Corollary 1.16. *The contraction of a covariant index with a contravariant index gives a (coordinate independent) scalar*

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}' \cdot \mathbf{v}'$$

One easy way to see this is

$$\underbrace{\mathbf{u}' = \mathbf{u}\Lambda}_{covariant\ row} \quad , \quad \mathbf{v} = \Lambda \mathbf{v}' \Leftrightarrow \underbrace{\mathbf{v}' = \Lambda^{-1}\mathbf{v}}_{contravar\ column}$$

from which the assertion readily follows.

1.3 Scalars, vectors, tensors, densities

The charge of a particle, its mass, or the length of a vector is a scalar. You do not need to decide onn coordinates to measure scalars. If you do use coordinates, the result should be independent of the choice of coordinates. This is the defining property of scalars.

Vectors are geometric objects and as such do not rely of a coordinate system either. But, their representation by covariant or contravariant components depend on the choice of coordinate system. The defining property is that their components have a single index and the rule of transformation is:

$$v^j = \Lambda^j{}_k (v')^k, \quad (v')_j = v_k \Lambda^k{}_j \tag{1.15}$$

Tensors are multi-index objects and the metric tensor is an example. The number of indices is called the rank of the tensor. Each index transforms according to whether it is up or down. For example, the second rank tensor $T^j{}_k$ transform like $v^j u_k$.

1.3.1 Symmetric and anti-symmetric tensors

Coordinate transformations respect the symmetry of tensors: If T is symmetric (anti-symmetric), i.e. $T_{jk} = \pm T_{kj}$, so is (T').

Exercise 1.17. *Show that if T_{jk} is symmetric (anti-smmetric) so is T^{jk} but in general ${T_j}^k \neq {T_k}^j$. (One finds instead ${T_j}^k = T^k{}_j$).*

1.3.2 Weights

There are interesting physical quantities that are neither scalars nor tensors. $\det g$ is and example. From Eq. 1.2

$$\det g' = \big(\det \Lambda\big)^2 \det g$$

Objects with such a rule of transformation are called weights. $\det g$ has weight -2.

1.3.3 Levi-Civita symbol

Suppose you are in two dimensions. The highest rank anti-symmetric tensor has rank two and it has one non trivial component which gives the sign of the permutation

$$\varepsilon^{11} = \varepsilon^{22} = 0, \quad \underbrace{\varepsilon^{12}}_{even} = \underbrace{-\varepsilon^{21}}_{odd} = 1$$

It is a tensor under coordinate transformations provided $\det \Lambda = 1$. This follows from:

$$\varepsilon^{12} = \Lambda^1{}_a \Lambda^2{}_b (\varepsilon')^{ab} = \det \Lambda \, (\varepsilon')^{12}$$

In general, when $\det \Lambda \neq 1$, it is a tensor density, i.e. has a weight.

The same idea works in any dimension: In three dimensions the highest rank of completely anti-symmetric tensor is three and it has one interesting component

$$\underbrace{\varepsilon^{123} = \varepsilon^{231} = \varepsilon^{312}}_{even\ permutations} = \underbrace{-\varepsilon^{321} = -\varepsilon^{213} = -\varepsilon^{132}}_{odd\ permutations} = 1$$

It is a tensor under cordinate transformations provided $\det \Lambda = 1$. This follows from

$$\varepsilon^{123} = \Lambda^1{}_a \Lambda^2{}_b \Lambda^3{}_c (\varepsilon')^{abc} = \det \Lambda \, (e')^{123}$$

and so, in general, is a tensor density.

Exercise 1.18. *Show that*

$$\frac{\varepsilon^{a\dots}}{\sqrt{g}}, \quad \sqrt{g}\varepsilon_{a\dots}$$

are both bona-fide tensors.

Remark 1.19. *The conventions for spherical coordinates is such that you get a right handed frame provided or order the coordinates r, ϕ, θ, i.e.*

$$\varepsilon^{r\phi\theta} = 1$$

Remark 1.20. *In even dimensions cyclic permutations are odd, while in odd dimensions cyclic permutations are even.*

Exercise 1.21. *Show that in n dimensions*

$$\varepsilon^{ij\dots}\varepsilon_{ij\dots} = n!$$

Exercise 1.22. *Show that (in 3 dimensions)*

$$\varepsilon^{ijk}\varepsilon_{iab} = \delta^j_a \delta^k_b - \delta^j_b \delta^k_a, \quad \varepsilon^{ijk}\varepsilon_{ijb} = 2\delta^k_b$$

1.3.4 Volume

In a 3-dimensional Euclidean space, with Cartesian coordinates x^j the volume element is

$$dV = dx^1 dx^2 dx^3$$

and, of course, $dx^1 dx^1 dx^2$ has no volume. If we also allow volume to take a sign according to the handedness of the frame, we naturally associate volume element with the components of the completely antisymmetric tensor:

$$dV = dx^1 \wedge dx^2 \wedge dx^3$$

and $\wedge$ means that order matters: $dx^i \wedge dx^j = -dx^j \wedge dx^i$. In Cartesian coordinates, where $\sqrt{g} = 1$, this can be re-written as

$$dV = \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3 = \frac{\sqrt{g}}{3!} \varepsilon_{ijk} dx^i \wedge dx^j \wedge dx^k \tag{1.16}$$

We have seen in Ex.1.18 $\sqrt{g}\varepsilon_{ijk}$ are the covariant components of a bona-fide (completely anti-symmetric third rank) tensor. Similarly, $dx^i \wedge dx^j \wedge dx^k$ are the conravariant components of a completely anti-symmetric third rank tensor. Contracting the two we get that dV behaves like a scalar under coordinate transformation. Hence, Eq. 1.16 holds in any curvilinear coordinate system, (provided $1, 2, 3$ is right handed frame). This works in any dimension.

Remark 1.23. *It is useful to assign signs to volume (and areas): Positive for right handed frames and negative for left handed frames.*

Exercise 1.24 (Spherical coordinates)**.** *Let (x, y, z) be Cartesian coordinates off Eucliden space with metric $d\ell^2 = (dx)^2 + (dy)^2 + (dz)^2$. Show that the usual spherical coordinates*

$$z = r\cos\theta, \quad x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi$$

have the metric tensor

$$d\ell^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta\,(d\phi)^2$$

and that the volume element is

$$dV = \sqrt{\det g}\ dr\, d\theta\, \mathrm{d}\phi = r^2\sin\theta\ dr\, d\theta\, \mathrm{d}\phi = -r^2\ dr\, d\cos\theta\, \mathrm{d}\phi$$

1.3.5 Areas

The area element dS_3 associated with the parallelogram (dx^1, dx^2) is naturally defined as

$$dV = dS_3 dx^3 = \underbrace{(dS_3\mathbf{e}^3)}_{area\ vector} \cdot (dx^3\mathbf{e}_3)$$

Exercise 1.25. *Verify the formula for the area for spherical area elements dS_r.*

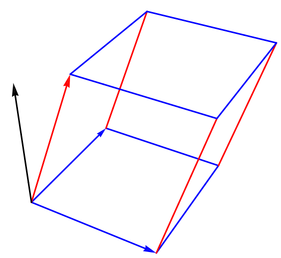

Figure 1.5: The area element, black arrow, directed along $\mathbf{e}^3$, is defined so that its scalar product with the red arrow, $\mathbf{e}_3$, gives the volume. The black arrow is perpendicular to the blue arrows $\mathbf{e}_1$ and $\mathbf{e}_2$.

1.4 Mirror

A mirror flips some things and does not flip others. It does not flip up and down but does flip right and left, it flips a right handed coordinate system to a left handed one etc.

The length of an object is a scalar. It does not flip in a mirror. However, when one integrates it makes a sign difference if you go right or left. So, for example, you want the volume element to flip a sign when reflected in a mirror.

For the sake of concreteness consider the 3-dimensional Euclidean space. Under coordinate inversion $(x')^j = -x^j$ the contravariant components of say a position or velocity vector flip sign. (The vector still points in the same direction.) Now consider the angular velocity. since you use your right hand to determine the direction of the vector of angular velocity, in a mirror you'll use your left hand. The vector of angular velocity now points in the opposite direction. It is a pseudo vector. Under inversion its components *do not* flip sign.

The cross product of two vectors is therefore a pseudo-vector. Since

$$(\mathbf{a} \times \mathbf{b})^i = \varepsilon^{ijk} a_j b_k$$

we conclude that the Levi-Civita symbol is also pseudo.

1.5 Isometries of Euclidean space

Euclidean space looks the same no matter where you are or how you are oriented: It is homogeneous and isotropic. These symmetries reflect invariance properties of the metric tensor of Euclidean space under suitable coordinate transformations. In Cartesian coordinates x^j a shift is:

$$(x')^j = x^j + a^j \Longrightarrow \Lambda = \mathbb{1} \Longrightarrow g' = g = \mathbb{1},$$

It leaves g invariant. This reflects the homogeneity of Euclidean space. Because of that, the components of vectors are invariant under translations: $(v')^j = v^j$.

This is what is meant by saying that vectors in Cartesian do not have a location.

Rotation keeps the origin fixed. This is also the case for general linear transformation,

$$(x')^j = \Lambda^j{}_a x^a \tag{1.17}$$

A linear transformation is a symmetry provided it leave the metric invariant. In Cartesian coordinates $g = \mathbb{1}$ and by Eq. (1.2)

$$\mathbb{1} = (g') = \Lambda^t g \Lambda = \Lambda^t \mathbb{1} \Lambda = \Lambda^t \Lambda$$

This says that Λ^t is the inverse of Λ:

$$\Lambda^t \Lambda = \mathbb{1}$$

which is the standard definition of orthogonal transformation in Cartesian coordinates. It follows that

$$\det \Lambda^2 = 1 \Longrightarrow \det \Lambda = \pm 1$$

Orthogonal transformations are associated with two types of symmetries of the Euclidean space: When $\det \Lambda = 1$ they represent rotations. When $\det \Lambda = -1$ they represent mirror symmetries.

Example 1.26 (Rotations)**.** *In three dimensions, rotation by θ about the x^3 axis is given by*

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Evidently, the inverse is the transpose

$$R(-\theta) = R^t(\theta) \tag{1.18}$$

Example 1.27. *In a two dimensional Euclidean space there are two second rank tensors that are invariant under rotations*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.19}$$

1.6 Tensorial equations are coordinate free

The nice thing about tensor equations is that once

$$T^{jk\cdots} = 0$$

holds in one (fixed) coordinate system, it hold in any other coordinate system. For example, Newton's equation

$$f^j = ma^j$$

is a tensor equation relating force and acceleration (m is a scalar). If it holds in one coordinate system it hold in any other.

1.7 Differential operators

The equations of motions of fields are partial differential equations. This forces us to mind how differential operators behave under change of coordinates. The general case is complicated and one needs to introduce the notion of covaraint derivatives. To see where the complication comes from consider the covaraint components of a vector field A_j in Euclidean space. Then, in the exercise 1.28 you are requested to show that $\partial_j A_k$ transforms like a second rank tensor provided x' is a linear function of x.

Exercise 1.28. *Show that under a change to general curvilinear coordinates*

$$\partial_j A_k = \Lambda^a{}_j \Lambda^b{}_k \, (\partial'_a A'_b) + \left(\partial^2_{jk}(x')^a\right) A'_a$$

A simplification however occurs for certain differential operators, and in particular for the three differential operators we need for Maxwell's equations: grad, div and curl. The following exercise shows where this simplification comes from

Exercise 1.29. *Show that the anti-symmetric second rank tensor*

$$F_{jk} = \partial_j A_k - \partial_k A_j$$

transforms like a tensor under general curvilinear coordinate transformations.

1.7.1 Grad

The chain rule

$$\frac{\partial}{\partial(x')^j} = \frac{\partial x^k}{\partial(x')^j}\frac{\partial}{\partial x^k} = \Lambda^k{}_j \frac{\partial}{\partial x^k}$$

says that partial derivatives behave like *covariant* components of a vector. In particular, if $\phi(x)$ is a scalar valued function then $\nabla\phi$ give the components of a covariant vector field.

Exercise 1.30. *Show that $\nabla\phi$ in spherical coordinates is*

$$\begin{aligned}\nabla\phi &= (\partial_r\phi)\,\mathbf{e}^r + (\partial_\theta\phi)\,\mathbf{e}^\theta + (\partial_\varphi\phi)\,\mathbf{e}^\varphi \\ &= (\partial_r\phi)\,\hat{\mathbf{r}} + \frac{\partial_\theta\phi}{r}\,\hat{\theta} + \frac{\partial_\varphi\phi}{r\sin\theta}\,\hat{\varphi}\end{aligned}$$

Here you see why covariant components often lead to simpler formulas than normalized coordinates.

1.7.2 Div

We shall show that

$$\nabla\cdot\mathbf{E} = \frac{1}{\sqrt{g}}\partial_j(\sqrt{g}E^j),$$

The formulas clearly hold in Cartesian coordinates where g is the identity.

The defining property of the divergence is Gauss law

$$\int_{Volume} dV(\nabla \cdot E) = \int_{surface} dS \cdot E$$

Consider a smal cube in the coordinates dx^j. The putative expression for div indeed satisfies Gauss law

$$\begin{aligned}
dV(\nabla \cdot E) &= \sqrt{g} dx^1 dx^2 dx^3 (\nabla \cdot E) \\
&= dx^2 dx^3 \sqrt{g}\, E^1 \Big|_i^f + \dots \\
&= dx^2 dx^3 \sqrt{g} \mathbf{e}^1 \cdot E^1 \mathbf{e}_1 \Big|_i^f + \dots \\
&= \underbrace{\sqrt{g}\, dx^2 dx^3 \mathbf{e}^1}_{dS} \cdot \mathbf{E} \Big|_i^f + \dots \\
&= d\mathbf{S} \cdot \mathbf{E}
\end{aligned}$$

Example 1.31. *In spherical coordinates div is*

$$\begin{aligned}
\nabla \cdot E &= \frac{1}{r^2 \sin\theta} \left(\partial_r(r^2 \sin\theta E^r) + \partial_\theta(r^2 \sin\theta E^\theta) + \partial_\phi(r^2 \sin\theta E^\phi)\right) \\
&= \frac{1}{r^2} \partial_r(r^2 E^r) + \frac{1}{\sin\theta} \partial_\theta(\sin\theta E^\theta) + \partial_\phi(E^\phi) \\
&= \frac{1}{r^2} \partial_r(r^2 E_{\hat{r}}) + \frac{1}{r \sin\theta} \Big(\partial_\theta(\sin\theta E_{\hat{\theta}}) + \partial_\phi(E_{\hat{\phi}})\Big)
\end{aligned}$$

The last line is in terms of the normalized coordinates.

1.7.3 Curl

The last differential operator we shall need to discuss is the curl:

$$(\nabla \times E)^i = \frac{\varepsilon^{ijk}}{\sqrt{g}} \partial_j E_k$$

The formula is evidently the standard definition in Cartesian coordinates. To see why it is true in general we take Stokes law as defining property of the curl:

$$\int dS \cdot (\nabla \times \mathbf{E}) = \int d\ell \cdot \mathbf{E}$$

The putative formula for curl indeed gives Stokes for dS a small square $dx^1 \times dx^2$.

$$\begin{aligned}
dS \cdot (\nabla \times E) &= dS_3 \, (\nabla \times E)^3 \\
&= \left(\sqrt{g} dx^1 dx^2\right) \left(\frac{\varepsilon^{3ij}}{\sqrt{g}} \, \partial_i E_j\right) \\
&= dx^1 dx^2 (\partial_1 E_2 - \partial_2 E_1) \\
&= dx^2 E_2 \Big|_i^f - \dots \\
&= (dx^2 \mathbf{e}_2) \cdot (E_2 \mathbf{e}^2) \Big|_i^f - \dots \\
&= (dx^2 \mathbf{e}_2) \cdot \mathbf{E} \Big|_i^f - \dots \\
&= d\ell \cdot \mathbf{E}
\end{aligned}$$

Example 1.32. *The ϕ components of curl in spherical coordinates* [2] *(recall Remark1.19) is :*

$$(\nabla \times \mathbf{E})^{\varphi} = \frac{1}{r^2 \sin\theta} \left(\partial_\theta E_r - \partial_r E_\theta\right)$$

and in normalized components

$$\begin{aligned}
(\nabla \times \mathbf{E})_{\hat{\varphi}} &= \frac{1}{r} \left(\partial_\theta E_r - \partial_r E_\theta\right) \\
&= \frac{1}{r} \left(\partial_\theta E_{\hat{r}} - \partial_r (r E_{\hat{\theta}})\right)
\end{aligned}$$

Exercise 1.33. *Compute the $(\nabla \times \mathbf{E})^r$ and $(\nabla \times \mathbf{E})_{\hat{r}}$.*

Exercise 1.34 (Vector identities)**.** *Show the vector identities*

$$\begin{aligned}
\nabla \times (\nabla \phi) &= 0 \\
\nabla \cdot (\nabla \times \mathbf{E}) &= 0
\end{aligned}$$

1.7.4 Laplacian

The Laplacian of a scalar function is defined by

$$\Delta \phi = \nabla \cdot \nabla \phi = \frac{1}{\sqrt{g}} \partial_j (\sqrt{g} g^{jk} \partial_k \phi) = \frac{1}{\sqrt{g}} \partial_j (\sqrt{g} \, \partial^j \phi)$$

and for a vector field by

$$\nabla \times (\nabla \times \mathbf{E}) = -\Delta \mathbf{E} + \nabla (\nabla \cdot \mathbf{E})$$

[2] Note that Wolfram Mathematica notation compares with mine by interchanging $\varphi \leftrightarrow \theta$

Example 1.35. $\Delta\phi$ *in spherical coordinates:*

$$\begin{aligned}\Delta\phi &= \frac{1}{r^2\sin\theta}\left(\partial_r(r^2\sin\theta\partial_r\phi) + \partial_\theta\left(r^2\sin\theta\,\frac{\partial_\theta\phi}{r^2}\right) + \partial_\varphi\left(r^2\sin\theta\frac{1}{r^2\sin^2\theta}\partial_\varphi\phi\right)\right)\\ &= \frac{1}{r^2}\partial_r(r^2\partial_r\phi) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\phi\right) + \frac{1}{r^2\sin^2\theta}\partial_{\varphi\varphi}\phi\end{aligned}$$

1.8 Bibliography

- S. Weinberg, Gravitation and Cosmology, gives all a physicist needs to know about tensors.
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Chapter 2

Minkowski space-time

Minkowski space-time gives a geometric description of special relativity. It encapsulates the facts: The velocity of light is the same in all inertial frames; Every inertial describes a homogeneous space-time; Every inertial frame frame admits a fixed time slice which is a Euclidean 3 dimensional space; and clocks can be synchronized in any fixed inertial frame (but disagree between frames).

2.1 The principle of relativity

Physicists unlike, say, lawyers, do not need to replace their textbooks when they relocate. Your physics library would still be useful even if you relocated to a different galaxy, receding from earth at large speed. You do not need to make an adjustment for the relative motion between earth and your new home (provided space-time is locally the same). Empty space, the vacuum, has no distinguished inertial frame; none distinguished as being at rest and none whose origin is the center of the universe.

The speed of light c is a property of the vacuum. It is a scalar, a constant of nature, which takes the same value in different inertial frames. This property of light was established around 1887 in experiments of Michelson and Morley. It conflicts with our common intuition about adding velocities much smaller than c.

Remark 2.1 (c is large)**.** *Human length scale, say the length of the foot, is* $\ell = 1[meter]$*. When you walk the foot behaves like a pendulum, and its period is* $2\pi\sqrt{\ell/g} \approx 2\ [s]$ *with* $g \approx 9.8\ [m/s^2]$*. A human speed is then* $\approx 1\ [m/s]$*. On this scale* $c \approx 3 \times 10^8\ [m/s]$ *is essentially infinite. I do not know who first entertained the thought that c may be finite but large, but the first to estimate c from astronomical data was the Danish astronomer Rømer (1644-1710) .*

2.1.1 Causality

Causality is a way to distinguish past from future: We remember the past but can't change it and we can affect the future but do not know it. When one takes into account the fact that c is a constant of nature, one finds that one needs to reconsider what one means by the notion of "future" , "now" and "past". Einstein said that if you could break the speed of light, you could use this to send a message to your dead grandmother. An alternative characterization of c is therefore the ultimate speed at which information propagates.

2.2 Space-time

Space-time is the stage on which events happen. An event, like my typing this text, is something that happens in space and time and is labeled by 4-coordinates x^μ, $\mu \in 0, 1, 2, 3$ where $x^0 = ct$ with t time. It is natural to give space and time the same dimension. This is what we do when we say that the sun is about 8 light-minutes away from earth.

Remark 2.2. *We shall use the conventions that Greek indices* μ, ν *run from* 0 *to* 3*, while Roman indices* j, k *from 1 to 3.*

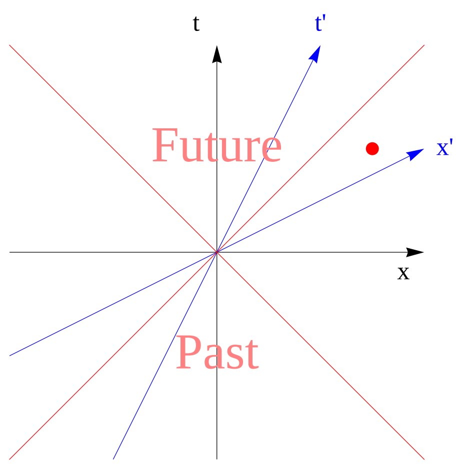

Figure 2.1: Space-time: The red disk is an event. The t axis is the world line of a black clock that sits at the origin $x^1 = x^2 = x^3 = 0$. The x axis represents a Euclidean 3-space. The blue line is the world line of a blue clock moving at constant speed $c/2$. In the frame of the blue clock t' is the time axis. The red lines represent the future and past light cones relative associated with a signal that is emitted (or absorbed) at the origin. The Euclidean distance on the paper reflects badly $(d\ell)^2$ in Minkowski space.

The "distance squared" between two nearby events in space-time is called *interval* and is given by

$$(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu$$

with g a real, symmetric tensor. The notation is taken from geometry where $(ds)^2 \geq 0$ but is potentially misleading because we allow $(ds)^2$ to have either sign: When the two events are simultaneous in an inertial frame the interval is indeed the distance squared and $(ds)^2 > 0$. When two events are separated in time, but occur at the same point the interval is the time elapsed on a clock squared, but $(ds)^2 < 0$. Let us motivate this choice of signs.

We can always choose coordinates so that g is diagonal at a given point. (Take Λ an orthogonal transformation.) By scaling we can then make the diagonal entries ± 1. If all the entries are $+1$ we have a something that looks locally like 4-dimensional Euclidean space, not space-time. Space-time has three spatial coordinates, naturally associated with the three entries $+1$. The remaining time coordinate is different. It comes with the -1. This gives a 3+1 space-time manifold[1] .

Definition 1. *A space-time is Minkowski if it admits coordinates so that $g = \eta$ everywhere with*

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.1}$$

A time slice $x^0 = const$ gives a Euclidean 3-space: homogeneous and isotropic. Minkowski space-time is a good local approximation of physical space-time. It fails at cosmological distances and at distant times or when great accuracy, such as in GPS, is needed. Asher Peres said that physics is not an exact science but the science of approximation. A more accurate model of space-time in the vicinity of star (planet) of mass M is

$$(ds)^2 = -(1-\Phi)(cdt)^2 + (1+\Phi)\, d\mathbf{x} \cdot d\mathbf{x}, \quad \Phi(\mathbf{x}) = \frac{2GM}{c^2 r}$$

G is Newton constant. $(t, \mathbf{x})$ are space-time coordinates. A clock at a fixed location ticks at rate

$$d\tau = \sqrt{1+\Phi}\, dt$$

Far from the star, the clock rate coincides with the coordinate time rate.

Exercise 2.3. *Compute Φ at the surface of the earth. (Answer: $\Phi = 1.4 \times 10^{-9}$)*

Exercise 2.4. *Compute the correction terms to Minkowski, $\frac{2V}{c^2}$ on earth due the sun. (Answer: $\Phi = 2 \times 10^{-8}$)*

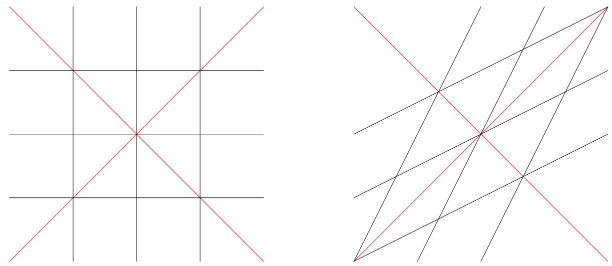

Figure 2.2: Two Cartesian coordinates in Minkowski space. In both the t axis and the x axis are Minkowsi orthogonal: The light-cone (red) bisects the angles between the t and x axes. If you have installed Mathematica cdf player you can view a simulation courtesy of Slava Pollak.

2.2.1 Cartesian coordinates and Inertial frames

A given Euclidean space accommodates many Cartesian coordinate system. In all of them the metric $g = \mathbb{1}$. Similarly, Minkowski space accommodates many coordinate systems where $g = \eta$. A coordinate system where $g = \eta$ is called *Cartesian*. As we shall different inertial frames are associated with different Cartesian coordinates.

In Cartesian coordinates the (unit) vector tangent to the wold line at the origin, $\mathbf{t} = (1, 0, 0, 0)$, is Minkowski orthogonal to the space-like vector $\mathbf{x} = (0, 1, 0, 0)$. Now consider the blue (unit) vector tangent to the world line of an inertial observer moving at velocity $|v| < c$ (in the $\mathbf{x}$ direction) of Fig. 2.1. Its contravariant components (in the black frame) are

$$\mathbf{t}' = \gamma(1, v/c, 0, 0), \quad \gamma = \frac{1}{\sqrt{1-(v/c)^2}} \geq 1$$

What is the direction of the $\mathbf{x}'$ of the blue inertial frame? Since the blue frame is inertial, $\mathbf{x}'$ and $\mathbf{t}'$ must be Minkowski orthogonal and the contravariant components of $\mathbf{x}'$ must be

$$\mathbf{x}' = \gamma(v/c, 1, 0, 0) \Longrightarrow t'_\mu x'^\mu = \gamma^2(v/c - v/c) = 0$$

Example 2.5. *In spherical coordinates (ct, r, θ, ϕ) Minkoski metric is*

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{2.2}$$

[1]This would also be the case with three -1 and one $+1$.

2.2.2 Events, world lines, light cones, etc.

An event is an objective happening and does not depend on the choice of coordinates. However, its representation in terms of coordinates $x = (x^0, x^1, x^2, x^3)$ depends of the choice of coordinates.

A line in space-time is called a world line. For example, the collection of events associated with an observer—a clock moving at subluminal speed—make a world line.

The light cone is a three-dimensional (conic) surface in space-time. It describes the collection of events x related to a given event, x_0, by a light signal. If we denote $dx = x - x_0$ then dx is a vector with zero length

$$0 = d\mathbf{x} \cdot d\mathbf{x} - c^2(dt)^2 = \eta_{\mu\nu} dx^\mu dx^\nu = dx_\mu dx^\mu$$

Light-cone is made of vectors whose Minkowski length is *zero*.

Definition 2. *A vector dx is called space like if $dx_\mu dx^\mu > 0$; It is called time-like if $dx_\mu dx^\mu < 0$ and light-like if $dx_\mu dx^\mu = 0$*

If the vector connecting two events is time-like, there is a clock, moving at subluminal speed, whose world line passes through the two events. The time elapsed on *this* clock $d\tau$ measures the interval

$$(cd\tau)^2 = -(d\ell)^2 = -dx_\mu dx^\mu \tag{2.3}$$

Clearly it does not depend on the choice of coordinates. $d\tau$ is known as the *proper time*.

Exercise 2.6 (Measuring space like intervals with a clock)**.** *To measure space-like intervals you would normally use meter sticks. Wigner found a clever trick to measure space like intervals using a single clock and two light signals connecting the clock with the space-like event. This is illustrated in Fig. 2.3. The interval is the product of two time intervals a and b measured by an inertial clock.*

$$(x_O - x_S)^\mu (x_O - x_S)_\mu = ab$$

2.3 Everything is relative

The ct axis represents the world line of a clock at rest at the origin of space: $x^j = 0$. $x^j = 0$ is 'here' for the clock and the world line is $(ct, 0, 0, 0)$. A blue clock is moving at constant velocity v whose world line is $(ct, vt, 0, 0)$. This world line defines ct' axis in Fig. 2.4. It is the origin the blue inertial frame, $(x')^j = 0$. The notion of 'here' is different for the two clocks: Being at the same place is not objective but frame (and coordinate) dependent.

Now is relative

Fig. 2.4 shows that notion of now is relative. In the black coordinate system events separated by $\mathbf{x}$ occur simultaneously, in the blue ones events separated by $\mathbf{x}'$ are simultaneous.

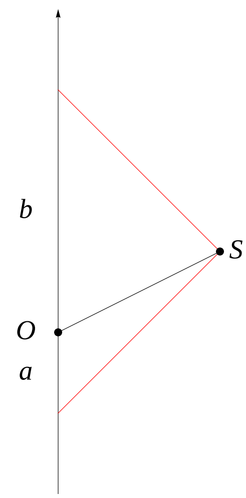

Figure 2.3: The interval between the two space like events O and S is related to the clock readings of an inertial clock. The red lines in the figure are light-like vectors and a and b are the time intervals measured by the clock at O.

2.3.1 Time dilation

When you meet your high school buddies after a long time you usually note how everybody aged and wonder, did I too age that much. This is psychology, and I have nothing to say about it. But there is an analog objective property of time: your clock is slowest: . If you travel fast enough and far enough when you come back, all your friends will be dead.

Consider the interval $d\tau$ between the two time ticks of the black clock. The time registered on a single physical clock is called *proper time*. It is represented by the (cyan) vector $(cd\tau), 0)$ in Fig. 2.5. In the blue frame the clock is moving at v. The cyan vector has blue coordinates

$$dx' = -vdt' \Longrightarrow (c, -v)dt'$$

Since the interval is a scalar, $d\tau$ and dt' are related

$$(c\,d\tau)^2 = (c^2 - v^2)(dt')^2 = \frac{(c\,dt')^2}{\gamma^2},$$

where v is the (instantaneous) velocity of the clock. Proper time is the smallest time γ is the ratio between increments in coordinate-time dt and proper time $d\tau$:

$$\gamma = \frac{dt}{d\tau}$$

Exercise 2.7. *Suppose that you have a factory at the origin that makes identical clocks. Explain how you can distribute the clocks while keeping them synchro-*

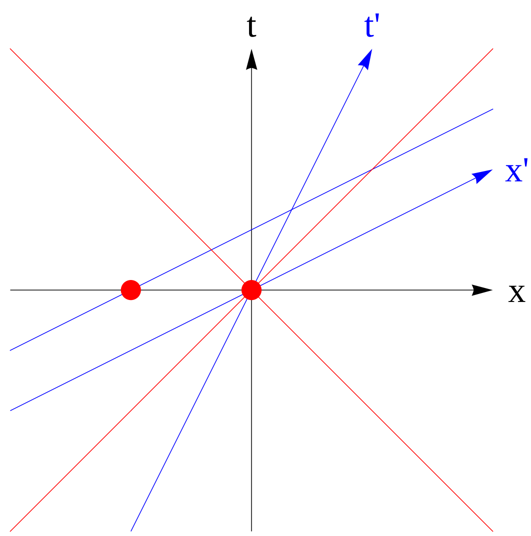

Figure 2.4: The black coordinate system is inertial. So is the blue coordinate system. In both the t and x axes are Minkowski orthogonal. The two red dots are two events which are simultaneous in the black inertial frame but not simultaneous in the blue frame. The light cone is a coordinate independent entity.

nized. (Hint: What happens to time delay if you half the speed and double the travel time?)

When Paul Krugman, A Nobel Laureate in economics, was a young assistant professor he wrote funny article where he applies time-dilation to economics. A pleasant diversion.

2.3.2 Length contraction

Consider a rod at rest in the black frame whose length there is ℓ. In the blue frame the rod is moving at velocity v. A measurement of the rod in the blue frame must be made simultaneously at the two ends. The difference of the two events is represented by the vector $(0, dx')$, the green interval in Fig. 2.6.The t' axis direction in the black coordinates is $(1, v/c)$. The green vector pointing in the x' direction has black coordinates

$$(v/c, 1)\ell$$

(since the length of the rod, measured by x^1, in the frame where it is stationary is ℓ). The interval, being a scalar, relates the proper length ℓ with the apparent

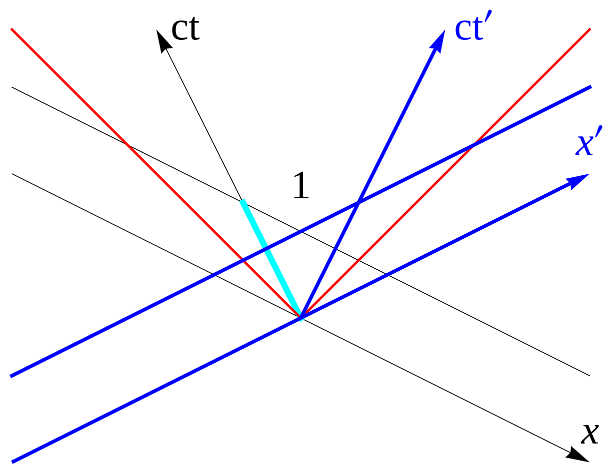

Figure 2.5: The blue frame moves to the right at u and the black frame moves to the left at $-u$. Because of the symmetry, the length on the paper along $\mathbf{t}$ and $\mathbf{t}'$ correctly reflect the interval. The thick cyan line is one second measured by a black clock. Its end point is on the line $t = 1$. The line of events $t' = 1$ in the blue frame intersects the cyan line: The moving clock is slow– Time dilates.

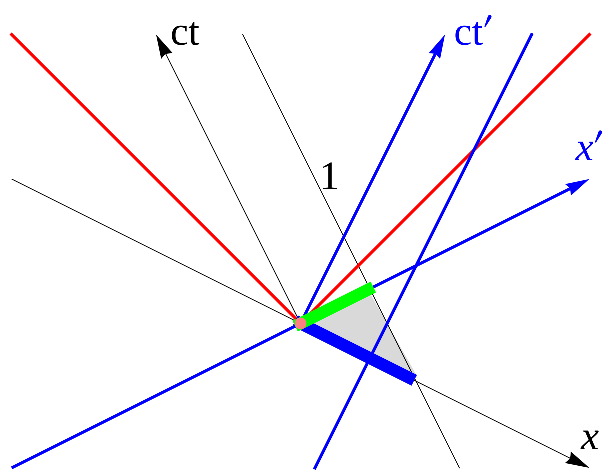

Figure 2.6: The thick blue interval is a meter stick at rest in the black frame. The black frame moves left and the blue frames righht at equal speeds. This makes the Euclidean lengths along the x and x' axes proportional to the interval. The blue thick interval is longer than the green interval: The meter stick has shorter length in the blue frame. Moving objects contract.

length dx'

$$(dx')^2 = \frac{\ell^2}{\gamma^2}, \quad \gamma \geq 1$$

You are biggest in your own rest frame.

2.3.3 Other coordinates

Rotating frames

The rotating earth is a non-inertial frame. What is the structure of space-time in such a frame. Let Ω be the angular frequency and $(ct', \mathbf{x}')$ be the coordinate of an inertial fame and $(ct, \mathbf{x})$ the coordinates in a rotating frame. In cylindrical coordinates the inertial and rotating frames are related by

$$t' = t, \quad \rho' = \rho, \quad z' = z, \quad \phi' = \phi + \Omega t$$

The Euclidena metrics are the related by

$$\begin{aligned} d\mathbf{x}' \cdot d\mathbf{x}' &= (d\rho')^2 + (dz')^2 + \rho'^2 (d\phi')^2 \\ &= (d\rho)^2 + (dz)^2 + \rho^2 (d\phi + \Omega dt)^2 \\ &= d\mathbf{x} \cdot d\mathbf{x} + 2\rho^2 \Omega d\phi dt + \rho^2 \Omega^2 (dt)^2 \end{aligned}$$

Consequently, the Minwoski metric in the inertial frame is

$$(-cdt')^2 + d\mathbf{x}' \cdot d\mathbf{x}' =$$
$$-(dt)^2 \Big(c^2 - \underbrace{\Omega^2 \rho^2}_{centrifugal} \Big) + \underbrace{2\Omega\rho^2 \, dt d\phi}_{Sangac\ effect} + d\mathbf{x} \cdot d\mathbf{x}$$

Exercise 2.8. *Earth actually rotates pretty fast. To get and idea compute $\Omega\rho/c$ at the equator .* (Answer: 1.5×10^{-6})

In the case of earth the centrifugal correction can normally be neglected, but the Sangac term is important and to a good approximation

$$-(cdt)^2 + 2\Omega\rho^2 \, dt d\phi + d\mathbf{x} \cdot d\mathbf{x} \tag{2.4}$$

Exercise 2.9 (Coordinate times and clock times)**.**

1. *Compare the change in coordinate time dt in the rotating earth frame with the proper-time $d\tau$ measured by a clock at a fixed location in the rotating frame*

2. *Compare the change in coordinate time dt' in the inertial frame with the change in coordinate time dt in the rotating coordinates*

3. *A clock is taken for a one year trip around earth equator. Show that the time lag relative to a clock that stayed is*

$$\Delta\tau = \pm \frac{2A\Omega}{c^2}, \quad A = \pi R_e^2$$

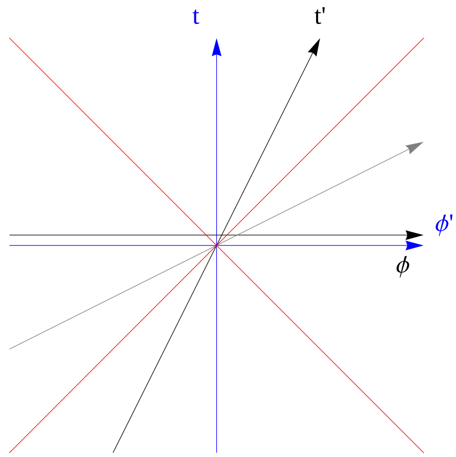

Figure 2.7: The coordinates in the inertial frame are drawn blue and the coordinates in the rotating frame black. The light cone in red. The gray line is the synch line in a local inertial frame.

where R_e is the earth radius and the $\pm$ depends on whether the trip was towards the east or towards the west.

4. *Compute $\Delta\tau$. (Answer: 207 [ns])*
5. *Explain why the result implies that one can not synchronize clocks on earth.*
6. *Is it still true that $u_\mu u^\mu = -c^2$ in a rotating frame?* Yes)
7. *What does get modifies is γ. Show that*

$$\gamma^{-2} = 1 - (\mathbf{v}/c)^2 + 2\Omega\dot{\phi}\rho^2/c^2 - \Omega^2\rho^2/c^2$$

Light-cone coordinates

Light cone coordinates in Minkowski space are

$$\sqrt{2}u = x - ct, \quad \sqrt{2}v = x + ct, \quad y = y, \quad z = z$$

Exercise 2.10 (Metric in light cone coordinates)**.** *Show that the Minkowski metric tensor in light-cone ordinates is*

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

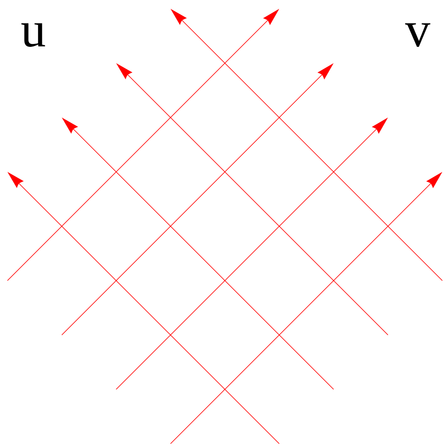

Figure 2.8: Light-cone coordinates. This is not a Cartesian frame since $\mathbf{u}\cdot\mathbf{v} = 1$.

Rindler coordinates

Rindler coordinates are the analog of polar coordinate system in a two dimensional Minkowski space time

$$x = \rho \cosh \tau, \quad t = \rho \sinh \tau$$

Note that with $\rho \geq 0$ the coordinates cover 1/4 of space time.

Exercise 2.11. *Show that*

$$(dx)^2 - (dt)^2 = (d\rho)^2 - \rho^2 (d\tau)^2$$

2.4 Lorentz transformations

Definition 3 (Lorentz transformations are isometries)**.** *We call Λ a Lorentz transformation if it leaves the Minkowski metric η invariant*

$$\eta'_{\mu\nu} = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta \eta_{\alpha\beta} = \eta_{\mu\nu}$$

As an equation for matrices this reads

$$\eta' = \Lambda^t \eta \Lambda = \eta \tag{2.5}$$

In particular, for any Lorentz transformation

$$\det \Lambda = \pm 1$$

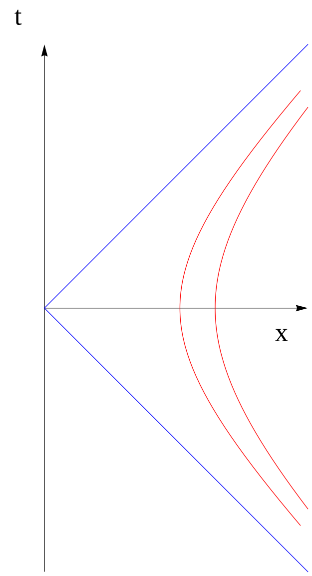

Figure 2.9: The red lines shows the ρ Rindler coordinate.

2.4.1 Space-time translations

Space time translations are given by

$$(x')^\mu = x^\mu + a^\mu$$

This gives

$$\Lambda^\mu{}_\nu = \frac{\partial (x')^\mu}{\partial x^\nu} = \delta^\nu_\mu \Longleftrightarrow \Lambda = \mathbb{1}$$

which is the trivial Lorentz transformations. This expresses the homogeneity of Minkowski space time.

2.4.2 Generators of Lorentz transformations

Suppose L is a 4×4 real matrix so that

$$\eta L + L^t \eta = 0 \tag{2.6}$$

We can use L to generate a one parameter family of 4×4 matrices $\Lambda(t)$ as a solution of the differential equation

$$\dot{\Lambda}(t) = L\Lambda(t), \quad \Lambda(0) = \mathbb{1}$$

The $\Lambda(t)$ genertaed in this way is a Lorentz transformation for any t. This follows from the following exercises

Exercise 2.12. *Use Eq. 2.6 and the differential equation to show that* $\det \Lambda(t) = 1$.

Exercise 2.13. *Show that*

$$\eta(t) = \Lambda(t)^t \eta \Lambda(t)$$

is t independent and hence $\eta(t) = \eta(0) = \eta$. (Hint: Differentiate)

Exercise 2.14. *Show that:*

1. *if λ is an eigenvalue of Λ so is λ^*.*

2. *If λ is an eigenvalue so is* $1/\lambda$ (Hint: Use $\Lambda^{-1} = \eta\Lambda^t\eta$ to show that $\det(\Lambda - \lambda) = \det(\Lambda^{-1} - \lambda)$)

As we shall now see, there are 6 (linearly independent) generators of Lorentz transformations: 3 are associated with (Euclidean) rotations and 3 with boosts that relate different inertial frames.

2.4.3 Rotations

Rotation by θ about the x-axis and its generator are given by

$$\Lambda_{yz}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \quad L_{yz} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Similarly for rotations about the x, y space axes. The isometry expressses the isotropy of space in of Minkowski space-time.

Remark 2.15. *An airplane has three rotation controls: Stick, for pitch, rudder for yaw, and ailerons for roll. The three are* lineraly *independnet, but non-linearly dependent: You can always generate one from the other two.*

Exercise 2.16. *Show that*

$$[L_{yz}, L_{zx}] = -L_{xy},$$

Exercise 2.17. *Calculate the residual rotation of (pairwise-cancelling) rotations*

$$\Lambda_{yz}(\pi/2)\Lambda_{zx}(\pi/2)\Lambda_{yz}(-\pi/2)\Lambda_{zx}(-\pi/2)$$

Show that this is a rotation about the $(-1, 1, 1)$ *axis.*

2.4.4 Boosts

Lorentz transformations relating different inertial frames are called as boosts. A boost in the x direction and its generator are

$$\Lambda_{tx}(\phi) = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_{tx} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.7}$$

If you remember your quantum mechanics, then an easy way to see that Eq. (2.6) is to recall that for Pauli matrices $\sigma_x\sigma_z + \sigma_z\sigma_x = 0$. and so are isometries of Minkowski space. The isometry expresses the fact that Minkowski space looks the same in all inertial frames.

Exercise 2.18. *Show that the commutator of two boosts is a rotation:*

$$[L_{tx}, L_{ty}] = L_{xy},$$

What can yous conclude from that? (Thomas precession).

2.4.5 Rapidity

ϕ of Eq. (2.7) is called the *rapidity*. To related the rapidity to the relative velocity between the frames consider the origin in the primed frame $x' = 0$.

$$0 = (x')^1 = \Lambda^1{}_\mu x^\mu = -(\sinh\phi)\, ct + (\cosh\phi)\, x$$

From this we conclude that

$$v = c\tanh\phi \tag{2.8}$$

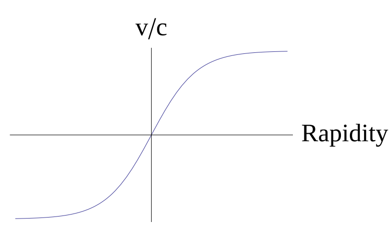

Figure 2.10: The velocity as function of the rapidity

Exercise 2.19 (Rapidity add)**.** *Show that rapidities (in the same direction) add*

$$\Lambda(\phi_1)\Lambda(\phi_2) = \Lambda(\phi_1 + \phi_2)$$

Exercise 2.20 (Galilean transformations)**.** *Show that for small rapidities Lorentz boosts reduce to Galilean transformation:*

$$t' = t + O(c^{-2}), \quad x' = x - vt + O(c^{-2})$$

2.5 4-Velocity

The proper time $d\tau$ is a Lorentz scalar and is non-zero for a clock that travels slower than light. In this case we can define the velocity as a 4-vector

$$u^\mu = \frac{dx^\mu}{d\tau} \tag{2.9}$$

The length of u is always -c^2 , essentially by definition,

$$u_\mu u^\mu = \frac{dx_\mu dx^\mu}{(d\tau)^2} = -c^2 \tag{2.10}$$

The 4-velocity is a time-like vector. It lies in the forward light cone. It is related to the usual velocity by

$$(c, v) = \frac{dx^\mu}{dt} = \frac{dx^\mu}{d\tau}\frac{d\tau}{dt} = \frac{u^\mu}{\gamma}$$

Remark 2.21 (Old fashioned velocities)**.** *Whereas the 4-velocity transforms like any 4-vector under Lorentz transformations, the usual velocity has complicated (bad) transformation properties. This is because both the numerator and the denominator are components of a vector.*

It is convenient to think of the path $x^\mu(\tau)$ as parametrized by proper time. Since the 4-velocity is normalized we can always write it as

$$u = c\,(\cosh\phi, \mathbf{n}\,\sinh\phi), \quad \mathbf{n}\cdot\mathbf{n} = 1$$

$\mathbf{n}$ is the direction, which may depend on τ and $\phi = \phi(\tau)$. Evidently

$$\gamma = \cosh\phi$$

from which is follows that ϕ is the rapidity defined in the previous section as $v = c\tanh\phi$.

Exercise 2.22. *Show this*

2.5.1 4-momentum

Define the 4-momentum

$$p^\mu = mu^\mu = m\gamma(c, \mathbf{v}) = (E/c, \mathbf{p})$$

The associated scalar is

$$p_\mu p^\mu = -(mc)^2$$

It is a time-like vector for massive particles. This encapsulates the most famous equation in physics, Einstein equation for the energy in the rest frame, $E = mc^2$.

Remark 2.23 (Tachyons)**.** *Tachyons are (fictitious) particles with whose mass is (fittingly) imaginary. The 4-velocity is space like, $u_\mu u^\mu = (mc)^2 > 0$. A Tachyon can not be a point particle: there is a Lorentz frame where the particle is spread on a line at fixed t. If Tachyons would interact with ordinary particles (tardyons), you could go back in time and kill your grandfather.*

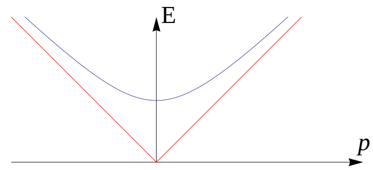

Figure 2.11: The dispersion relation, energy as function of momentum, for a massive free particle.

2.5.2 4 Acceleration:

The 4-acceleration can be similarly defined as

$$a^\mu = \frac{du^\mu}{d\tau} \tag{2.11}$$

It is always Minkowski orthogonal to the velocity

$$u_\mu a^\mu = 0 \tag{2.12}$$

(Since the Minkowski length of the velocity is -1). It follows that *The 4-acceleration is always a space like vector.*

Example 2.24 (Constant acceleration)**.** *The 4-velocity along some fixed direction is*

$$u = c\,(\cosh\phi, \mathbf{n}\,\sinh\phi)$$

where $\phi(\tau)$ is a parametrization of the path and $\mathbf{n}$ *is a fixed unit vector. The 4-acceleration is then*

$$a = c\,\dot\phi(\sinh\phi, \mathbf{n}\cosh\phi)$$

Evidently

$$a_\mu a^\mu = c^2\dot\phi^2$$

Constant acceleration g corresponds to linear dependence of ϕ on the proper-time τ

$$\phi(\tau) = \frac{g\tau}{c}$$

The red lines in Fig. 2.9 show paths of constant acceleration.

Remark 2.25 (Coincidence)**.** *An amusing phenomenological coincidence is that the year, the gravitational acceleration on earth g, and c are simply related:* $g \times year/c = 1.03$

Exercise 2.26. *What fraction of the velocity of light would you reach in this case.* Answer: $\tanh 1.03 = 0.77$

Exercise 2.27 (Space travel)**.** *You may worry that since c is the ultimate speed man, living for, say 80 years, can explore at most a smallish neighborhood of 80 light-years around earth. This is wrong. A space traveler who lives for n years, in a space ship which accelerates at g will travel a distances,* $\cosh n$, *measured in light years. The visible universe has radius of about* 10^{11} *light years. So you will get there is about 26 years.*

Example 2.28 (Motion with fixed velocity but changing direction)**.** *The 4-velocity of a particle moving with fixed velocity but changing direction is*

$$u^\mu = c(\cosh\phi, \mathbf{n}(\tau)\sinh\phi)$$

where ϕ is fixed. The acceleration is

$$a^\mu = c\sinh\phi(0, \dot{\mathbf{n}})$$

Exercise 2.29. *Find the orbit $x^\mu(\tau)$ describing a plane circular motion with constant angular velocity.*

2.5.3 Horizons

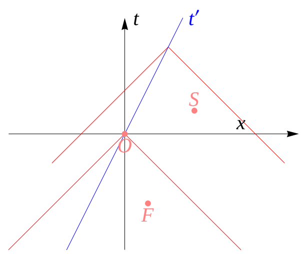

Figure 2.12: An inertial observer that lives long enough sees all the events in Minkowski space-time

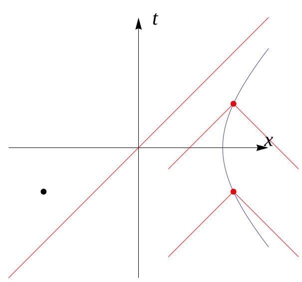

Figure 2.13: An accelerating sees only half the events in Minkowski space-time. He never sees the black dot on the left. The red line is his horizon.

2.6 GPS

Every time you use your GPS and find, with relief, that the GPS really knows where you are, you are testing special and general relativity. It took a century to turn Einstein's conceptual revolution into a palm gadget.

GPS works like that: There are about 24 GPS satellites orbiting earth at a radius of about $26,000\,[km]$ and period of about half a day. Their orbit are known (and monitored) with great precision (few centimeters). On each satellite there is an atomic clock that measures the proper time with great precision. Each satellite radios the coordinates of the event of transmission: The b satellite radios the event $X^\mu_b(\tau_b)$ and the event of reception if x^μ. Since the transmission is by electromagnetic wave that propagate at c, the two events are light-like. To determine the four unknown coordinates x^μ of teh reception event you need 4 equation. You need to see 4 GPS satellites and record 4 transmission events all light-like separated from you. This gives 4 equations with 4 unknowns.

The GPS system is sophisticated and involved: It takes into account special and general relativity; atmospheric effects on the veldocity off light, and the fact you insist on having your location in a non-inertial coordinate system attached to a rotation earth. The point I want to make here is that relativistic corrections are large: Ignoring relativity would degrade the the accuracy to about 10 km and make GPS useless. Relativity is regularly and routinely tested. If you want to know more about that, then the article of Neal Ashby in Living Reviews of General Relativity is a good place to learn. Wikipedia is, as usual, quite good as well.

Instead, I will consider a caricature of GPS to illustrate one idea. Namely, how one can use 4-clocks, with known position, and the fact the speed of light is a constant of nature, to determine the reception event in space time.

Exercise 2.30 (Orders of magnitudes)**.**

- *What is a typical velocity of GPS satellite? (Anser: $3.8\ [km/s]$)*
- *Compute the difference between the coordinate time and the self-time of a GPS clock after one day.* ($\Delta t \approx \pi R v/c^2 \approx 3.6\times 10^{-6}\ [s]$)
- *What is the resulting positioning inaccuracy?*

2.6.1 Two dimensional space time

Consider a toy GPS problem in 1+1 dimensions, shown in the Fig. 2.14. If you see 2 satellites in 1+1 dimensions, this means that one is on your right and the other on your left, as in Fig. 2.14. (Otherwise one would eclipse the other.) The light-cones intersect.

Exercise 2.31 (GPS in 1+1 dimensions)**.** *Two satellites with known orbits ,$\big(a^0_b(\tau), a^1_b(\tau)\big)$, $b = 1, 2$, emit signals at τ_a and τ_b respectively. Assuming that $a^\mu - b^\mu$ is space like, show that light-cone intersect at*

$$2x^1 = \pm(a^0_1 - a^0_2) + a^1_1 + a^1_2, \quad 2x^0 = (a^0_1 + a^0_2) \pm (a^1_1 - a^1_2)$$

(One solution is in the past and the other in the future.)

This toy model tests:

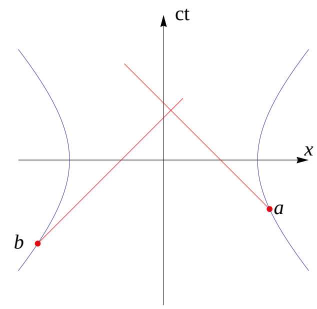

Figure 2.14: The world line of the two satellites are the blue lines. The intersection of the light cones is the event whose coordinates we seek. Since the orbits of the satellites are known, the events (a^0, a^1) and (b^0, b^1) are known given the proper times τ_a and τ_b.

1. Space-time is approximately Minkowski
2. Electromagnetic waves propagate at c
3. The velocity of the satellite at the transmission event is irrelevant
4. The velocity of the lost tourist at the reception event is irrelevant

2.6.2 3+1 dimensions

We have four satellites with known orbits $a_b^\mu(\tau),\ b \in 1,2,3,4$ as functions of their proper time. The GPS of the lost tourist receives the data $\tau_b,\ b \in 1,2,3,4$ and reveals the 4 events ${a_b}^\mu(\tau_b)$. All these events lie on the backward light cone of the event of reception x^μ. This makes all 6 pairs $a_b(\tau_b) - a_{b'}(\tau_{b'})$ space-like. (The satellites do not eclipse each other.) In a Minkowski space-time the event x^μ satisfy the 4 light-cone equations

$$\eta_{\mu\nu}(x^\mu - a_b^\mu)(x^\mu - a_b^\mu) = 0, \quad b \in 1,2,3,4$$

This is a system of (multivariate) quadratic equations.

Exercise 2.32. *Show that if the two events x_A^μ and x_B^μ both lie* on *the future light cone of the origin (i.e. both are light-like) then $x_A - x_B$ is space like (or light like).*

It can't be solved analytically, but Mathematica solves it numerically without a problem. If the data are physical, $a_b(\tau_b) - a_{b'}(\tau_{b'})$, are all space-like, its spits two events x^μ, one in the future and one in the past. The first is the physical one. If the data are not physical, x^μ will, in general be complex valued.

Remark 2.33. *If some of the satellites are close to eclipsing, it will be difficult to solve the equations accurately. You'd better look for a fifth satellite. Optimally, you'd want the "volume' of the space time tetrahedron $e^{\mu\nu\alpha\beta}(a_1^\mu - x^\mu)(a_2^\nu - x^\nu)(a_3^\alpha - x^\alpha)(a_4^\beta - x^\beta)$ to be large.*

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1. B. F. Schutz, A first course in general relativity, geometric exposition of relativity.
2. Neal Ashby in Living Reviews of General Relativity

Chapter 3

The electromagnetic tensor

In space-time the electric and magnetic fields $(\mathbf{E}, \mathbf{B})$ are amalgamated into a second rank, anti-symmetric, tensor, $F_{\mu\nu}$. $F_{\mu\nu}$ is derived from the 4-potential A_μ. This leads to the homogeneous Maxwell equations and a neat way to write Coulomb and Lorentz laws.

3.1 Amalgamating E and B

The basic objects of mechanics, such as velocity and accelerations can be viewed as the 3 dimensional shadows of 4-vectors in Minkowski space-time. In both cases the spatial 3-vectors, gained a zero component which then gives nice behavior under Lorentz transformations.

What about the electromagnetic fields **E** and **B**? What are they shadows of? From a Euclidean perspective **E** and **B** are different entities. This can be seen from their definition via the Coulomb Lorentz force law in the (defunct) c.g.s. (Gaussian) units which we shall use here:

$$\mathbf{f} = \underbrace{e\mathbf{E}}_{Coulomb} + \underbrace{e\frac{\mathbf{v}}{c} \times \mathbf{B}}_{Lorentz} \tag{3.1}$$

The electric field is that part of the force which is independent of the velocity of the particle and the magnetic field is the part of the force is linear in the velocity. Of course, the partition into electric and magnetic field is different in different inertial frames. Observers in different inertial frames do not agree on the partition. E and B mix under change of frame.

Remark 3.1 (SI). *Eq. (3.1) appears to say the magnetic forces are a relativistic correction to the electric forces. This is misleading because it depends on what values we take for the wo fields. In Gaussian units the unit of electric field is* 300 $[V/cm]$ *and the unit of magnetic field is Gauss. In SI units the unit of electric field is much five orders of magnitudes smaller,* 1 $[V/m]$*, and the unit* *of magnetic field is four orders of magnitude larger: Tesla* $= 10^4$ *Gauss. In SI units*[1] *the Coulomb-Lorentz law is*

$$\mathbf{f} = e\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

The force of electric field of $1\,[volt/cm]$ *and the magnetic force of* 1 $[Gauss]$ *have comparable magnitudes at velocities of* $1000\,[km/sec]$.

Exercise 3.2 (Galilei invariance)**.** *Galilean transformations are the non-relativistic limit of Lorentz transformations. In Newtonian mechanics the force is Galilean invariant:* $\mathbf{f} = \mathbf{f}'$ (Why)*. Show that the transformation rules for the fields under Galinean transformations with relative velocity* $\mathbf{v}$ *is*

$$\mathbf{E}' = \mathbf{E} + \frac{1}{c}\,\mathbf{v} \times \mathbf{B}\,, \quad \mathbf{B}' = \mathbf{B} \tag{3.2}$$

The mixing of $\mathbf{E}$ and $\mathbf{B}$ under the change of inertial frames suggests that they come from a single entity in space-time. This entity *is not* a 4-vector since we need 6 slots and a 4-vector has too few. It is not a general second rank tensor, since it has too many components–16. It is not a symmetric second rank tensor since this too has too many components–10. However an anti-symmetric rank 2 tensor

$$F_{\mu\nu} = -F_{\nu\mu} \tag{3.3}$$

has just the right number of components–6.

Exercise 3.3 (Symmetry invariance)**.** *Show that symmetry is a tensorial invariant, e.g. if* $F_{\mu\nu}$ *is anti-symmetric so is* $(F')_{\mu\nu} = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta F_{\alpha\beta}$ *under arbitrary change of coordinates (and Lorentz transformation in particular). As a consequence if* F *is anti-symmetric in Cartesian coordinates it is also anti-symmetric in spherical coordinates.*

This leaves us with the problem of how to put the two Euclidean vectors $(\mathbf{E}, \mathbf{B})$ in $F_{\mu\nu}$. We need to identify slots in F that behave like vectors. Assuming Cartesian coordinates, write

$$F_{\mu\nu} = \begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & 0 & y^3 & -y^2 \\ -x_2 & -y^3 & 0 & y^1 \\ -x_3 & y^2 & -y_1 & 0 \end{pmatrix} \tag{3.4}$$

We shall show that $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y^1, y^2, y^3)$ transform like Euclidean 3-vectors. (The reason for putting some indexes up and some down will become apparent below). This will leave us with the problem of matching the pair $(\mathbf{x}, \mathbf{y})$ with $(\mathbf{E}, \mathbf{B})$.

[1] And also if one takes units where $c = 1$.

3.1.1 Euclidean Rotations

Consider a 3×3 rotation matrix R of Euclidean space so $R^{-1} = R^t$. In Euclidean space contravariant components are the same as covariant components. It will be convenient to write the indexes as one of each

$$R\underbrace{{}_j}_{row}{}^{\overbrace{k}^{column}} = (R^t)^{\overbrace{k}^{row}}\underbrace{{}_j}_{column}$$

The lift of R to a rotation in Minkowski space is

$$\Lambda_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix}, \tag{3.5}$$

(The identity on the right expresses the fact that Euclidean rotations correspond to orthogonal transformation). The (covariant) components

$$F_{0j} = x_j$$

transfrom by

$$\begin{aligned} (x')_j = (F')_{0,j} &= \Lambda_0{}^\mu \Lambda_j{}^\nu F_{\mu\nu} \\ &= R_j{}^k F_{0k} \\ &= R_j{}^k x_k \end{aligned} \tag{3.6}$$

which is the rule of transformation of Euclidean 3-vectors under rotations.

Rotations of F_{jk}

Now consider the triplet $2y^j = \varepsilon^{jmn} F_{mn}$. The Levi-Civita symbol is invariant under (proper) rotations since $\det R = 1$. Hence

$$\begin{aligned} (2y')^j &= (\varepsilon')^{jmn} (F')_{mn} \\ &= \varepsilon^{jmn} R_m{}^a R_n{}^b F_{ab} \\ &= \varepsilon^{jmn} \varepsilon_{abk} R_m{}^a R_n{}^b y^k \\ &= (R^{-1})_k{}^j \,(2y)^k \end{aligned} \tag{3.7}$$

In the last step I used the formula for inverse of a 3×3 matrix R

$$(R^{-1})_k{}^j = \frac{1}{2 \det R}\, \varepsilon_{kab} \varepsilon^{jmn} R_m{}^a R_n{}^b$$

and the fact that $\det R = 1$ for a rotation. It remains to get rid of the inverse. To do that write

$$(R^{-1})_k{}^j = (R^t)_k{}^j = R_j{}^k$$

This gives

$$\begin{aligned}(2y')^j &= (R^t)_k{}^j\,(2y)^k \\ &= R^j{}_k\,(2y)^k\end{aligned} \tag{3.8}$$

One sees that **x** and **y** both transform as vectors. More precisely, **x** is a vector while **y** is a pseudo-vector as the transformation rule relied on the use of Levi-Civita (and $\det R = 1$).

We can use these facts to identify the slots for **E** and **B**. The force **f** is a vector. By the Coulomb law **E** must be a vector. In contrast, the Lorentz force involves a cross product, which uses a hand rule: **B** is a pseudo-vector.

3.1.2 The overall of sign

It remains to choose the overall sign. This depends on choice of the Minkowski metric[2]: If $\eta = (-1, 1, 1, 1)$. the right choice turns out to be (see the next section)

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B^z & -B^y \\ E_y & -B^z & 0 & B^x \\ E_z & B^y & -B^x & 0 \end{pmatrix} \Longleftrightarrow F_{k0} = E_k, \quad F_{jk} = \varepsilon_{ijk} B^i \tag{3.9}$$

Note that the Cartesian co and contravariant components of Euclidean vectors coincide.

Exercise 3.4 (Contravariant components)**.** *Verify that*

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \tag{3.10}$$

Exercise 3.5 (Mixed components)**.** *The matrix associated with the mixed tensor F is neither symmetric nor anti-symmetric. Verify that*

$$F^{\mu}{}_{\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \tag{3.11}$$

Remark 3.6 (Coordinate free form)**.** *In a coordinate free form*

$$\mathbf{F} = F_{\mu\nu}\,\mathbf{e}^{\mu} \otimes \mathbf{e}^{\nu}$$

Exercise 3.7 (Coulomb in spherical coordinates)**.** *Using the rules of tensor calculus, show that F for Coulomb field in spherical coordinates has $F_{0r} = -F_{r0} = \frac{e}{r^2}$ and all other components are* 0*.*

[2] And also on the covariant form of Couloms-Lorentz law, see the next section

Example 3.8. *The covariant components of the field tensor in cylindrical coordinates* (ct, ρ, ϕ, z) *are*

$$F_{cylind} = \begin{pmatrix} 0 & -E_x c - E_y s & \rho(-E_y c + E_x s) & -E_z \\ \dots & 0 & \rho B_z & -B_y c + B_x s \\ \dots & \dots & 0 & \rho(B_x c + B_y s) \\ \dots & \dots & \dots & 0 \end{pmatrix}$$

where $c = \cos\phi$ *and* $s = \sin\phi$*. In normalized coordinates*

$$\begin{pmatrix} 0 & -E_x c - E_y s & -E_y c + E_x s & -E_z \\ \dots & 0 & B_z & -B_y c + B_x s \\ \dots & \dots & 0 & B_x c + B_y s \\ \dots & \dots & \dots & 0 \end{pmatrix}$$

Exercise 3.9 (Magnetic field of currnet line)**.** *A line of current* I *along the z-axis creates a magnetic field* $\mathbf{B} = \frac{2I}{c\rho}\hat{\theta}$ *in cylindrical coordinates. Show that in cylindrical coordinates* $F_{\rho z} = -F_{z\rho} = \frac{2I}{c\rho}$ *and all other componnets vanish. (*Hint: It is simpler to use properties of the basis vectors $\mathbf{e}^\rho, \mathbf{e}^z$ rather than compute the transformation matrix.)

3.1.3 Covariant formulation

It is natural to expect that the force $\mathbf{f}$ is a 3-dimensional shadow of a 4-force vector. The Coulomb-Lorentz law generates this force from an anti-symmetric second rank tensor F and from the 4-velocity u. There is essentially one way to do that, namely

$$f_\mu = \frac{e}{c} F_{\mu\nu} u^\nu$$

Note that the 4-force has the desirable property that it is automatically Minkoski orthogonal to the 4-velocity, no matter what the fields are:

$$f_\mu u^\mu = \frac{e}{c} F_{\mu\nu} u^\mu u^\nu = 0$$

Exercise 3.10. *Explain why the last identity is true and why it is desirable.*

Since this is a tensorial identity, it holds in any coordinate system. In particular, it holds in all inertial frames.

We can now use this to verify that the signs we picked in Eq. (3.9) are the good signs: To check the sign of E consider a particle at rest: $u^\mu = c(1, 0, 0, 0)$. Then

$$f_\mu = \frac{e}{c} F_{\mu\nu} u^\nu = e F_{\mu 0} \Longrightarrow \mathbf{f} = e\mathbf{E}$$

as it should. To check the sign for B consider a particle moving in a magnetic field (without electric field) with 4-velocity $u = \gamma(c, \mathbf{v})$. Then

$$\begin{aligned} f_k &= \frac{e}{c} F_{kj} u^j \\ &= \frac{e}{c} \gamma F_{kj} v^j \\ &= \frac{e}{c} \gamma\, \varepsilon_{kjm} B^m v^j = \frac{e}{c} \gamma\, \varepsilon_{kjm} v^j B^m \\ &= \frac{e}{c} \gamma\, (v \times B)_k \end{aligned}$$

For a slow particle $\gamma \approx 1$ we recover Lorentz force.

3.2 The electromagnetic potential

The homogeneous Maxwell equation:

$$\nabla \cdot B = 0$$

says that *there are no magnetic monopoles.* In Euclidean space this statement is equivalent to the fact that $\mathbf{B}$ is derived from a vector potential $\mathbf{A}$:

$$\mathbf{B} = \nabla \times \mathbf{A}$$

A second (vector valued) Maxwell equation is Faraday law of induction[3]

$$\nabla \times \mathbf{E} = -\frac{1}{c} \partial_t \mathbf{B} \Longrightarrow \nabla \times \left(\mathbf{E} + \frac{\partial_t \mathbf{A}}{c} \right) = 0 \tag{3.12}$$

and this says that $\mathbf{E} + \frac{\partial_t \mathbf{A}}{c}$ is the gradient of a scalar potential $-\phi$. $\mathbf{E}$ and $\mathbf{B}$ are therefore derived from potentials

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla\phi - \frac{1}{c}\, \partial_t \mathbf{A} \tag{3.13}$$

3.2.1 Covarinat formulation

The scalar and vector potentials ϕ and $\mathbf{A}$ can be amalgamated to a 4-vector potential

$$A_\mu = (-\phi, \mathbf{A}) \tag{3.14}$$

(recall $\eta = (-1, 1, 1, 1)$). Given a 4-vector field, we can construct from it the anti-symmetric tensor field

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.15}$$

In components:

$$E_j = F_{j0} = \partial_j A_0 - \partial_0 A_j = -\frac{1}{c}\, \partial_t A_j - \partial_j \phi,$$

[3] In SI units c is absorbed in B.

and

$$B^i = \tfrac{1}{2}\varepsilon^{ijk}F_{jk} = (\nabla \times \mathbf{A})_i$$

We have reproduced Eq. 3.13.

Exercise 3.11. *Show that if A is a 4-vector field then F defined through the 4-potential transforms like a second rank tensor under arbitrary (possibly curvilinear) coordinate transformation.*

3.2.2 Gauge freedom

The field tensor F does not determine the 4-potential A uniquely. For any (scalar) function Λ of space-time

$$A_\mu \to A_\mu + \partial_\mu \Lambda \tag{3.16}$$

share the same fields. In particular, the vector potential $\partial_\mu \Lambda$ generates no fields:

$$\partial_\mu(\partial_\nu \Lambda) - \partial_\nu(\partial_\mu \Lambda) = (\partial_\mu \partial_\nu)\Lambda - (\partial_\nu \partial_\mu)\Lambda = 0$$

since derivatives commute. Fields can be measured at point in space time and so have a direct physical meaning. Potentials are tools for computations.

Exercise 3.12. *Suppose that someone tells you that $A_\mu(x)$ is time-like (or space-like) at a given point x. What information does this give on the fields.* (Answer: None)

3.2.3 Non-local gauge invariants Lorentz scalars

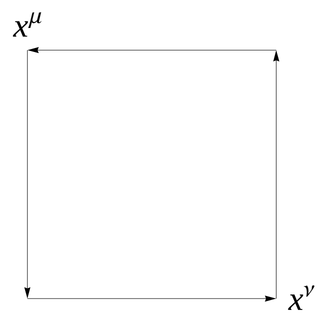

Figure 3.1: Loop and surface element for Stokes.

Although A is not gauge invariant, line integral of A over closed loops in space-time is a gauge invariant scalar. By Stokes:

$$\oint A_\mu dx^\mu = \int F_{\mu\nu} dS^{\mu\nu}$$

where dS is the area element spanned by the loop. Both sides are manifestly Lorentz scalars, and the right hand side is manifestly gauge invariant.

Exercise 3.13. *Prove Stokes for the square planar loop in Fig 3.1.*

A familiar, special case of the formula is a closed loop at fixed time, $dx^\mu = (0, dx^k)$, where

$$\oint A_k dx^k = \oint \mathbf{A} \cdot d\ell = \int \nabla \times \mathbf{A} \cdot dS = \int \mathbf{B} \cdot dS = \Phi$$

gives the magnetic flux through the loop.

Exercise 3.14. *Suppose that γ is a curve in Euclidean space and consider the surface S spanned by the curve for $t \in [0, t_0]$. Show that $\int F dS$ is the emf action, i.e. $\int \mathcal{E}_{emf}\ dt$.*

Quantum mechanics gives a fundamental unit of magnetic flux $\Phi_0 = 2 \times 10^{-15}$ $[Weber]$. In SI $[Weber] = [\hbar/e]$. The unit is such that there are about 10 quantum flux quanta flux through a 1 $[\mu^2]$ area of the earth magnetic field. So for a bacterium a quantum flux is a natural magnetic flux scale. It is interesting, and even mysterious, that when quantum mechanics meets special relativity, the Lorentz scalars give rise to quantized objects. Here are some examples whey Φ_0 shows up: It is the quantized magnetic flux in vortices that thread a superconductor; The charge of magnetic monopoles; and it is the emf action $\int \mathbf{E} \cdot dt$ that shows up in the quantum Hall effect.

Exercise 3.15. *What is $\Phi_0 = 2 \times 10^{-15}$ $[Weber]$ in c.s.g in terms of ¯,e and c .*

3.2.4 What does a voltmeter measure?

Fields are measurable. Potentials are not. No instruments can measure the potential at a point. What does a voltmeter measure?

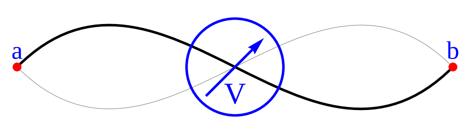

Figure 3.2: Voltmeter measures the electromotive force: $\mathcal{E}_{ab} = -\int \mathbf{E} \cdot d\ell$, which is gauge invariant but, in general, path dependent (and non-local). In contrast with A_0 which is gauge dependent, but path independent (and local). It is path independent provided the loop defined by the path (black) and its variation (gray) enclose no time-dependent magnetic fields.

The emf $\mathcal{E} = \int_\gamma \mathbf{E} \cdot d\ell$ is manifestly gauge invariant and so, in principle, measurable. In general, it is non-local[4] and path dependent. It is not a property of a single event but the pair (a, b) and, in general, depends on the path connecting them. The integral form of Faraday induction law is

$$\oint \mathbf{E} \cdot d\mathbf{x} = -\frac{\dot{\Phi}}{c}$$

where Φ is the magnetic field enclosed by a loop (at a given time slice). If the variation of the path γ does not enclose any time dependent magnetic fields Faraday induction law, Eq. 3.12, says the emf is (at least locally) path independent. We may then define $\mathbf{E} = -\nabla\phi$. The difference in ϕ is what a voltmeter measures.

3.2.5 Generalizations

Manifolds

Since we wrote the equations in a tensorial form, they have a natural translation to space-time manifolds that are only locally Minkowski.

$d + 1$ dimensions

It may well be that the apparent dimension of space-time as 3+1 is the dimension we perceive on macroscopic scales whereas the dimension is different (larger) in the microscopic scale. It is a nice feature of the formalism that one can formulate electrodynamics in d+1 space time dimensions. The vector potential A_μ has d+1 components, and the field F is still a second rank antisymmetric tensor. It has $\binom{d+1}{2}$ components. d components "electric" and the remaining are "magnetic". In $d = 3$ the number of electric and magnetic component coincide. In lower dimension there are more electric components andd in higher dimensions more magnetic.

Exercise 3.16 (1+1 dimension)**.** *How many components does the electric field have in one dimension and how many the magnetic field?. Show that the electric field is a scalar.*

Non Abelian gauge fields

QCD, and other nonabelian gauge theories, are the non-commutative generalizations of electrodynamics in the sense that the (real, commutative) 4-vector potential A_μ, is replaced by a Hermitian (matrix valued) 4-vector potential.

[4] To measure the emf oen needs to measure $\mathbf{E}$ simultaneously along the Euclidean path γ. One could argue that a simple instrument, like a AVOmeter can't possibly do that exactly. It is at best an idealization.

3.3 Lorentz scalars

One can construct interesting Lorentz scalars from the field tensor F.

3.3.1 Local scalars

By local scalars I mean scalars one can construct from F at a given event. There are no interesting scalars that are linear in the field at a given point since

$$F_{\mu\nu}\eta^{\mu\nu} = F_\mu{}^\mu = 0$$

We can however construct interesting quadratic scalars:

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= F_{0j}F^{0j} + F_{j0}F^{j0} + F_{jk}F^{jk} \\ &= -2E_jE_j + 2B_iB_i \\ &= -2(\mathbf{E}^2 - \mathbf{B}^2) \end{aligned}$$

You might be worried that we have made a sign error: $\mathbf{E}^2 + \mathbf{B^2}$ is proportional to the energy density of the field. Why the minus sign? This is the same minus sign you find in Lagrangian mechanics: The Lagrangian is the difference of the kinetic and potential energies, not their sum.

3.3.2 Dual: F^*

Duality, denoted by $*$, is an operation whose square in the identity: $** = \mathbb{1}$. This means that taking a dual involves no loss of information. In n dimensions, the Levi-Civita tensor allows us to define a duality for anti-symmetric tensors of rank r and anti-symmetric tensors of rank $n-r$. Note first that anti-symmetric tensors of rank r make a linear space whose dimension $\binom{n}{r}$ (the number of independent components). The contraction of the Levi-Civita with an anti-symmetric tensor of rank r gives an anti-symmetric tensor of rank $n-r$. Since $\binom{n}{r} = \binom{n-r}{r}$ the operation can be used to define a duality.

In 4-dimensions the dual of an anti-symmetric second rank tensor is a second rank tensor. The dual of F is then defined by

$$(F^*)^{\mu\nu} = \frac{\varepsilon^{\mu\nu\alpha\beta}}{2\sqrt{|g|}}F_{\alpha\beta} \Longleftrightarrow (F^*)_{\mu\nu} = \frac{\sqrt{|g|}\varepsilon_{\mu\nu\alpha\beta}}{2}F^{\alpha\beta} \tag{3.17}$$

In Lorentz-Cartesian coordinates, $g = \eta$ and $|g| = 1$, of course. To avoid writing ugly formulas involving $\sqrt{|g|}$ is shall sacrifice generality for transparency and write formulas in Lorentz-Cartesian coordinates.

Exercise 3.17. *Show that $*$ is indeed a duality. Namely, show first that*

1.

$$\varepsilon_{\alpha\beta\gamma\delta}\varepsilon^{\alpha\beta\mu\nu} = 2(\delta_\alpha{}^\mu\delta_\beta{}^\nu - \delta_\alpha{}^\nu\delta_\beta{}^\mu)$$

2. *Using this show that*

$$F^{**} = F$$

F^* effectively interchanges $\mathbf{E}$ and $-\mathbf{B}$. In Minkowski-Cartesian coordinates:

$$(F^*)^{0i} = \tfrac{1}{2}\varepsilon^{0ijk}F_{jk} = \varepsilon^{ijk}F_{jk} = B^i = B_i$$

Similarly,

$$(F^*)^{jk} = \tfrac{1}{2}\varepsilon^{jk\alpha\beta}F_{\alpha\beta} = \varepsilon^{jk0i}F_{0i} = \varepsilon^{0jki}F_{0i} = -\varepsilon^{jki}E_i$$

3.3.3 Second Lorentz scalar: $\mathbf{E}\cdot\mathbf{B}$

Evidently,

$$(F^*)^{\mu\nu}F_{\mu\nu} = 2(F^*)^{0j}F_{0j} + (F^*)^{jk}F_{jk} = -4E\cdot B \tag{3.18}$$

is a Lorentz scalar.

It follows that there are nine classes of Lorentz distinct fields:

$$\mathbf{E}^2 - \mathbf{B}^2 = \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases} \qquad \mathbf{E}\cdot\mathbf{B} = \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}$$

For example, the field of a charge moving at uniform velocity has $\mathbf{E}^2 - \mathbf{B}^2 > 0$ and $\mathbf{E}\cdot\mathbf{B} = 0$. Similarly, The field of a plane electromagnetic wave has $\mathbf{E}^2 - \mathbf{B}^2 = \mathbf{E}\cdot\mathbf{B} = 0$, in any frame.

3.4 The homogeneous Maxwell equations

The 4 homogeneous Maxwell's equations express the fact that F is derived from the potential A. Write

$$\partial_\beta A_\alpha = \tfrac{1}{2}\underbrace{(\partial_\beta A_\alpha - \partial_\alpha A_\beta)}_{anti-symmetric} + \tfrac{1}{2}\underbrace{(\partial_\beta A_\alpha + \partial_\alpha A_\beta)}_{symmetric}$$

Since Levi-Civita is completely anti-symmetric, any[5] vector field, A_μ, trivially satisfies

$$0 = 2\varepsilon^{\mu\nu\alpha\beta}\partial_{\nu\beta}A_\alpha = \varepsilon^{\mu\nu\alpha\beta}\partial_\nu F_{\beta\alpha}$$

since $\partial_{\mu\nu}$ is symmetric. Since the Levi-Civita symbols are fixed numbers we can rewrite this as

$$0 = \partial_\nu\left(\frac{\varepsilon^{\mu\nu\alpha\beta}}{2}F_{\beta\alpha}\right) = \partial_\nu\left(\sqrt{|g|}(F^*)^{\mu\nu}\right) \tag{3.19}$$

by the definition of the dual 3.17.

[5] Twice differentiable

Exercise 3.18 (Alternate form)**.** *Show that 4 equations (3.19) can also be written as*

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$$

Exercise 3.19. *Show that if F is an anti-symmetric second rank tensor then $\partial_\mu \left(\sqrt{|g|}(F^*)^{\mu\nu}\right)$ transforms like a 4-vector under curvilinear coordinate transformations.*

In mechanics the price for using the a non-inertial coordinate system, such as earth, leads to the price of the emergence of new forces: Coriolis and Centrifugal. You may wonder if there is an analog in electrodynamics. The next exercise explains why there is none.

Exercise 3.20. *Consider the coordinate system attached to the rotating earth introduced in the previous chapter.*

1. *Compute* $\det g$ *for the earth rotating coordinate system.* (Answer: $\det g = -1$)
2. *What does this imply about the homogeneous Maxwell equations in the earth frame?*

3.4.1 Maxwell form

The 0 component of Eq. (3.19) is the statement that magnetic fields are sourceless:

$$0 = \partial_\mu (F^*)^{0\mu} = \partial_j (F^*)^{0j} = -\nabla \cdot \mathbf{B} \tag{3.20}$$

The spatial components give Faraday law:

$$0 = \frac{1}{c}\,\partial_t \mathbf{B} + \nabla \times \mathbf{E} \tag{3.21}$$

The two equations have different physical character. The first equation can be viewed as a constraint equation on the admissible magnetic fields at any given time. The second may be viewed as an equation that determines the evolution in time of the magnetic fields (given the electric field). This system must be consistent. Namely, if $\mathbf{B}$ starts divergence-less, it must evolve in a way that its stays divergence-less. This is indeed the case:

$$\frac{1}{c}\,\partial_t \big(\nabla \cdot \mathbf{B}\big) = -\nabla \cdot \big(\nabla \times \mathbf{E}\big) = 0 \tag{3.22}$$

Exercise 3.21 (Harmonic A)**.** *Given the vector potential*

$$A_\mu(x) = A_\mu e^{ik_\nu x^\nu}$$

Compute F. Show that

$$F_{\mu\nu}F^{\mu\nu} = -2\left(k_\mu k^\mu\, A_\nu A^\nu - (k_\mu A^\mu)^2\right), \quad F_{\mu\nu}(F^*)^{\mu\nu} = 0$$

Chapter 4

Variational principle

The equations of motion of a relativistic (charged) particle are formulated through a variational principle .

4.1 The action begins

Lagrangian mechanics reformulates Newtonian mechanics: Newton equations are interpreted as the Euler-Lagrange equations for the minimizers of the action. One advantage of the Lagrangian formulation is that it facilitates implementing basic symmetry principles, such as Lorentz invariance.

The property of being minimizer does not depend of the the choice of coordinates and so the Lagrangian formulation guarantees the tensorial character of the equations. Since Lorentz transformations are special coordinate transformations of Minkowski space, a theory is guaranteed to be Lorentz invariant once the action is a Lorentz scalar. The action S associates a scalar to an orbit.

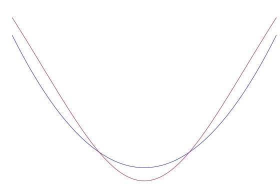

Figure 4.1: The blue curve shows $S(x) = x^2$. The black curve shows the action under a change of coordinate $x \to \tan x$, a change of scale and shift $S \to 2S - 1/10$. The action changes, but the physical point p where the minimum occurs is the same. In the example also the coordinates of the points are the same: $x = 0$.

Requiring that the action be a Lorentz scalar which reduces to the Newtonian form for non-relativistic motions fixes the action uniquely.

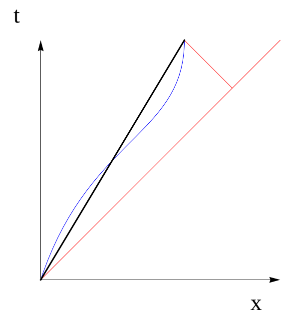

Figure 4.2: The world line is required to have time-like tangents–otherwise the action is complex. The blue curve represents the variation of a world line with fixed end points. The black line maximizes the proper-time τ. The red lines are light-like. The red path between the end points has zero proper-time.

4.1.1 Action for a free particle

Let us start with a free particle with mass $m > 0$ (a scalar). A natural Lorentz scalar associated to the orbit in the elapse off proper time τ. Since the action has units of $[p][x]$, we append scalars c and m to fix the dimension:

$$S_p = -mc^2 \int \underbrace{d\tau}_{proper\ time} = \int L \underbrace{dt}_{coordinate\ time} \tag{4.1}$$

and

$$L = -\frac{mc}{\gamma}\sqrt{-g_{\alpha\beta}(x)\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}} = -\frac{mc^2}{\gamma}\sqrt{-g_{00}c^2 - 2g_{0j}cv^j g_{jk}(x)v^j v^k}$$

In particular, in Minkowski space-time with metric

$$g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{jk} & \\ 0 & & & \end{pmatrix}$$

and a particle moving non-relativistically, $\gamma \to 1$ and

$$L \to -\underbrace{mc^2}_{const} + \underbrace{\frac{m}{2} g_{jk}(x)v^j v^k}_{kinetic\ energy}$$

This is the same as the classical Lagrangian of free particle, up to a (large) negative constant $-mc^2$. Adding a constant to the action does not affect the minimizing path, of course.

Example 4.1. *In rotating (earth) coordinates, 2.3.3,*

$$\frac{1}{\gamma} \approx 1 - \frac{1}{2c^2}\left(\mathbf{v}^2 - \Omega^2\rho^2 + \Omega\,\dot\phi\rho^2\right)$$
$$= 1 - \frac{1}{2c^2}\left(\underbrace{\mathbf{v}^2}_{Kinetic} - \underbrace{(\boldsymbol{\Omega}\times\mathbf{x})^2}_{centrifugal} + \underbrace{2\,\boldsymbol{\Omega}\cdot\mathbf{v}\times\mathbf{x}}_{Coriolis}\right)$$

4.1.2 Interaction

The action in Mechanics is constructed from the potentials, not the fields. The electromagnetic potential is a 4-vector, which we can pair with the path element dx^μ to form a scalar. Electromagnetic interaction is, of course, proportional to the charge. In c.g.s we also need c to fix the units and the overall sign is fixed by the choice of the metric $\eta = (-1, 1, 1, 1)$ so

$$S_{int} = \frac{e}{c}\int A_\mu dx^\mu = -e\int \phi(x)dt + \frac{e}{c}\int \mathbf{A}(x)\cdot\mathbf{v}\,dt \tag{4.2}$$

This is *precisely* the terms one adds to the Kinetic energy in classical mechanics to describe the interaction with **E** and **B**.

4.1.3 Gauge invariance

Since the action is constructed from the potentials, one should worry about the gauge invariance of the minimizing orbit. Under change of gauge

$$A'_\mu = A_\mu + \partial_\mu\Lambda \tag{4.3}$$

This leads to

$$S'_{int} = S_{int} - \frac{e}{c}\int (\partial_\mu\Lambda)\,dx^\mu = S_{int} - \frac{e}{c}\big(\Lambda(x_f) - \Lambda(x_i)\big) \tag{4.4}$$

This means that although S_{int} changes under a gauge transformation, its variation δS_{int} does not (so long as the end points are fixed). This guarantees that the Euler Lagrange equaltions are gauge invariant.

4.2 Variation of the action

The action is a function on paths: It associates a number with a given path $x^\mu(\tau)$. We may think of the path as parametrized by its proper time. The action is assumed to have the form

$$S = \int_{x_i}^{x_f} f(x,u)d\tau$$

i.e. it is a function of the position and velocity. The end point events x_i and x_f are fixed. The action associates a number for every path $x(\tau) = \{x^0(\tau), \dots x^3(\tau)\}$. We shall denote the variation of the path by

$$\delta x = \{ \underbrace{\delta x^0(\tau), \dots \delta x^3(\tau)}_{infinitesimal\ functions} \}$$

The rule of the game is that the end points are fixed: δx vanishes at the end points events. (The events which are fixed *not the proper-times at the end points.*) The strategy is to use integration by parts to bring δS to the form

$$\delta S = h_\mu(x,u)\delta x^\mu\Big|_{x_i}^{x_f} + \int_{x_i}^{x_f} g_\mu(x,u,\dot u)\delta x^\mu d\tau$$

Since δx^μ vanish at the end points the first term drops. And since δx^μ are arbitrary,[1] the only way for the integral to vanish is if

$$g_\mu(x,u,\dot u) = 0$$

These are differential equation that the optimal path must satisfy. They are known as Euler-Lagrange. Lets consider examples.

4.2.1 Variation of S_p

In Minkowski Cartesian coordinates[2], $g = \eta$, the variation in the proper-time with is given by

$$\delta(cd\tau)^2 = 2c^2 d\tau\delta(d\tau) = \delta(-dx_\mu dx^\mu) = -2dx^\mu\,\delta(dx_\mu)$$

This can be written in terms of the 4-velocity as

$$c^2\delta(d\tau) = -u^\mu\,\delta(dx_\mu) = -d\big(u^\mu\,\delta x_\mu\big) + du^\mu\,(\delta x_\mu)$$

The variation of the free action is then

$$-\,\delta S_p = -m\,u_\mu\delta x^\mu\Big|_{end\ pts} + m\int \dot u_\mu\,\delta x^\mu\,d\tau \tag{4.5}$$

By assumption, the variation vanishes at the boundary so the first term drops. since δx is arbitrary, so long as it is small, if follows that $\delta S_p = 0$ is equivalent to conservation of 4-momentum:

$$\dot p_\mu = m\dot u_\mu = 0$$

[1]Viewed as functions of τ the variations satisfy one constraint: $(d\delta x_\mu)(d\delta x^\mu) = -(cd\tau)^2$

[2]The case of a general metric $g(x)$ is treated in a supplement

4.2.2 Variation of S_{int}

For S_{int} write

$$\delta(A_\mu dx^\mu) = (\delta A)_\mu dx^\mu + A_\mu \delta(dx)^\mu = (\partial_\nu A_\mu)\,(\delta x^\nu) dx^\mu + A_\mu \delta(dx)^\mu$$

The basic idea in the calculus of variation is to use integration by parts to get rid of terms of the form δdx. Hence, rewrite the last term

$$A_\mu \delta(dx)^\mu = d\big(A_\mu \delta x^\mu\big) - (dA)_\mu \delta x^\mu = d\big(A_\mu \delta x^\mu\big) - (\partial_\nu A_\mu) dx^\nu\, \delta x^\mu$$

Combining the two expressions and changing summation indices where needed we find

$$\begin{aligned}\delta(A_\mu dx^\mu) &= d\big(A_\mu \delta x^\mu\big) + (\partial_\mu A_\nu)\,(\delta x^\mu) dx^\nu - (\partial_\nu A_\mu) dx^\nu\, \delta x^\mu \\ &= d\big(A_\mu \delta x^\mu\big) + \Big(\partial_\mu A_\nu - \partial_\nu A_\mu\Big) u^\nu\, d\tau\, \delta x^\mu \\ &= d\big(A_\mu \delta x^\mu\big) + F_{\mu\nu}\, u^\nu\, d\tau\, \delta x^\mu\end{aligned}$$

Hence,

$$\delta S_{int} = \frac{e}{c}\big(A_\mu \delta x^\mu\big)|_{bdry} + \frac{e}{c}\int F_{\mu\nu}\, u^\nu\, d\tau\, \delta x^\nu \tag{4.6}$$

4.2.3 Euler-Lagrange equation

The variation of the total action vanishes for any δx^μ provided the integrand in $S_{free} + S_{int}$ add to zero. This gives the Euler-Lagrange equation

$$m\,\dot u_\mu = \frac{e}{c} F_{\mu\nu} u^\nu \tag{4.7}$$

4.2.4 The non-relativistic limit

By general principles, we know that Eq. 4.7 must reduce to the standard formulas of non-relativistic classical mechanics. It is also easy to check this directly: For a slow particle has

$$u^\mu \approx (c, \mathbf{v}), \quad \dot u_j = a_j$$

(recall that $\eta = (-1, 1, 1, 1)$). The Euler-Lagrange equations reduce to Newton equations of motions

$$m\mathbf{a} = e\mathbf{E} + \frac{e}{c}\mathbf{v} \times \mathbf{B} \tag{4.8}$$

Remark 4.2 (Sign conventions)**.** *You can use the non-relativistic limit to fix and verify the signs convention in 3.9*

4.2.5 Existence and uniqueness

In general the action does not have an honest minimizer and when it does it not be unique. The (non-relativistic) Harmonic oscillator is an example. The Lagrangian is

$$2L = \dot{x}^2 - x^2 \tag{4.9}$$

Now consider paths that starts and terminates at the origin in half the period, $t = \pi$. There are many that satisfy the Euler-Lagrange equation

$$x_n(t) = A \sin t$$

for arbitrary amplitude A. All solve the equation of motion $\ddot{x} = -x$ and so are local minimizers. For all the action vanishes:

$$S = \int_0^\pi L dt = A^2 \int_0^\pi (n^2 \cos^2 nt - \sin^2 nt) dt = 0,$$

At the same time there is no nice minimizer connecting the origin and any other point at half the period.

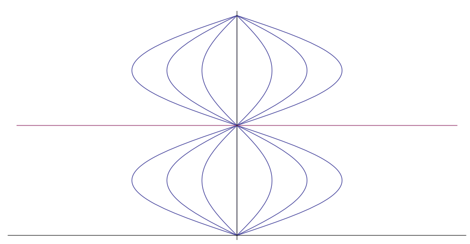

Figure 4.3: There are infinitely many paths connecting the origin when the time difference is half the period. But there is no honest minimizer connecting the origin to any other point on the red line at half the period. The minimizer "goes through infinity".

Exercise 4.3. *Consider the family of paths parametrized by $A > x_0 \geq 0$*

$$x_A(t) = \begin{cases} x_0 & 0 < \sin t < x_0/A \\ A \sin t, & \sin t > x_0/A \text{ and } t \leq \pi \end{cases}$$

Note that that the second segment is a solution of the Euler-Lgrange equation for the Harmonic oscillator Eq. 4.9. Compute the action of the two segments for $A/x_0 \gg 1$. Show that when $A \to \infty$, the total action $S(x_A) \approx -x_0 A$ and hence diverges to $-\infty$ if $x_0 \neq 0$.

Convexity and uniqueness

A function S is called convex if

$$S(\lambda x + \lambda' y) \leq \lambda S(x) + \lambda' S(y), \quad \lambda + \lambda' = 1, \quad 1 \geq \lambda, \lambda' \geq 0$$

For example, the functions in Fig. 4.1 are convex. It is evident that if a function is convex its minimum is unique. (It may, however, lie at infinity).

In the case at hand x, the argument of S, is itself a function–a path. Functions naturally form a vector space so the notion of convex function also extends to this case. γ is a convex function of $\mathbf{v}$. This then implies that the minimizer for a free relativistic particle is unique.

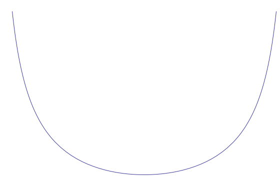

Figure 4.4: γ is a convex function of $\mathbf{v}$. This implies that S is a convex functional of the path.

4.2.6 Consistency check

The 4- acceleration is *always* (Lorentz) perpendicular to the 4-velocity (by the constancy of $u_\mu u^\mu$). This is indeed respected by the equation of motion since $F_{\mu\nu}$ is antisymmetric:

$$m\,\dot u_\mu u^\mu = \frac{e}{c} F_{\mu\nu} u^\mu u^\nu = 0$$

Exercise 4.4 (Charged particle in a constant fields)**.** *Solve the equations of motion of a charge particle in constant parallel electric and magnetic fields*

Exercise 4.5 (Charged particle in a radiation field)**.** *Show that the equations of motion of a charge particle in the radiation field of a circularly polarized plane wave admit solutions that are circular orbit in the plane orthogonal to the direction of propagation of the light.*

4.3 Supplement

4.3.1 Fermat principle

The mother of variational principles is Fermat principle. It formulates geometric optics at the the minimizer of the time of propagation between two points. The ray propagates in a medium with index of refraction $n(\mathbf{x})$. The propagation time dt is $c\,dt = n(\mathbf{x})|d\mathbf{x}|$. We can think of n as inducing a metric in Euclidean space–one that measures the propagation time:

$$(c\,dt)^2 = n^2(\mathbf{x})\,d\mathbf{x}\cdot d\mathbf{x} = n^2(\mathbf{x})\,(d\ell)^2$$

Such a metric is called conformal.

Consider a ray $\mathbf{x}(\ell)$ parametrized by it Euclidean length: $d\ell = \sqrt{d\mathbf{x} \cdot d\mathbf{x}}$. The tangent to the ray

$$\mathbf{t} = \frac{d\mathbf{x}}{d\ell}$$

is a unit vector. For a variation of the path $\delta\mathbf{x}$:

$$\delta(d\ell) = \frac{d\mathbf{x} \cdot d\delta\mathbf{x}}{d\ell} = \mathbf{t} \cdot d\delta\mathbf{x}, \quad \delta n = (\delta\mathbf{x} \cdot \nabla)n$$

The corresponding time variation is (the integral of)

$$\delta(c\ dt) = (\delta\mathbf{x} \cdot \nabla)nd\ell + n\mathbf{t} \cdot d\delta\mathbf{x} = \delta\mathbf{x} \cdot \Big(\nabla n\, d\ell - d\big(n\mathbf{t}\big)\Big) + d\big(n\mathbf{t} \cdot \delta\mathbf{x}\big)$$

The integral of variation vanishes provided the brackets (and boundary terms) vanish:

$$\frac{d(n\mathbf{t})}{d\ell} = \nabla n$$

In particular, in a region where the refraction index is a constant, $\nabla n = 0$, the ray keeps it direction of propagation: $\mathbf{t}$ is a constant.

Exercise 4.6. *Show that the equation of motion is consistent with the* $\mathbf{t}$ *being a unite vector.*

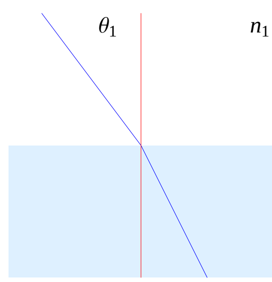

Figure 4.5: The change in direction of a ray when n jumps is ddetermined by Snell's law

Exercise 4.7. *Derive Snell's law, fig.4.5,*

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

from Fermat principle.

4.3.2 Rainbow

The simplest features of the rainbow can be understood from Snell's law.

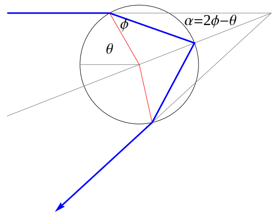

Figure 4.6: The blue line shows a ray undergoing one internal reflection in a drop of water. The impact angle θ is defined in the figure. The outgoing ray is focused near the maximum $2\phi - \theta$. This partial focusing is called a caustic. This gives the direction of the rainbow.

Exercise 4.8. *Use Snell law and show that a light ray in air ($n_a = 1$) hitting a water droplet, $n_w > 1$, at lattitude θ is reflected back at angle $2\alpha(\theta) = 4\phi(\theta) - 2\theta$, see figure. The function $\phi(\theta)$ is defined by Snell law: $n_w \sin\phi = \sin\theta$.*

The intensity of the light reflected at angle $I(2\alpha)$ is proportional to the intensity of the incoming light:

$$d(\sin\theta) = I(2\alpha)\,|d\alpha|$$

A computation gives

$$\frac{d\alpha}{d\theta} = -1 + 2\frac{\cos\theta}{\sqrt{n^2 - \sin^2\theta}}$$

The derivative vanishes for

$$3\cos^2\theta = n^2 - 1$$

which gives a real value for θ provided $1 < n < 2$. This gives the maximal value of 2α. Evidently, $I(2\alpha) = \infty$ there. The divergence implies focusing of the reflected light. This is called *caustic* in geometrical optics.

Exercise 4.9. *Show that for water ($n = 1.33$) the caustic occurs for $2\alpha = 42°$. This is the main angle of the rainbow, first found by Bacon in 1268. (Different colors have slightly different angles due to the slight frequency dependence of n).*

4.3.3 Geodesics in Curved space-time

This section is an aside. We set $c = 1$. In a general space-time (not necessarily Minkowski) the notion of proper-time is defined, as usual, by

$$(d\tau)^2 = -g_{\mu\nu}(x)\,dx^\mu dx^\nu$$

The 4-velocity is

$$u^\mu = \frac{dx^\mu}{d\tau}$$

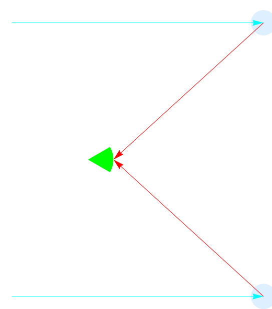

Figure 4.7: The cyan arrows represent light rays from the sun. The two small light-blue balls represent two water droplets. The red arrow are the reflected light rays in the direction of the rainbow caustics. The green eye represents the observer. Pilots sometimes see rainbow that are circular.

which is normalized to -1 since

$$(d\tau)^2 = -g_{\mu\nu}(x)\, u^\mu u^\nu\, (d\tau)^2 = -u_\mu u^\mu\, (d\tau)^2$$

Exercise 4.10 (Acceleration)**.** *When the metric is posiytion dependent it is not true that the acceleration is orthogonal to the velocity. Show that*

$$2\dot{u}^\mu u_\mu + (\partial_\alpha g_{\mu\nu}) u^\mu u^\nu u^\alpha = 0$$

We want to find the path that *minimizes* the action (equivalently, maximizes the proper time)

$$S = -\int d\tau$$

The variation of $(d\tau)^2$ is

$$\delta(d\tau)^2 = 2(d\tau)\delta(d\tau) = -(\delta g_{\mu\nu})\, dx^\mu dx^\nu - 2g_{\mu\nu}\, \delta(dx^\mu) dx^\nu$$

Hence

$$-2\delta(d\tau) = (\delta g_{\mu\nu})\, u^\nu\, dx^\mu + 2g_{\mu\nu}\, u^\nu\, \delta(dx^\mu)$$

Rewrite the first therm as

$$(\delta g_{\mu\nu})\, u^\nu dx^\mu = (\partial_\alpha g_{\mu\nu})\, u^\nu dx^\mu\, \delta x^\alpha = (\partial_\mu g_{\alpha\nu})\, u^\nu dx^\alpha\, \delta x^\mu$$

The second term can be rewritten as

$$\begin{aligned} g_{\mu\nu}\, \delta(dx^\mu) u^\nu &= d(g_{\mu\nu}\, \delta x^\mu u^\nu) - d(g_{\mu\nu}\, u^\nu)\, \delta x^\mu \\ &= d(g_{\mu\nu}\, \delta x^\mu u^\nu) - (\partial_\alpha g_{\mu\nu})\, dx^\alpha\, u^\nu\, \delta x^\mu - g_{\mu\nu}\, du^\nu\, \delta x^\mu \end{aligned}$$

collecting

$$-\delta(d\tau) = d(g_{\mu\nu}\, \delta x^\mu u^\nu) + \Big(\tfrac{1}{2}(\partial_\mu g_{\alpha\nu})\, u^\nu dx^\alpha\ - (\partial_\alpha g_{\mu\nu})\ u^\nu\, dx^\alpha - g_{\mu\nu}\, du^\nu\Big)\delta x^\mu$$

The first term is a boundary term so vanishes upon integration. The vanishing of the variation gives the Euler-Lagrange which is of the form

$$\dot{u}^\mu + \Gamma^\mu_{\alpha\beta}\, u^\alpha\, u^\beta = 0 \tag{4.10}$$

and $\Gamma^\mu_{\alpha\beta}$ may be assumed to be symmetric in α, β without loss. One then finds

$$\Gamma^\mu_{\alpha\beta} = \tfrac{1}{2} g^{\mu\nu} \Big(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta} \Big) \tag{4.11}$$

aka as the Christoffel symbol.

Exercise 4.11. *Show that great circles on the sphere are geodesics.*

Exercise 4.12 (Geodesic equation in covariant components)**.** *Show that the geodesic equation for the covariant components satisfies the equation*

$$2\dot{u}_\mu = -\Big(\partial_\mu g^{\alpha\beta}\Big)\, u_\alpha\, u_\beta$$

Chapter 5

Maxwell Equations

The non-relativistic notions of charge and currents is amalgamated into a single notion in space-time the 4-current and the inhomogeneous Maxwell equations are derived from a variational principle.

5.1 Technical preliminaries

5.1.1 Space-time volume element

In Riemannian geometry the volume element is[1]

$$dV = \sqrt{g} \prod dx^j, \quad g = \det g$$

We stick with the same definition in Minkowski geometry paying the small price $g \to |g|$, taking care of the fact that the metric is non-definite. We shall denote by $d\Omega$ the space-time volume element.

Lorentz transformations are isometrics of Minkowski space-time. In Cartesian coordinates $\det \eta = -1$. It follows that the Cartesian expression for the space-time volume is a Lorentz scalar:

$$d\Omega = \sqrt{|\eta|} dx^0\, dx^1\, dx^2\, dx^3 = \underbrace{dx^0\, dx^1\, dx^2\, dx^3}_{Minkowski\ cartesian} = (cdt)\, dV \tag{5.1}$$

in agreement with the fact that time-dilation and space contraction have compensating factors.

5.1.2 Densities

Under a change of coordinates $x' \leftrightarrow x$ a scalar function $\varphi(x)$ has the transformation rule

$$\varphi'(x') = \varphi(x)$$

[1] This is clearly the right notion of volume in orthogonal coordinates where g is diagonal. The tensorial properties then guarantee that this expression holds in general.

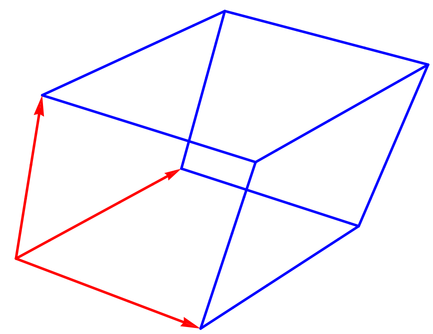

Figure 5.1: The volume of the parallelepiped spanned by the vectors $\mathbf{v}_j$ is their triple product.

It retains the value at the image point. Densities, $\rho(x)$ are different. The associated scalar is $\rho(x)dV$. It may count the charge in the volume, or the number of particles there. The rule is therefore

$$\rho'(x')dV' = \rho(x)dV$$

But dV' is related to dV be the Jacobian of the transformation. As a consequence $\rho(x)$ does not transform like a scalar, but like a density:

$$\frac{\rho}{\sqrt{|g|}}$$

transforms like a scalar.

5.1.3 Distributions

We want to introduce Dirac delta function in Minkowsky space-time. The defining property is

$$f(0) = \int d\Omega f(x)\delta^{(4)}(x) \tag{5.2}$$

for any smooth function (scalar) f.

Remark 5.1. *In Minkowski-Cartesian coordinates $d\Omega = cdtdV$ is a Lorentz scalar. It follows that $\delta^{(4)}(x)$ is a Lorentz scalar.*

Exercise 5.2. *Write the delta function $\delta^3(\mathbf{x}-\mathbf{x}_0)$ in spherical coordinates.*

Remark 5.3 (Functios and Distributions)**.** *Dirac delta function is an example distributions. Distributions and functions are distinct objects:*

- *Smooth functions form an algebra: You can add and multiply them in the obvious way. Distributions are a vector space: You can add them but not multiply. There is not such thing as a square of a delta function, or a root of a delta function.*

- *Distributions are naturally viewed as the dual vector space to the space of smooth and localized functions. With a pair we associate the number given by integration.*

- *A distribution $D(x)$ is the zero distribution if for any smooth and localized function $f(x)$*

$$\int f(x)D(x)dx = 0$$

An example is $x\delta(x)$. In contrast, the useful notion of the trivial (zero) function in the theory of integration is: $f(x) = 0$ for all x except for a set of measure zero.

- *All linear operations on distributions are allowed. In particular, distributions D can be differentiated according to the rules of integration by parts, as many times as one pleases*

$$\int f(x)D^{(n)}(x)\,dx = (-)^{n-1}\int f^{(n)}(x)D(x)dx$$

where f is a smooth (infinitely differentiable) function. In contrast, functions may or may not be differentiable. For example, the derivative of the step function, θ, is the Dirac distribution.

Exercise 5.4 (Delta of f)**.** *Suppose $f(x)$ is a nice function, such that $f(x_j) = 0$ and $f'(x_j) \neq 0$. Show that*

$$\delta\big(f(x)\big) = \sum_j \frac{\delta(x - x_j)}{|f'(x_j)|}$$

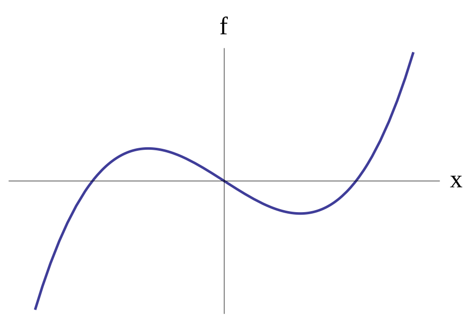

Figure 5.2: The function $f(x)$ associated with exercise 5.4.

5.2 Charge densities and currents

For a charges e moving on the trajectories $\xi(t)$ in 3 dimensions, the non-relativistic notion of charge density ρ and current $\mathbf{j}$ is defined by

$$\rho(\mathbf{x},t) = e\delta^{(3)}\big(\mathbf{x}-\xi(t)\big), \quad \mathbf{j}(\mathbf{x},t) = e\delta^{(3)}\big(\mathbf{x}-\xi(t)\big)\mathbf{v}(t) \tag{5.3}$$

where $\mathbf{v} = \dot{\xi}$ is the velocity. For several particles, of charges e_a, moving of different trajectories ξ_a, the generalization is obviously

$$\rho(\mathbf{x},t) = \sum_a e_a\delta^{(3)}\big(\mathbf{x}-\xi_a(t)\big), \quad \mathbf{j}(\mathbf{x},t) = \sum_a e_a\delta^{(3)}\big(\mathbf{x}-\xi_a(t)\big)\mathbf{v}_a(t) \tag{5.4}$$

We would like to amalgamate ρ and $\mathbf{j}$ into a n option of a 4-current-density. However, neither $\mathbf{v}$ nor $\delta^{(3)}(\mathbf{x})$ are natural objects in space-time and, a-priori, you may well worry that the non-relativistic notions of charge and currents are, at best, a non-relativistic approximations so we will need to tinker with them before being able to amalgamate them. It will turn out that these expressions are fine as they stand. (We still need to adjust dimensions so we can fit both ρ and $\mathbf{j}$ in a 4-vector with identical dimensions.)

It is clear that is is enough to formulate the notions of current and density for a single particle. This simplifies the notation.

5.2.1 4-current-density

Consider a point particle whose trajectory is given $\xi^\mu(\tau)$ as a function of its proper-time. For the sake of simplicity, we work in *Minkowski Cartesian* coordinates. We can make a 4-current density using only scalars and 4-vectors, objects that behave nicely under Lorentz transformations:

$$j^\mu(x) = ec\int d\tau\ \delta^{(4)}\big(x-\xi(\tau)\big)\dot{\xi}^\mu(\tau) \tag{5.5}$$

where dot is a derivative with respect to the proper time. The scalar factor ec fixes the dimensions to the dimensions of current density.

To relate this expression to Eq. (5.4) integrate over τ, getting rid of one of the delta functions. Since ξ is a real orbit, there is a 1-1 correspondence between coordinate time $\xi^0(\tau)/c$ and the proper time τ. Changing variables from $c\tau$ to ξ^0

$$\begin{aligned} c\int d\tau\ \delta^{(4)}\big(x-\xi(\tau)\big)\dot{\xi}^\mu(\tau) &= c\int d\xi^0\ \delta^{(4)}\big(x-\xi\big)\dot{\xi}^\mu\frac{d\tau}{d\xi^0} \\ &= \int d\xi^0\ \delta^{(3)}\big(\mathbf{x}-\xi\big)\delta(ct-\xi^0)v^\mu(\xi^0) \\ &= \delta^{(3)}\big(\mathbf{x}-\xi(t)\big)v^\mu(t) \end{aligned}$$

where t' is the solution of $ct' = \xi^0(\tau)$ and $v^\mu = (c,\mathbf{v})$. The result is a pleasant surprise because it coincides with

$$(c\rho, \mathbf{j})$$

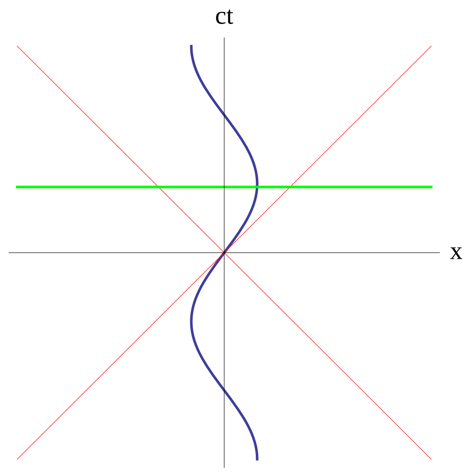

Figure 5.3: The parametrized orbit $\xi^\mu(\tau)$.

derived before we knew anything about Lorentz invariance. There are no relativistic corrections one needs to make to the classical formulas for charge densities and currents. $\delta^{(3)}$ is not a density in Minkowsky space, and v^μ is not a 4-vector. However, together they conspire to give the 4-vector (density) j^μ.

5.2.2 Charge conservation

The total charge, at any given instant,

$$\int dV \rho(\mathbf{x}, t) = e \int dV \delta^{(3)}(\mathbf{x} - \xi(t)) = e$$

is independent of the time slice t[2]. This is charge conservation.

The local, differential equations, expressing this is the continuity equation which we now derive: Consider the rigid transport of a bump function: $\rho(\mathbf{x} - \xi(t))$. Write $\dot{\xi} = \mathbf{v}$. Then

$$\partial_t \rho\big(\mathbf{x} - \xi(t)\big) = -(\mathbf{v} \cdot \nabla)\rho\big(\mathbf{x} - \xi(t)\big)$$

Evidently,

$$\nabla \cdot \big(\mathbf{v}(t)\rho(\mathbf{x} - \xi(t))\big) = (\mathbf{v} \cdot \nabla)\rho\big(\mathbf{x} - \xi(t)\big)$$

and thus

$$0 = \partial_t \underbrace{\rho}_{density} + \nabla \cdot \underbrace{(\mathbf{v}\rho)}_{current} = \partial_\mu j^\mu, \quad j^\mu = (c, \mathbf{v})\rho$$

A point charge is the limit $\rho \to \delta$.

[2] And in which frame the slice is taken. You may take this to be the definition of $\delta^{(3)}$.

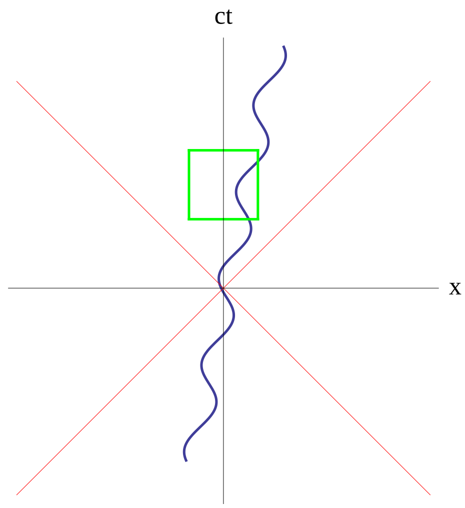

Figure 5.4: Charge conservation expresses the fact that the orbit is a continuous curve which does not terminate and moves always into the future. If it enters a box in space-time it also leaves it. If the orbit enters the box at the bottom leaves it at the top we say that the charge in the box is conserved. If it leaves and enters on the sides we say that incoming current balances the outgoing current.

Once this equation holds for one charge, it holds for any number. When we consider huge numbers of charges with poor spatial resolution we may then think of $j^\mu(x)$ as a smooth function on space time, which satisfies the continuity equation.

Example 5.5 (Orders of magnitudes)**.**

1. *A current of 1 Ampere transports* 6×10^{18} *electrons per second*

2. *A copper wire of cross section* $S = 1\,[mm^2]$ *has* $S8 \times 10^{20}\,[cm^{-1}]$ *atoms per unit length. Since it copper has valence 2, the number of electrons per unit length is* $Sn \approx 1.6 \times 10^{21}\,[cm^{-1}]$*. If yo use the classical formula for the current* $I = env$ *you find that the velocity associated with ampere in such a wire is very small, about* $40\ [\mu/sec]$*. In reality, electrons move in copper with Femi velocity which is of the order of* $1/137$ *of the speed of light. The small velocity we have computed may then be interpreted as representing the tiny displacement of the Fermi sphere from the origin.*

5.2.3 Current conservation and gauge invariance

We want to generalize the expression for S_{int} form a finite collection of charges to a continuous distribution. For a single charge the action representing the interaction is

$$\begin{aligned} S_{int} &= \frac{e}{c}\int A_\mu(\xi)u^\mu(\tau)\,d\tau \\ &= \frac{e}{c^2}\int A_\mu(\xi)u^\mu(\tau)\,\delta^{(3)}(\mathbf{x}-\xi(\tau))\;dV\,d(c\tau) \\ &= \frac{e}{c^2}\int A_\mu\big(\xi^0,\mathbf{x}\big)v^\mu(\xi^0)\,\delta^{(3)}(\mathbf{x}-\xi(t))\;dV\,d\xi^0 \\ &= \frac{e}{c^2}\int A_\mu\big(x\big)v^\mu(t)\,\delta^{(3)}(\mathbf{x}-\xi(t))\;d\Omega \end{aligned}$$

The middle two terms are the 4-current, hence

$$S_{int} = \frac{1}{c^2}\int A_\mu(x)j^\mu(x)\,d\Omega \tag{5.6}$$

describing both smooth and discrete 4-current distributions.

5.2.4 Gauge invariance and the continuity equation

We have seen that under a change of gauge S_{int} for a single charge changed by a boundary term. This then implied the gauge invariance of the Euler-Lagrange equations. It is interesting to reconsider this issue for smooth current distributions. We will lean something. Under a change of gauge

$$A_\mu \to A_\mu + \partial_\mu\Lambda$$

and

$$S_{int} \to S_{int} - \frac{1}{c^2}\int(\partial_\mu\Lambda)\,j^\mu d\Omega$$

Allowing for curvilinear coordinates, the integrand can be rearranged as

$$\sqrt{|g|}(\partial_\mu\Lambda)\,j^\mu = \partial_\mu(\sqrt{|g|}\Lambda\,j^\mu) - \Lambda\,\partial_\mu\left(\sqrt{|g|}j^\mu)\right)$$

The first term gives a boundary term and vanishes if $\Lambda \to 0$ at infinity. S_{int} is therefore guaranteed to be gauge invariance provided the current satisfies the continuity equation holds

$$\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}j^\mu\right) = 0 \tag{5.7}$$

5.3 Lagrangian field theory

In Lagrangian mechanics the basic object is the Lagrangian, $L(q_j, \dot{q}_j, t)$, a function of the "generalized coordinates" q_j and their velocities $\dot{q}_j$ and j labels the degrees of freedom. Lagrangian field theory can be viewed as a generalization of Lagrangian mechanics to infinitely many degrees of freedom where the discrete index j is replaced by $\mathbf{x}$, a point in space. The Lagrangian is then of the form $L = \int d\mathbf{x}\,\mathcal{L}_F$ where $\mathcal{L}_F$ is a suitable Lagrangian density for the field: a function of the fields and their time derivatives.

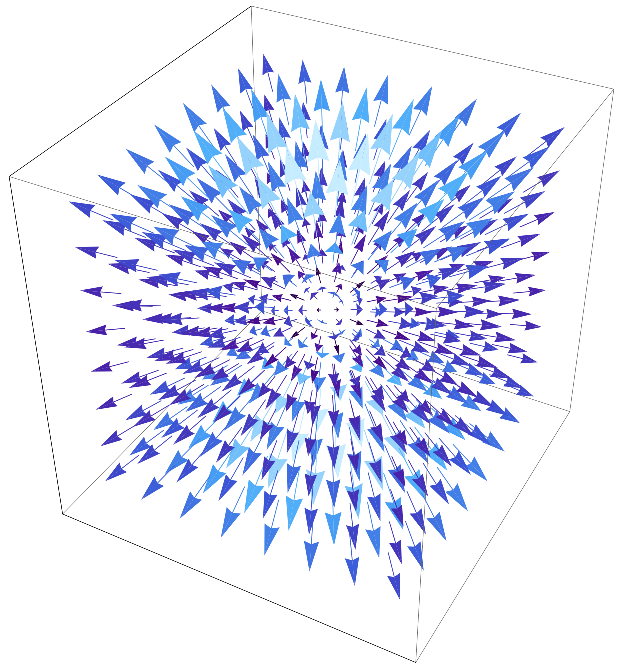

Figure 5.5: The action associated a number with a given field configuration and a box in space-time. We allow variation of A^μ inside the box: The variation vanishes outside the box and on its boundary. This is the analog of what we do when we vary the path.

5.3.1 The Lagrangian of the electromagnetic field

Now we come to deciding what replaces the q_j and $\dot{q}_j$ for the electromagnetic field. Two natural choices are $F_{\mu\nu}$ and A_μ. The right choice is, of course, the one that reproduces Maxwell equations and turns out to be

$$q_j \leftrightarrow A^\mu(x), \quad \dot{q}_j \leftrightarrow \dot{A}^\mu(x)$$

This agrees with S_{int} being a function of A not F.

Lorentz invariance of the Euler-Lagrange equations is automatically guaranteed if the action was a Lorentz scalar. We have at our disposal two Lorentz scalars

$$F \cdot F = F_{\mu\nu} F^{\mu\nu}, \quad F \cdot F^* = F_{\mu\nu} (F^*)^{\mu\nu},$$

whose dimensions are energy density. Since the volume element in space-time $d\Omega$ is a Lorentz scalar and since the action must[3] have dimension $[Et]$, two candidates for the field action are suitable numerical multiples of

$$\frac{1}{c} \int \underbrace{d\Omega}_{volume\ element} F \cdot F, \quad \frac{1}{c} \int d\Omega F^* \cdot F$$

[3] So we can add it to S_p and S_{int}

However, using the homogeneous Maxwell equation

$$\begin{aligned}(F^*)\cdot F &= 2(F^*)^{\mu\nu}(\partial_\mu A_\nu)\\ &= 2\partial_\mu\left((F^*)^{\mu\nu}\,A_\nu\right)\end{aligned}$$

This means that the associated action is a boundary term: Its variation vanishes identically.

We are left with the first candidate. We need first to decide on the sign so that the action will have a minimum rather than a maximum. Recall that in Lagrangian mechanics the kinetic energy comes with a positive sign. Now $\mathbf{E}$ is linear in $\dot{\mathbf{A}}$ so the $\mathbf{E}^2$ must come with a positive coefficients.

$$S_F = -\frac{1}{16\pi c}\int F\cdot F \underbrace{d\Omega}_{\sqrt{|g|}\prod_\mu dx^\mu} \tag{5.8}$$

The 16π is the choice[4] that gives Maxwell equations in c.g.s units and in particular leads to the Coulomb potential $\frac{e}{r}$.

5.4 Variation of the field: Rules of the game

The actions S_F assigns a number for any given field $A_\mu(x)$. The action is then a functions whose arguments are functions too. Such objects are sometimes called functionals.

We consider variation of the action due to variations δA_μ. We shall consider local variations only, namely, variations in a finite region of space time: $\delta A_\mu = 0$ outside some large space-time box, so we do not need to worry about infinite variations that can come with infinite boxes.

5.4.1 Variation of the field: Calculations

The variation of A causes a variation of $F\cdot F$ which is

$$\delta(F_{\mu\nu}F^{\mu\nu}) = 2F^{\mu\nu}\,\delta(F_{\mu\nu})$$

and

$$\delta(F_{\mu\nu}) = \partial_\mu\delta A_\nu - \partial_\nu\delta A_\mu$$

By the anti-symmetry of F

$$\delta(F_{\mu\nu}F^{\mu\nu}) = 2F^{\mu\nu}\left(\partial_\mu\delta A_\nu - \partial_\nu\delta A_\mu\right) = 4F^{\mu\nu}\left(\partial_\mu\delta A_\nu\right)$$

In variational calculation one wants to end up with an expression proportional to δA_μ: We need to get rid of terms of the form $\delta\partial A$ by integrating by parts.

[4] A natural choice that gets rid of the 4π which is special to 3-dimensions, is to replace 16π by 4 where Coulomb potential is $e/4\pi r$.

However, in the case of curvilinear coordinates, we must also pay attention to $\det |g|$ in the volume element[5]

$$\sqrt{|g|}\delta(F_{\mu\nu}F^{\mu\nu}) = 4\partial_\mu(\sqrt{|g|}F^{\mu\nu}\,\delta A_\nu) - 4(\sqrt{|g|}\partial_\mu F^{\mu\nu})\,\delta A_\nu$$

The first term looks like the divergence of the vector field $\sqrt{|g|}F^{\mu\nu}\,\delta A_\nu$ and by a Gauss type theorem can be converted to a 4-surface integral on the boundary of the box where $\delta A = 0$. Hence,

$$\delta S_F = bdry\ terms + \frac{1}{4\pi c}\int \underbrace{d\Omega \frac{1}{\sqrt{|g|}}}_{\Pi_\alpha\, dx^\alpha} (\partial_\mu \sqrt{|g|}F^{\mu\nu})\,\delta A_\nu \tag{5.9}$$

5.4.2 Variation of the interaction

We have already determined the action associated with the interaction when we studied the dynamics of relativistic charged particles as

$$S_{int} = \frac{1}{c^2}\int A_\nu j^\nu\, d\Omega$$

To get the field equations we consider variations δA for a given source term j. This variation gives

$$\delta S_{int} = \frac{1}{c^2}\int \delta A_\nu j^\nu\, d\Omega \tag{5.10}$$

5.4.3 The inhomogeneous Maxwell equations

The Euler-Lagrange equations for the fields are those that minimize the action $S_{field} + S_{int}$. The minimizer is the stationary point of the variation:

$$0 = \delta S_{field} + \delta S_{int} = \frac{1}{4\pi c}\int d\Omega \left(\partial_\mu F^{\mu\nu} + \frac{4\pi}{c}j^\nu\right)\delta A_\nu\, d\Omega$$

This will vanish for arbitrary variation δA provided

$$\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}F^{\nu\mu}\right) = \frac{4\pi}{c}j^\nu \tag{5.11}$$

These are the 4-inhomogeneous Maxwell equations in curvilinear coordinates. The space-time formalism encapsulate the inhomogeneous Maxwell's equations in a neat a concise form.

[5] g is a function of the coordinats. It is not a function of A and is not affected by the variation.

5.4.4 Current conservation

We have derived Maxwell equations as the Euler-Lagrange equations for the field A_μ for a given source term j^μ. This derivation did not assume that the source j^μ is a reasonable physical current and did not explicitly require that it be a conserved current. However, a-posterior, Maxwell equation enforce current conservation on j as a direct consequence of the fact that F is an antisymmetric tensor:

$$0 = \underbrace{\partial_{\mu\nu}\left(\sqrt{|g|}F^{\mu\nu}\right)}_{0\ by\ symmetry} = \frac{4\pi}{c}\partial_\nu\left(\sqrt{|g|}j^\nu\right)$$

in accordance wirh Eq. 5.7. If the source j was not current conserving, Maxwell equations would not form a consistent set of equation.

5.4.5 3-D form

To translate back Maxwell equations from their covariant space-time form to 3-D form, consider first the $\nu = 0$ equation $\partial_\mu F^{\mu 0} = j^0$. Since

$$-E_j = F_{0j} = F^{j0} \quad j^0 = c\rho$$

we get Gauss-Coulomb law

$$\nabla \cdot \mathbf{E} = 4\pi\rho \Longleftrightarrow \partial_\mu F^{0\mu} = \frac{4\pi}{c} j^0 \tag{5.12}$$

The spatial components are:

$$\partial_\mu F^{j\mu} = \partial_0 F^{j0} + \partial_k F^{jk} = -\frac{1}{c}\,\partial_t E_j - \partial_k(\varepsilon_{ikj}B_i) = \frac{4\pi}{c} j^j$$

Using

$$\partial_k(\varepsilon_{ikj}B_i) = -\varepsilon_{jki}\partial_k B_i = -(\nabla \times \mathbf{B})_j$$

This gives Ampere-Maxwell equation:

$$-\dot{\mathbf{E}} + \nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} \Longleftrightarrow \partial_\mu F^{k\mu} = \frac{4\pi}{c}\, j^k \tag{5.13}$$

5.4.6 Time reversal

Time-reversal of the orbits of the sources $\xi_a(t) \mapsto \xi_a(-t)$, sends $\rho(t, \mathbf{x}) \mapsto \rho(-t, \mathbf{x})$ but flips the currents $\mathbf{J}(t, \mathbf{x}) \mapsto -\mathbf{J}(-t, \mathbf{x})$. It follows that solutions to Maxwell's equations transfrom under time-reversal as $\mathbf{E}(t, \mathbf{x}) \mapsto \mathbf{E}(-t, \mathbf{x})$ and $\mathbf{B}(t, \mathbf{x}) \mapsto -\mathbf{B}(-t, \mathbf{x})$. We say that $\mathbf{E}$ is even under time reversal and $\mathbf{B}$ is odd.

5.4.7 Maxwell equations: Structure

The traditional form of Maxwell' equations, as equations for $\mathbf{E}$ and $\mathbf{B}$, is a set of two scalar equations and two vector equations. The two scalar equations are Gauss laws:

$$\underbrace{\nabla\cdot\mathbf{E}=4\pi\rho}_{Gauss}\,,\quad \nabla\cdot\mathbf{B}=0 \tag{5.14}$$

The two vector equations are Faraday and Maxwell-Ampere laws

$$\underbrace{\dot{\mathbf{E}}=\nabla\times\mathbf{B}-\frac{4\pi}{c}\mathbf{J}}_{Ampere}\,,\quad \underbrace{\dot{\mathbf{B}}=-\nabla\times\mathbf{E}}_{Faraday} \tag{5.15}$$

and dot denotes *partial* derivative with respect to ct. The vector equations are written in the form of first order evolution equations that allow to propagate $\mathbf{E}$ and $\mathbf{B}$ in time, given their initial values and the source $\mathbf{J}$.

In total, there are 8 Maxwell equations for the 6 unknown fields. This looks like an over constrained system. It is better to view them as two evolution vector equation for two vectors and view the scalar equations as a constraint on the initial data. This constraint is preserved by the evolution provided $(\rho,\mathbf{J})$ satisfy the continuity equation.

Exercise 5.6. *Show that the evolution respects the constraint.*

Example 5.7 (Current carrying wire)**.** *An electrically neutral, infinitely long, metallic straight wire (along the z-axis) with circular cross section of radius a carries a stationary current I. Suppose Ohm's law in the form $\mathbf{J}=\sigma\mathbf{E}$ with σ a constant in the wire. Find the profiles of the electric and magnetic fields inside and outside the wire. Assume cylindrical symmetry, translational symmetry in the z direction and stationarity. Analyze the problem in cylindrical coordinates (ρ,θ,z).*

By Gauss and the symmetry $E_\rho=0$ Since the magnetic field is assumed stationary, by Faraday and the symmetry $E_\theta=0$. You might then be tempted to say that outside the wire, *one should have $E_z=0$. This, however, leads to a contradiction: Combinig Gauss and Faraday*

$$0=\nabla\times\mathbf{E}\Rightarrow 0=\nabla\times\big(\nabla\times\mathbf{E}\big)=-\Delta\mathbf{E}+\nabla\big(\nabla\cdot\mathbf{E}\big)\Rightarrow\Delta\mathbf{E}=0$$

Which says that $\mathbf{E}$ is harmonic everywhere. Hence, if it is zero outside the wire, it is zero everywhere. This, together with Ohm's law, contradicts the assumption that the wire carries current.

Let us then retreat to the next line of defense and take $\mathbf{E}=E_0\hat{\mathbf{z}}$ with E_0 a constant. This is still harmonic By the integral form of Ampere

$$\mathbf{B}=\frac{2I}{c\rho}\hat{\theta}\times\begin{cases}1 & \rho>a\\ \left(\frac{\rho}{a}\right)^2 & \rho<a\end{cases}$$

where I is the total current. We have used the fact that inside the wire, the constancy of $\mathbf{E}$ implies the constancy of $\mathbf{J}$.

It may be a little shocking at first that a neutral current carrying wire bundles with it an electric field that does not decay as you get far from the wire. This is a pathology due to the assumed infinite length of the wire.

5.5 Dielectric and magnetic media

So far we considered the action of the field in Minkowski space where the action was essentially determined by requiring Lorentz invariance. A medium breaks Lorentz invariance: The dielectric and permeability tensors that characterize a medium are Lorentz invariant only in the trivial case where both are the identity tensors. These tensors represent microscopic charge densities and currents in the medium which may not be directly accessible. I want to outline the underlying theory.

The homogeneous Maxwell equations express the fact that $\mathbf{E}$ and $\mathbf{B}$ are derived from the potential and as such, are oblivious to the: They remain intact:

$$\nabla \cdot \mathbf{B} = 0, \quad \dot{\mathbf{B}} + \nabla \times \mathbf{E} = 0$$

For the inhomogeneous equations we want to encode the (possible unknown) sources in tw fields which called polarization $\mathbf{P}$ and magnetization $\mathbf{M}$. They are defined as the solutions of equations which are formally like Gauss and Ampere law up to funny normalization[6] :

$$\rho = -\nabla \cdot \mathbf{P}, \quad \mathbf{J} = \nabla \times \mathbf{M} - \dot{\mathbf{P}}$$

As these are supposed to represent the local sources in the material, we impose the boundary condition

$$\mathbf{P}(\mathbf{x}) = \mathbf{M}(\mathbf{x}) = 0, \quad \mathbf{x} \in \{\textit{outside body}\}$$

(We allow for currents and surface charges.) Define the auxiliary fields $\mathbf{D}$ and $\mathbf{H}$

$$\mathbf{D} = \mathbf{E} + 4\pi \mathbf{P}, \quad \mathbf{H} = \mathbf{B} - 4\pi \mathbf{M}$$

They satisfy the homogeneous Gauss and Ampere equations:

$$\nabla \cdot \mathbf{D} = 0, \quad -\dot{\mathbf{D}} + \nabla \times \mathbf{H} = 0$$

So far, we have not done anything: We made no approximation and no assumption. We have just recast the source terms in terms of the polarization and magnetization field and introduced the auxiliary fields $\mathbf{D}$ and $\mathbf{H}$.

Exercise 5.8. *Show that the if you treat Gauss law as a constraint, Ampere's evolution law respects the constraint.*

[6] The choice of normalization is unit dependent. The choice made here is standard in cgs.

The sources respond to the fields and vice versa. If, in the absence of external driving fields, the body in question is in equilibrium, one would expect a linear relation between cause and effect. Moreover, if we are only interested in macroscopic length and memoryless response, the relation should be local:

$$D^j = \varepsilon^{jk} E_k, \quad B^j = \mu^{jk} H_k \tag{5.16}$$

This replace the unknown microscopic sources by material specific functions ε and μ. For a homogeneous system, these are constant matrices. For isotropic bodies, they are material specific constant numbers. Eq. 5.16 is known as *constitutive relation.*

Exercise 5.9. *Derive Maxwell's equation from the variation of the action*

$$S = \frac{1}{16\pi c} \int d\Omega (D^j E_j - B^j H_j)$$

subject to the constitutive relations.

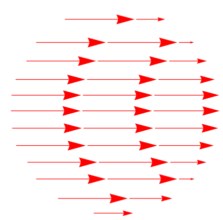

Figure 5.6: Sphere with constant polarization

Exercise 5.10. *Explain why one might expect ε to be a positive matrix (i.e. real symmetric with positive eigenvalues) and why there is no similar expectation from μ.*

Exercise 5.11. *Show that the charge distribution of a homogeneously polarized sphere of radius a is concentrated on the surface with surface density*

$$\mathbf{P} \cdot \hat{\mathbf{x}}\ \delta(|\mathbf{x}| - a)$$

Exercise 5.12. *Show that the current distribution of a homogeneously magnetized sphere of radius a is concentrated on the surface with surface density*

$$\mathbf{M} \times \hat{\mathbf{x}}\ \delta(|\mathbf{x}| - a)$$

5.6 New Physics

Lagrangian field theory is a framework that allows one to repackage existing theories such as Maxwell's electrodynamics in an elegant formalism, but perhaps more importantly, allows to explore new theoretical models. Most models are, at the end, just models, and their best application is as questions in homework sets and exams. But occasionally, some turn out to capture new physics. Here are three examples.

5.6.1 The quantum Hall effect and Chern-Simon action

The Integer Quantum Hall effect, was discovered by Klaus von Klitzing in 1980. This discovery ushered a new era of research now called the study of *topological phases*. These phases are are intrinsically quantum and labeled by integers[7]. In the case of the integer quantum Hall effect the discovery of phases labeled a quantized value of conductance. In two dimensions that conductance is, in general, a matrix. If, in addition, the system is isotropic the conductance matrix is necessarily of the form

$$\mathbf{J} = \begin{pmatrix} \sigma & \sigma_H \\ -\sigma_H & \sigma \end{pmatrix} \mathbf{E}$$

Exercise 5.13. *Show this*

The diagonal part is the dissipative conductance and the off-diagonal is the Hall conductance. von Klitzing found that in certain two dimensional systems, at sufficiently low temperatures, and with sufficiently strong magnetic field, the system is characterized by non-dissipative topological phases where

$$\sigma = 0, \quad \sigma_H \in \mathbb{Z}\frac{e^2}{h}$$

Planck constant is an indication that the phenomenon is quantum.

A field theory that encapsulates the Hall effect relies on the Chern Simon action. In 2+1 dimensions we can construct a the scalar $F^* \cdot A$. The Chern Simon Lagrangian density:

$$\mathcal{L}_{CS} = -\frac{\nu\sigma_0}{2c^2}\,(F^*)^\alpha A_\alpha = -\frac{\nu\sigma_0}{4c^2}\,\varepsilon^{\alpha\beta\gamma}F_{\beta\gamma}A_\alpha, \quad \sigma_0 = \frac{e^2}{h}$$

α runs over $0, 1, 2$. ν is a dimensionless number h is Planck constant. σ_0 is the quantum unit of conductance with dimensions of velocity (in c.g.s) . So the prefactor the same dimensions as in the corresponding Maxwell term. This guarantees that the action has dimensions $[Energy \times time]$. Planck constant naturally appears since the origin of the term is quantum.

Exercise 5.14. *Show that* $F^* = (B, -E_2, E_1)$.

[7] A rational number is a pair of integers.

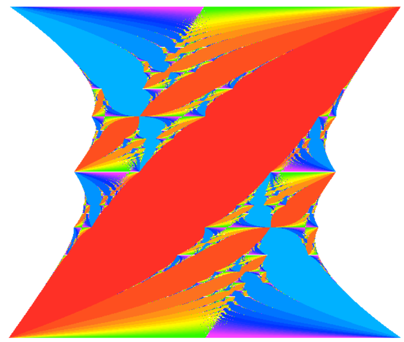

Figure 5.7: The phase diagram for the Integer quantum Hall effect for the Hofstadter model on the triangular lattice at $T = 0$. The vertical axis is the magnetic flux through the unit cell. The horizontal axis is the chemical potential. Figure made by Gal Yehoshua for an undergrad project.

Exercise 5.15. *Explain why $\mathcal{L}_{CS}$ has the right dimensions.*

You may worry that the Chern Simon Lagrangian density is not gauge invariant. This causes no problem for the variation of the action since:

Exercise 5.16. *Show that under a change of gauge $A_\mu \to A_\mu + \partial_\mu \Lambda$, the Chern-Simon Lagrangian density $\mathcal{L}_{CS}$ changes by a boundary term.*

The variation of the action is

$$\begin{aligned}\delta(F^* A) &= \varepsilon^{\alpha\beta\gamma}(\delta\partial_\beta A_\gamma)A_\alpha + (F^*)^\alpha \delta A_\alpha \\ &= -\varepsilon^{\alpha\beta\gamma}(\delta A_\gamma)(\partial_\beta A_\alpha) + (F^*)^\alpha \delta A_\alpha + \partial_\beta(\dots) \\ &= 2(F^*)^\alpha \delta A_\alpha + \partial_\beta(\dots)\end{aligned}$$

As a model for the quantum Hall effect take the Lagrangian density

$$\mathcal{L}_{CS} + \frac{1}{c^2} A_\alpha j^\alpha$$

The corresponding Euler-Lagrange equations are

$$-\frac{\nu\sigma_0}{c^2}(F^*)^\alpha + \frac{1}{c^2} j^\alpha = 0 \Longrightarrow \nu\sigma_0 F^* = j$$

Unlike Maxwell's equations, this is not a differential equation, but an algebraic relation between the fields and sources. In components

$$\nu\sigma_0 B = c\rho, \quad -\nu\sigma_0 E_2 = j_1 \quad \nu\sigma_0 E_1 = j_2$$

We have reproduced some of the known features of the Hall effect, namely

- Ohms law: The current is proportional to the fields

- There is no dissipation
- The conductivity tensor is isotropic
- The magnetic field determines the charge density

To explain why ν must be quantized one needs to input quantum mechanics, which is good, but also somewhat artificial assumptions about the topology of space time. Here is a sketch.

In quantum mechanics one is not looking at the extremal actions, but rather one allows the system to explore all path configurations, weighted by

$$e^{iS/\hbar}$$

Gauge invariance then means that under a gauge transfromation Λ

$$\frac{\nu\sigma_0}{c^2}\int_{bdry} F^*_\alpha \Lambda dS^\alpha = 0 \quad Mod\ (2\pi\hbar)$$

Now comes a shady trick. We assume that time is a circle we take for Λ the transformation corresponding to unit of emf-action: A jump in Λ across the time cut (also, the quantum unit of flux)

$$\Delta\Lambda = \frac{hc}{e}$$

The quantization condition is then reduced to spatial integration

$$\frac{\nu\sigma_0}{ce}\int F^*_0 dS^0 = \frac{\nu\sigma_0}{ce}\int B\ dxdy = 0 \quad Mod\ 1$$

Now, by an argument of Dirac, if space is a close manifold, say a torus then the total flux is quantized

$$\int B\ dxdy = m\frac{hc}{e}, \quad m \in \mathbb{Z}$$

We finally get

$$\nu m\,\sigma_0\,\frac{h}{e^2} = m\nu = 0\ Mod\ 1$$

which quantizes ν to a rational number.

5.6.2 Axion electrodynamics

In 3+1 dimensions $F^* \cdot F$ is a boundary term, and as such it does not affect the equations of motion. However, this terms can do something interesting if its coupling constant is replaced by a function. The function is called the Axion field $\phi(x)$:

$$\mathcal{L} = -\frac{1}{16\pi c}\,F\cdot F - \frac{\sigma_0}{8\pi}\,\phi(x)F^*\cdot F$$

Evidently, this Lagrangian is gauge invariant[8]. Since

$$\phi F^* \cdot F = 2\phi\,(F^*)^{\mu\nu}\partial_\mu A_n = 2\phi\partial_\mu\big((F^*)^{\mu\nu}A_\nu\big) = -2(\partial_\mu\phi)\,(F^*)^{\mu\nu}A_\nu + \partial_\mu(\dots)$$

we can then replace it by (a formally gauge dependent Lagrangian density)

$$\mathcal{L} = -\frac{1}{16\pi c}F\cdot F - \frac{\sigma_0}{4\pi}(\partial_\mu\phi)(F^*)^{\mu\nu}A_\nu$$

Up to boundary terms, the variation of the action is

$$4\pi c\,\delta\mathcal{L} = (\partial_\mu F^{\mu\nu})\,\delta A_\nu + \alpha\,(\partial_\mu\phi)(F^*)^{\mu\nu}\delta A_\nu,$$

Exercise 5.17. *Determine* α (Answer:$\alpha = -3\frac{e^2}{hc}$)

The Euler-Lagrange equations are

$$(\partial_\mu F^{\mu\nu}) = -\alpha\,(\partial_\mu\phi)(F^*)^{\mu\nu}$$

It is instructive to write the equations in terms of $\mathbf{E}$ and $\mathbf{B}$. Gauss law (without external sources) now takes the form

$$\nabla\cdot\mathbf{E} = \alpha(\nabla\phi)\cdot\mathbf{B}$$

Ampere law

$$\partial_0\mathbf{E} - \nabla\times\mathbf{B} = \alpha\dot{\phi}\mathbf{B} + \alpha\nabla\phi\times\mathbf{E}$$

Exercise 5.18. *Verify.*

When ϕ is a constant one recovers the sourceless Maxwell equations. In general, $\partial_\mu\phi$ acts like a source term in Maxwell equations.

5.6.3 Quantum interface

Axion electrodynamics started as a speculative model of an elementary particle: The Axion. A different perspective was taken by Qi et. al. who proposed looking at the interface between topologically distinct quantum phases. In the bulk of the two insulators Maxwell theory applies. This says that ϕ is constant in each. The constant is quantized to be 0 or π, by an gauge invariance argument similar to the one in CS theory of the quantum Hall effect. By definition, the two insulators are topologically distinct if the constant is different in each. This means that $\nabla\phi = \delta^{(2)}(\mathbf{x})\mathbf{n}$. Gauss law is replaced by

$$\nabla\cdot\mathbf{E} = \alpha\delta^{(2}(\mathbf{x})\mathbf{n}\cdot\mathbf{B}$$

The magnetic field on the surface acts as if there was a charge on the interface. This is something we have already encountered in the CS theory of the quantum Hall effect.

[8] Since $\mathbf{E}$ is even and $\mathbf{B}$ odd under time reversal the Lagrangian breaks time-reversal unless ϕ is also odd under time reversal. The notion of time reversal in the quantum case is subtle.

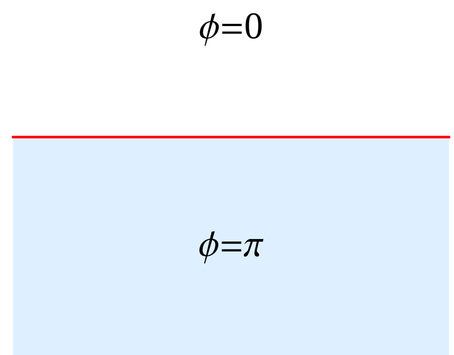

Figure 5.8: The interface between two insulators that are topologically distinct gives rise to a singular Axion field.

Ampere law is replaced by

$$\partial_0 \mathbf{E} - \nabla \times \mathbf{B} = \alpha \delta^{(2)}(\mathbf{x}) \mathbf{n} \times \mathbf{E}$$

This means that electric field on the surface acts as if there were currents at the interface. In particular in Axion electro-Magneto-statics

$$\nabla \cdot \mathbf{E} = \alpha \delta^{(2}(\mathbf{x}) \mathbf{n} \cdot \mathbf{B}, \quad -\nabla \times \mathbf{B} = \alpha \delta^{(2)}(\mathbf{x}) \mathbf{n} \times \mathbf{E}$$

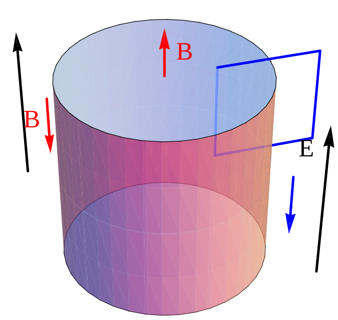

Figure 5.9: A cylinder of a (non-trivial) topological insulator is immersed in vacuum (trivial insulator). A uniform electric field **E** in the axial direction leads to response in a magnetic field inside the cylinder

5.6.4 Magnetic response to an electric field

Consider a non-trivial insulator in the form of an infinitely long cylinder of radius a immersed in the trivial vacuum and z-oriented. Take uniform electric field everywhere and uniform magnetic field inside the cylinder

$$\mathbf{E} = E_0\hat{\mathbf{z}}, \quad \mathbf{B} = B_0\,\theta(a-\rho)\,\hat{\mathbf{z}}$$

Clearly

$$\nabla\cdot\mathbf{E} = 0, \quad \mathbf{n}\cdot\mathbf{B} = 0$$

so Axion Gauss law is satisfied.

$\nabla\times\mathbf{B}$ vanishes inside the cylinder and outside the cylinder, but has a delta jump on the boundary. The magnetic field in the wire is proportional to the constant electric field:

Exercise 5.19. *Use Stokes theorem for the (blue) rectangle shown in the figure to show that*

$$B_0 = \alpha E_0$$

5.6.5 Phantom monopoles

In electrostatics an electric charge near a (grounded) conductor has an oppositely charged image. I want to show that in Axion electrodynamics you can create image which is a magnetic monopole. We want to find consistent solution

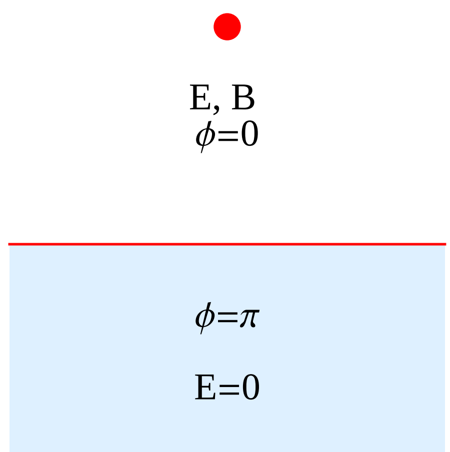

Figure 5.10: A electric charge, (red dot) is placed near a different topological insulator with zero fields. On the left, the physical setup. On the right the image method.

as if there was a magnetic monopole in the lower half space. Namely

$$\mathbf{B}(\mathbf{x}) = g\frac{\mathbf{x}+d\hat{\mathbf{z}}}{|\mathbf{x}+d\hat{\mathbf{z}}|^3}\,\theta(z) + (yet\ unknown\ function)\theta(-z)$$

Exercise 5.20. *Explain why* $\nabla \cdot \mathbf{B} = \nabla \times \mathbf{B} = 0$ *in the half space* $z > 0$

The magnetic provides a source term for the electric field. The source term is precisely the same as the source term in the corresponding electrostatic image charge problem provided

$$g\alpha = 2e$$

Now, if we add to this field the electric field given by electric monopole of charge e above the x-y plane we obtain the same electric field configuration as in the electrostatic image charge problem, everywhere, i.e.

$$\mathbf{E} = e\left(\frac{\mathbf{x} - d\hat{\mathbf{z}}}{|\mathbf{x} - d\hat{\mathbf{z}}|^3} - \frac{\mathbf{x} + d\hat{\mathbf{z}}}{|\mathbf{x} + d\hat{\mathbf{z}}|^3}\right)\theta(z)$$

This describes the electric field everywhere. It remains to see what values $\mathbf{B}$ takes in the lower half-space. Now $\mathbf{E} \cdot \mathbf{n} = 0$ on the boundary and so we see that $\mathbf{B}$ is the solution of

$$\nabla \times \mathbf{B} = \nabla \cdot \mathbf{B} = 0$$

everywhere subject to the boundary condition that fixes $\mathbf{B}$ on the plane $z = 0$.

We introduce a scalar potential for $\mathbf{B}$ in the lower half pace

$$\mathbf{B} = \nabla\phi, \quad \Delta\phi = 0$$

subject to the boundary condition $\nabla\phi = \mathbf{B}$ on the plane $z = 0$. Evidently

$$\phi(x, y, z = 0) = \frac{g}{\sqrt{x^2 + y^2 + d^2}}, \quad B_z = \partial_z \phi = g\frac{d}{|x^2 + y^2 + d^2|^{3/2}}$$

The problem then reduces to solving Laplace equation with two types of boundary conditions.

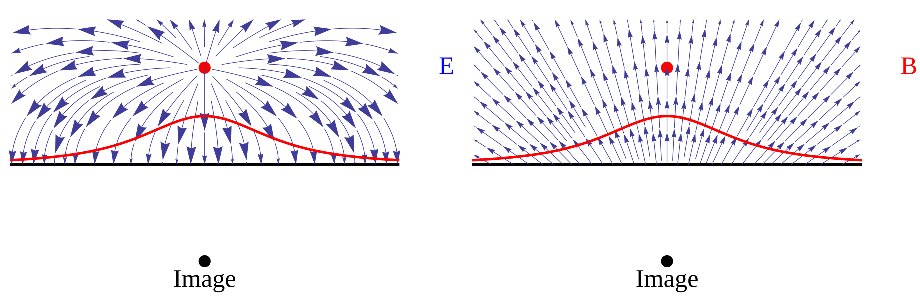

Figure 5.11: The red curve shows the surface charge density that allows the field to terminate at the surface. On the right one sees the response in the form of a magnetic field that seems to have a magnetic monopole at the image point. There is no real magnetic monopole anywhere, of course.

5.6.6 Electrodynamics in 1+1 dimensions

It is not possible to explicitly solve Maxwell's equation for the fields for general motion of the sources. However, in 1+1 dimensions this is possible.

In 1+1 dimensions F is an anti-symmetric 2×2 matrix. Its single entry is the electric field, which is a Lorentz scalar.

Exercise 5.21 (Solution of Maxwell equations for arbitrary motion of a point charge)**.** *Show that for a charged particle with a given, arbitrary, orbit, the solution of Maxwell equations for E takes two constant values in the space-time plane separated by the world line of the particle. Determine the jump across the world line.*

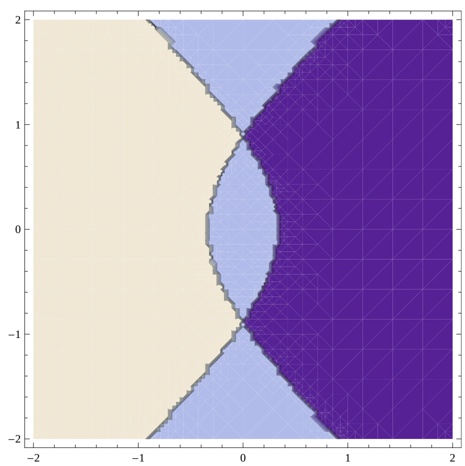

Figure 5.12: Maxwell's equation in 1+1 dimensions can be solved geometrically for arbitrary motion of the source. The figure illustrates the solution for two point sources undergoing constant acceleration. The field takes constant values in the different regions delineated by the orbits of the charges.

In Maxwell's theory the source term is a the vector field of currents j^μ. In 1+1 dimensions there is a different option for a source term, namely a scalar field $\phi(x)$ [9] :

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \phi(x) \varepsilon^{\mu\nu} F_{\mu\nu}$$

This looks first like a different theory, but it is actually equivalent to Maxwell's. The variation of A gives, up to boundary terms,

$$\delta\mathcal{L} = (\partial_\mu F^{\mu\nu}) \delta A_\nu - \partial_\mu \phi(x) \varepsilon^{\mu\nu} \delta A_\nu$$

The Euler-Lagrange equations for this model are then

$$\partial_\mu F^{\mu\nu} = j^\nu, \quad j^\nu = (\partial_\mu \phi) \varepsilon^{\mu\nu} \tag{5.17}$$

[9] Frank Wilczek, who invented this field, called it Axion field.

Note that $\partial_\nu j^\mu = 0$ so the current is conserved. We have recovered Maxwell theory except that the current is interpreted as the gradient of a scalar.

Bibliography Xiao-Liang Qi, et al, Inducing a Magnetic Monopole with Topological Surface States, Science 323, 1184 (2009);

Chapter 6

Conservation laws and the Stress-Energy tensor

The energy, momentum and angular momentum of the electromagnetic field are identified as the conservation laws associated with symmetries of space-time. Maxwell stress tensor is derived from variation of action due to variations of the metric.

6.1 Maxwell stress energy tensor

Our aim here is to identify the energy, momentum, and angular momentum of the electromagnetic field. They express conservation laws. The local form of the conservation law is

$$\partial_\mu T^{\mu\nu} = 0 \tag{6.1}$$

where T is known as Maxwell stress-energy tensor

$$T^{\alpha\beta} = \frac{1}{4\pi}\Big(F^{\alpha\mu}F^{\beta}{}_{\mu} - \tfrac{1}{4}g^{\alpha\beta}F\cdot F\Big) \tag{6.2}$$

T is chosen so that its physical dimension is [*energy density*], or, equivalently [*pressure*]. Explicitly, T is

$$4\pi T = \frac{\mathbf{E}^2+\mathbf{B}^2}{2}\,\mathbb{1} \underbrace{-}_{relative\ sign} \begin{pmatrix} 0 & \mathbf{B}\times\mathbf{E} \\ \mathbf{B}\times\mathbf{E} & E_iE_j + B_iB_j \end{pmatrix}$$

In the first row and columns you recognize the energy density and Poynting vector. This explains the 4π normalization. We shall discuss the other terms below.

6.1.1 Symmetric and traceless

The tensor is symmetric and traceless. Symmetry is obvious from the definition, as g is symmetric. Traceless follows from the fact that $g^\alpha{}_\alpha = 4$ and so

$$T^\alpha{}_\alpha = \frac{1}{4\pi}\Big(F^{\alpha\mu}F_{\alpha\mu} - \tfrac{1}{4}g^\alpha{}_\alpha F^{\mu\nu}F_{\mu\nu}\Big) = 0$$

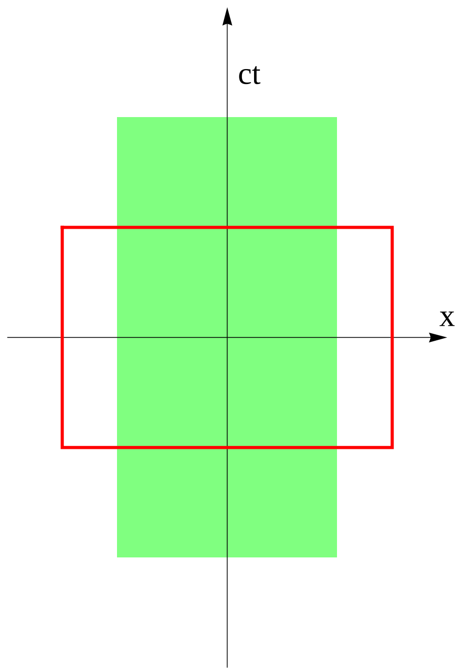

Figure 6.1: The green rectangle represent the field. The red box is a box in space-time. The symmetries of Minkowski space allow to shift (and rotate the box) and fields without affecting the action. For infinitesimal shifts of physical fields, this triviality statement translates to a conservation law.

6.1.2 Conservation laws

Consider a space-time box as in Fig. 6.1. Suppose the spatial box is large enough to embrace all the fields at any given time. Then

$$0 = \int d\Omega\,\partial_\mu T^{\mu\nu} = \int dS_\mu T^{\mu\nu} = \int dS_0 T^{0\nu}\Big|_{t_1}^{t_2} = \int dV\,T^{0\nu}\Big|_{t_1}^{t_2}$$

It follows that the 4 energy momentum vector

$$\int dV\,T^{0\nu} = (\mathcal{E}, c\mathcal{P}^j)$$

is conserved. This identifies the first row (and column) with the energy (momentum) densities.

6.1.3 Nöther currents

Eq. 6.1 is a consequence of the (free) Maxwell equations. However, this alone does not offer insight.

Exercise 6.1. *Show that Eq.6.1 follows from Maxwell's*

The missing insight is that conservation laws expresses the homogeneity of space time, a relation due to Nöther. If you compute the action of a given field configuration in a box, you get the same number if you translate (and rotate) the coordinates. This is a trivial statement, following from the homogeneity of Minkowski space-time. and you would not expect to get any interesting identities from it. The genius of Nöther was to realize that if the field is not an arbitrary field configuration, but one that solves the Euler Lagrange equations, then the statement reduces to statement about the fields on the boundaries of the box. This is what you expect form a conservation law: What comes in through one boundary must leave through another.

To do that Nöther introduced a technical step which is amusing: She split the real operation of coordinate shift into two virtual operations: One that shifts the box–but not the fields–and one that shifts the fields–but not the box. Think of moving your bag in two steps: First you move the content of the bag and second you move the empty bag. For infinitesimal shifts, for fields that satisfy Euler-Lagrange equations, these contributions live on the boundary.

6.1.4 Shifting the field

Consider the change in action due to a shift of the field without a shift of the box. The variation of the action S_F due to arbitrary variation δA_μ of the fields in a *a fixed box*, has been computed in Eq. 5.9 and was found to be

$$4\pi c\ (\delta S_F) = -\underbrace{\int d\Omega\, \partial_\mu (F^{\mu\nu}\delta A_\nu)}_{bdry\ term} + \underbrace{\int d\Omega (\partial_\mu F^{\mu\nu})\delta A_\nu}_{E-L} \tag{6.3}$$

It follows that for fields that satisfy the Euler-Lagrane equation, the variation is a boundary term

$$4\pi c\ (\delta S_F) = -\int d\Omega\, \partial_\mu (F^{\mu\nu}\delta A_\nu) \tag{6.4}$$

A uniform space-time shift by $\delta\xi^\alpha$ leads to variation in the fields

$$\delta A_\mu(x) = -(\partial_\alpha A_\mu)\delta\xi^\alpha \tag{6.5}$$

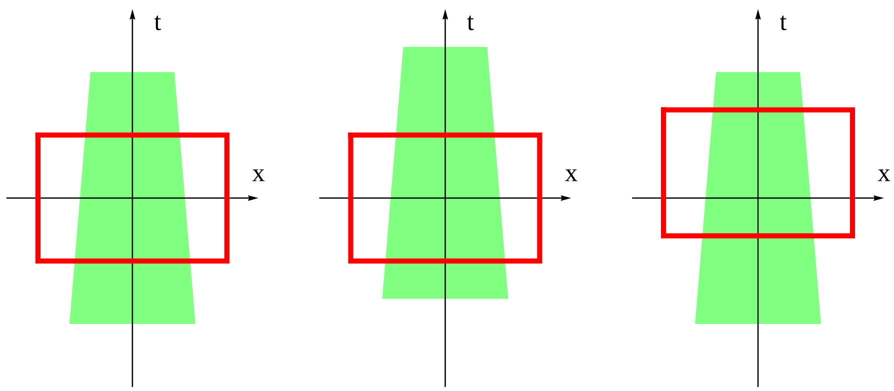

Figure 6.2: The action remains the same when the fields and the integration box are both shifted in space-time. For a small shift the change in action can be split into two virtual shifts: A shift of the field with the box held fixed, shown in the middle figure, and a shift of the box with the field held fixed field, shown on the right.

Exercise 6.2 (Signs-Sigh)**.** *Explain the minus sign.*

Inserting the variation into Eq. (6.3) and using Maxwell equation[1] one finds for the integrand

$$\begin{aligned}\partial_\mu(F^{\mu\nu}\delta A_\nu) &= \partial_\mu(F^{\mu\nu}\partial_\alpha A_\nu)\delta\xi^\alpha \\ &= \partial_\mu\big(F^{\mu\nu}F_{\alpha\nu}\big)\delta\xi^\alpha + \partial_\mu\big(F^{\mu\nu}\partial_\nu A_\alpha\big)\delta\xi^\alpha \\ &= \partial_\mu\big(F^{\mu\nu}F_{\alpha\nu}\big)\delta\xi^\alpha + \underbrace{\partial_{\mu\nu}}_{symm}\big(\underbrace{F^{\mu\nu}}_{anti}A_\alpha\big)\delta\xi^\alpha \\ &= \partial_\mu\big(F^{\mu\nu}F_{\alpha\nu}\big)\delta\xi^\alpha\end{aligned}$$

The change in action has been reduced to a boundary term

$$4\pi c\,(\delta S_F) = \int d\Omega\,\partial_\mu\big(F^{\mu\nu}F_{\alpha\nu}\big)\delta\xi^\alpha = \int \underbrace{dS_\mu}_{bdry} F^{\mu\nu}F_{\alpha\nu}\delta\xi^\alpha \tag{6.6}$$

6.1.5 Shifting the box

As a warmup consider the variation of a one dimensional integral upon shifting boundary points by $\delta\xi$

$$\delta\left(\int_a^b f(x)dx\right) = \delta\xi\big(f(b) - f(a)\big)$$

[1] Note that ξ^α is a constant, not a function.

The case at hand is the multidimensional version of this.

Shifting the box without shifting the fields changes the action by boundary terms of the form

$$\begin{aligned} 4\pi c\,(\delta S_F) &= -\tfrac{1}{4}\left(\int dS_\alpha\, F\cdot F\right)\delta\xi^\alpha \\ &= -\tfrac{1}{4}\int d\Omega\,\partial_\alpha\,(F\cdot F)\,\delta\xi^\alpha \end{aligned}$$

6.1.6 Joint box and field shift

For the joint shift we get

$$\begin{aligned} 0 &= \int d\Omega\,\Big(\partial_\mu\big(F^{\mu\nu}F_{\alpha\nu}\big) - \tfrac{1}{4}\partial_\alpha\,(F\cdot F)\Big)\delta\xi^\alpha \\ &= \int d\Omega\,\partial_\mu\Big(F^{\mu\nu}F_{\alpha\nu} - \tfrac{1}{4}g^\mu_\alpha\,(F\cdot F)\Big)\delta\xi^\alpha \\ &= \int d\Omega\,\Big(\partial_\mu T^\mu{}_\alpha\Big)\delta\xi^\alpha \end{aligned}$$

Since this is supposed to hold for any (infinitesimal) box and any shift, we get the conservation law.

6.1.7 The momentum of the field

Conservation of momentum reflects the homogeneity of space. A spatial shift and a large box is no shift at all. So that we only need to consider Eq. (6.6). Call the contribution from the surfaces at the final time $\mathcal{P}_j$. The contribution from the initial time must then be $-\mathcal{P}_j$, since the two cancel each other. We are still free to choose the normalization, and we choose

$$\begin{aligned} \frac{4\pi}{c}\mathcal{P}_j &= \int dS_0\, F^{0k}F_{jk} \\ &= \int dS_0\,\varepsilon^{ijk}E_k B_i \end{aligned}$$

which we recognize as the j-th component of the Poynting vector

$$\mathcal{P} = \frac{c}{4\pi}\int dV\,\mathbf{E}\times\mathbf{B} \tag{6.7}$$

Exercise 6.3. *Suppose* $\nabla\times\mathbf{E}=0$ *and* $\mathbf{B}=\nabla\times\mathbf{A}$ *show that*

$$\mathbf{E}\times\mathbf{B} = \nabla(\mathbf{E}\cdot\mathbf{A}) - (\mathbf{E}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{E}$$

Show that in the Coulomb gauge $\nabla\cdot\mathbf{A}=0$

$$\nabla\times(\mathbf{E}\times\mathbf{A}) = -\mathbf{A}(\nabla\cdot\mathbf{E}) + (\mathbf{A}\cdot\nabla)\mathbf{E} - (\mathbf{E}\cdot\nabla)\mathbf{A}$$

Exercise 6.4 (Courtesy of O. Kenneth)**.** *A charged point particle moves in a magnetic field* $\mathbf{B}$. *Show that, if* $\mathbf{A}$ *is the vector potential in the Coulomb gauge,* $\nabla \cdot \mathbf{A} = 0$. *Show that adding the field momentum to the particle kinetic momentum* $m\mathbf{v}$ *gives the minimal coupling expression for the particle momentum in the field*

$$m\mathbf{v} + \frac{1}{8\pi}\int \mathbf{E}\times\mathbf{B} = m\mathbf{v} + \frac{e}{c}\mathbf{A}$$

where $\mathbf{B} = \nabla \times \mathbf{A}$

6.1.8 The energy of a field

Energy is the conserved quantity associated to time translation. Since the box is finite in the time direction, need to pay attention to the fact that the box is shifted and the variation the action, and we need also to take care of Eq. (**??**). Again, for a large spatial box only the dS_0 surfaces contribute and combining Eq. 6.6 and **??** we get the conserved quantity[2]

$$\begin{aligned} 4\pi\mathcal{E} &= -\int dV\Big(F^{0j}\,F_{0j} - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\Big) \\ &= \int dV\Big(\mathbf{E}\cdot\mathbf{E} - \tfrac{1}{2}(\mathbf{E}\cdot\mathbf{E} - \mathbf{B}\cdot\mathbf{B})\Big) \\ &= \tfrac{1}{2}\int dV\Big(\mathbf{E}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{B})\Big) \end{aligned}$$

We then identify the energy of the field as

$$\mathcal{E} = \frac{1}{8\pi}\int dV\,\left(\mathbf{E}^2 + \mathbf{B}^2\right) \tag{6.8}$$

The integrand is the energy density that you recognize.

Example 6.5. *The magnetic field of earth is about* 1 $[Gauss]$. *The pressure associates with magnetic field of* 100 $[Gauss]$ *is the same as* $.5$ $[cm]$ *of water. In contrast, the pressure associated with* 10 $[KVolt/Cm]$ *is* 1 $[mm]$ *of water..*

Exercise 6.6. *Calculate the energy in that lies outside a ball of radius* r_0 *of charge* e *placed in its center. What should be the radius of the electron if its mass had purely electromagnetic origin?* (Answer: Energy: $\mathcal{E} = 4\pi e^2/r_0$. The classical radius of the electron is $r_0 = \frac{4\pi e^2}{m_e c^2} = 2.\times 10^{-15}$ $[m]$).

6.1.9 Angular momentum

The angular momentum of the field is the conserved quantity

$$\mathcal{J} = \frac{1}{4\pi c}\int dV\,\mathbf{x}\times(\mathbf{E}\times\mathbf{B}) \tag{6.9}$$

[2]Conserved quantities are fixed up to normalization. Here we picked different normalizations for the energy and the momentum so each has the appropriate dimensions. A more convincing normalization procedure will be given when we discuss Maxwell stress tensor.

This can be seen ass follows. Under infinitesimal spatial rotation by $\delta\theta$ (a vector) the coordinates transform like

$$x^0 \to x^0, \quad x^j \to x^j + R^j_k x^k, \quad R^j_k = \varepsilon^j{}_{km}(\delta\theta)^m$$

We are interested in the variation of $\delta A_n(x)$. The rotation acts both on the coordinates x^m and also on the vector A_m. Together, we have

$$\begin{aligned}\delta A_n &= R^j{}_n A_j + (R^j{}_k x^k)(\partial_j A_n)\\ &= R^j{}_n A_j + (R^j{}_k x^k)(F_{jn} + \partial_n A_j)\\ &= R^j{}_k x^k F_{jn} + R^j{}_n A_j + R^j{}_k \partial_n(x^k A_j) - R^j{}_k \delta^k_n A_j\\ &= R^j{}_k x^k F_{jn} + R^j{}_k \partial_n(x^k A_j)\end{aligned}$$

By Eq. (6.6) the conservation law is determined by

$$\begin{aligned}F^{0n}\delta A_n &= F^{0n}\left(R^j{}_k x^k F_{jn} + R^j{}_k \partial_n(x^k A_j)\right)\\ &= R^j{}_k x^k F^{0n} F_{jn} + \underbrace{R^j{}_k \partial_n(x^k F^{0n} A_j)}_{bdry\ term}\end{aligned}$$

and we have used Gauss law in the last step. Now

$$R^j_k x^k F^{0n} F_{jn} = \Big(\mathbf{x} \times (\mathbf{E} \times \mathbf{B})\Big) \cdot \delta\theta$$

from which the result follows.

Exercise 6.7. *Conservation of angular momentum is easier to see in cylindrical coordinates. Can you do that.*

Exercise 6.8. *Generalize the result to Lorentz transformations.*

6.2 T and variation of the metric

6.2.1 Blah

In the previous section we got the Maxwell tensor from conservation laws and symmetries: Deformations that did not affect the metric. It is interesting that one can also recover the tensor doing the complementary thing: Looking at how the action varies when we make a deformation that changes the metric. What should we expect?

When Maxwell constructed his theory the queen of science was, of course, mechanics. In particular, he understood well elasticity and fluid mechanics. In elasticity theory the concepts of stress and strain are important, and it was natural for Maxwell to ask what is their analogs in electrodynamics. One can think of a strain as a deformation of the metric. For example, the strain shown in the figure [3] can be represented by deformation of the Euclidean metric

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to g = \begin{pmatrix} 1+\epsilon & 0 \\ 0 & 1-\epsilon \end{pmatrix}$$

[3] The vector field is divergence-less and curl-free.

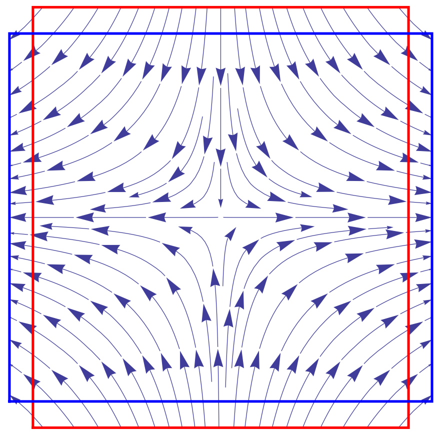

Figure 6.3: In the theory of elasticity a strain is described by a vector field. The figure shows the vector field associated with (uniform) contraction of y and dilation of x: Namely $(x, -y)$. The strain causes stress in the material. Energy is stored in it like in a compressed spring.

6.2.2 Variation of the metric in mechanics

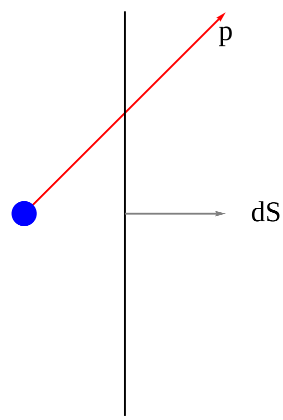

Figure 6.4: The momentum flux is a second rank tensor made from teh two vectors: Momentum and velocity.

The Lagrangian of a free classical particle is

$$L = \frac{m}{2} g_{ij} \dot{q}^i \dot{q}^j, \quad p_j = \frac{\partial L}{\partial \dot{q}^j} = m g_{jk} \dot{q}^k = m \dot{q}_j$$

The variation of the metric gives

$$\frac{\partial L}{\partial g_{ij}} = \tfrac{1}{2}\, m\dot{q}^i\dot{q}^j = \tfrac{1}{2}T^{ij}$$

The symmetric tensor

$$T^{ij} = \underbrace{p^i}_{momentum}\ \underbrace{\dot{q}^j}_{velocity}$$

is interpreted as the momentum flux.

6.2.3 Variation of the metric

The action depends on the metric in two places. First in the volume element. So let us reorganize

$$\underbrace{d\Omega}_{volume\ element} \quad F\cdot F = \sqrt{|g|}dx^0dV \quad \underbrace{F\cdot F}_{Lorentz\ scalar} \quad = \quad \underbrace{dx^0\,dV}_{metric\ indep}\ \underbrace{\sqrt{|g|}F\cdot F}_{Lorentz\ density}$$

The metric is also buried in the scalar product

$$F\cdot F = \ F^{\alpha\beta}g_{\beta\gamma}F^{\gamma\delta}g_{\delta\alpha} \tag{6.10}$$

Hence

$$\begin{aligned}
\delta\big(\sqrt{|g|}F\cdot F\big) &= \delta(\sqrt{|g|})\ F\cdot F + \sqrt{|g|}F^{\alpha\beta}F^{\gamma\delta}\delta(g_{\beta\gamma}g_{\delta\alpha})\\
&= \delta(\sqrt{|g|})\ F\cdot F + 2\sqrt{|g|}F^{\alpha\beta}F^{\gamma\delta}g_{\delta\alpha}\delta g_{\beta\gamma}\\
&= \delta(\sqrt{|g|})\ F\cdot F + 2\sqrt{|g|}F^{\alpha\beta}F^{\gamma}{}_{\alpha}\delta g_{\beta\gamma}
\end{aligned}$$

6.2.4 Matrix calculus

To compute the variation of $\det g$ we need tools from matrix calculus. Being symmetric g can be diagonalized

$$g = \sum P_\mu\gamma_\mu, \quad P_\mu = |\mu\rangle\,\langle\mu|$$

where γ_μ are its (real) eigenvalues and P_μ are orthogonal projections. By definition

$$\det g = \prod\gamma_\mu \Longrightarrow \log|g| = \sum\log\gamma_\mu$$

and so

$$\delta\log\sqrt{\det g} = \tfrac{1}{2}\delta\log(|g|) = \tfrac{1}{2}\sum\frac{\delta\gamma_\mu}{\gamma_\mu}$$

We want to express the right hand side in terms of g and its variationδg. To do that observe that, by the functional calculus of operators,

$$g^{-1} = \sum\frac{P_\mu}{\gamma_\mu}$$

and so

$$g^{-1}\delta g = \sum \frac{P_\mu \left(P_\nu \delta\gamma_\nu + \gamma_\nu \delta P_\nu\right)}{\gamma_\mu}$$

Exercise 6.9 (Projections). *Show that if P_μ are orthogonal projections, $P_\mu P_\nu = \delta_{\mu\nu} P_\nu$, follows that*

$$Tr(P_\mu \delta P_\nu) = 0$$

We have then showed that

$$\delta \log \sqrt{-\det g} = \tfrac{1}{2} Tr(g^{-1}\delta g) = \tfrac{1}{2} g^{\gamma\beta} \delta g_{\beta\gamma}$$

and so finally

$$\delta \sqrt{|g|} = \sqrt{|g|}\, \delta \log \sqrt{|g|} = \tfrac{1}{2}\sqrt{|g|} g^{\gamma\beta} \delta g_{\beta\gamma} \tag{6.11}$$

6.2.5 The stress tensor

Collecting terms we get

$$\frac{\partial(\sqrt{|g|} F \cdot F)}{\partial g_{\beta\gamma}} = \sqrt{|g|}\Big(F^{\beta\alpha} F^{\gamma}{}_{\alpha} - \tfrac{1}{4} g^{\beta\gamma} F^{\mu\nu} F_{\mu\nu}\Big) \tag{6.12}$$

We now shift back $\sqrt{g}$ into the volume element and recover the energy momentum tensor.

6.3 Energy momentum conservation

The most interesting property of T is that, *in the presence of sources*, it expresses the fact that the particles provide the source of energy and momentum to the fields. This is the content of the following elegant formula[4]

$$\partial_\alpha T^{\beta\alpha} = \frac{1}{c} j_\alpha F^{\beta\alpha} \tag{6.13}$$

Before discussing its content, let us comment on the derivation. Normally, identities have straightforward derivation. This one is tricky. The first step is easy enough

$$\begin{aligned} \partial_\alpha T^{\alpha\beta} &= \frac{1}{4\pi}\Big(\partial_\alpha F^{\alpha\mu} F^{\beta}{}_{\mu} + F^{\alpha\mu} \partial_\alpha F^{\beta}{}_{\mu} - \tfrac{1}{2} g^{\alpha\beta} F^{\mu\nu} \partial_\alpha F_{\mu\nu}\Big) \\ &= \frac{1}{c} j^\mu F^{\beta}{}_{\mu} + \frac{1}{4\pi}\Big(F^{\alpha\mu} \partial_\alpha F^{\beta}{}_{\mu} - \tfrac{1}{2} g^{\alpha\beta} F^{\mu\nu} \partial_\alpha F_{\mu\nu}\Big) \end{aligned}$$

The tricky part is to show that the brackets vanish. To see that note first that the homogeneous Maxwell equations implies

$$0 = \partial_\gamma (F^*)^{\gamma\nu} = -\tfrac{1}{2} \varepsilon^{\nu\gamma\alpha\beta} \partial_\gamma F_{\alpha\beta} \Longrightarrow \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \tag{6.14}$$

[4] In curvilinear coordinates replace the partial derivative with covariant derivative.

Using this we can now manipulate (twice the covariant β components) of the vector in the brackets

$$
\begin{aligned}
&2F^{\nu\mu}\partial_\nu F_{\beta\mu} - F^{\mu\nu}\partial_\beta F_{\mu\nu} \\
&= -F^{\mu\nu}\Big(2\partial_\nu F_{\beta\mu} + \partial_\beta F_{\mu\nu}\Big) \\
&= -F^{\mu\nu}\Big(\underbrace{\partial_\nu F_{\beta\mu} + \partial_\beta F_{\mu\nu}}_{use\ hom} + \partial_\nu F_{\beta\mu}\Big) \\
&= -F^{\mu\nu}\Big(-\partial_\mu F_{\nu\beta} + \partial_\nu F_{\beta\mu}\Big) \\
&= -F^{\mu\nu}\underbrace{\Big(\partial_\mu F_{\beta\nu} + \partial_\nu F_{\beta\mu}\Big)}_{\mu-\nu\ symmetric} \\
&= 0
\end{aligned}
$$

Exercise 6.10 (Plane waves)**.** *Show that the stress tensor for plane electromagnetic waves*

$$A_\mu = a_\mu e^{ik\cdot x}, \quad k^\mu A_\mu = 0, \quad k_\mu k^\mu = 0$$

is

$$4\pi T^{\mu\nu} = a\cdot a\ k^\mu k^\nu$$

6.3.1 The source term $j\cdot F$ and conservation of energy

The source term $j\cdot F$ is a 4-vector. Consider first its time-component

$$j_\alpha F^{0\alpha} = j_k F^{0k} = -j^k F_{0k} = \mathbf{J}\cdot\mathbf{E}$$

which we identify (up to signs) either as the power (per unit volume) that the field puts into accelerating the charges or alternatively, the energy that moving charges impart to the field. The time component of the conservation law gives

$$\partial_t\left(\frac{\mathbf{E}^2+\mathbf{B}^2}{8\pi}\right) + c\nabla\cdot\left(\frac{\mathbf{E}\times\mathbf{B}}{4\pi}\right) = \frac{1}{c}\,\mathbf{J}\cdot\mathbf{E}$$

The first term on the left is the (field) energy density. The second term on the left is the energy current, and the right hand side says that currents in a field are a source term for the equation: The field energy is not conserved, it can exchange energy with the particles.

You see that the Poynting vector admits two different interpretations: As the momentum density in the field adn as the energy current.

The spatial components of the source gives the force density

$$
\begin{aligned}
-\frac{1}{c}j_\alpha F^{m\alpha} &= -\frac{1}{c}j_0 F^{m0} - \frac{1}{c}j_k F^{mk} = -\rho E_m + \frac{1}{c}e_{mkj}j_k B_j = \\
&= -\left(\rho E + \frac{1}{c}J\times B\right)_m \qquad (6.15)
\end{aligned}
$$

Since a force causes a rate of momentum change, the rhs determines the rate at which momentum is transferred from the sources to the field.

6.3.2 The interpretation of T^{jk}: Stress

Eq. 6.13 allows us to give an interpretation of the remaining terms in $T^{\alpha\beta}$ and see why the tensor is also called Maxwell stress tensor. To do so, let us look at an example.

Simple settings are stationary, where both the fields and sources are time independent. Consider the attraction of two $\pm e$ charges. This, by itself, can not be a steady state since the two charges attract. One needs to apply non-electromagnetic force to keep the charges at rest. Think about this from the perspective of conservation of momentum. Bob holds the charge on the left for time T and Alice holds the charge on the right for time T. Since Bob is applying a constant force for time T he expects, by Newton, to transfer momentum

$$\delta\mathbf{P} = -F \times T\hat{\mathbf{x}}, \quad F = \frac{e^2}{4d^2}$$

to the system. The charge is at rest, so it is not the charge that is absorbing the momentum. It mus be the field.

Exercise 6.11. *Why minus?*

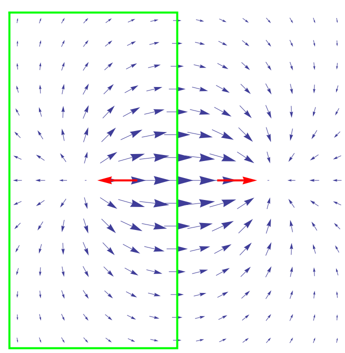

Figure 6.5: You need to apply a (non electromagnetic) force to hold two oppositely charged particles apart. This force can be computed from the surface integral of T^{11} on the relevant surface. The arrow, the force and the surface are all in the 1 direction.

We now use Eq.6.13 to see where the momentum of the field sits in T^{jk}. Consider the space-time box associated with the left half space of time duration T, i.e.

$$\Omega = \{t, \mathbf{x} | 0 < t < T, x_1 < 0\}$$

Now consider what Eq. 6.13 implies for this box. Lets focus on the 1 component:

$$\int_\Omega \partial_\alpha T^{1\alpha} d\Omega = \frac{1}{c}\int_\Omega j_\alpha F^{1\alpha}\, d\Omega = \frac{1}{c}\int_\Omega j_0 F^{10}\, d\Omega = -FT$$

The right hand side was easy to evaluate since there is a single, stationary a point charge in the box which only feels the electric field of the other side[5] This is interpreted the momentum Bob transfers to the box.

Exercise 6.12. *Check the sign on the right hand side.*

Now let us see how the field accommodates the momentum. For this consider the left hand side of the equation

$$\int_{\Omega} d\Omega \, \partial_\mu T^{1\mu} = \int_{\partial\Omega} dS_\mu T^{1\mu}$$

The momentum transfer of Bob to the box is expressed as an property of the fields alone one the boundary of the box. The boundary of the box is the three dimensional space time

$$\partial\Omega = \{t, \mathbf{x} | x_1 = 0, 0 < t < T\}$$

So

$$\int_{\partial\Omega} dS_\mu T^{1\mu} = \int_0^T dt \int dx^2 dx^3 T^{11}, \quad T^{11} = \frac{1}{8\pi}\Big(E_2^2 + E_3^2 - E_1^2\Big)$$

For the case at hand, $E_2 = E_3 = 0$ and $E_1 \neq 0$, but this is not crucial. The point is that we can now identify T^{11} with the momentum that is either leaving or entering the space-time box through its boundary. If we now focus on the spatial boundary of the box, namely, the plane, we get the interpretation that T^{11} is the force density on the boundary,

$$F_1 = \int dx^2 dx^3 T^{11}$$

Force per unit area is the standard notion of stress in elasticity. A similar interpretation applies to the other components by considering more complicated boxes.

6.3.3 Field lines as rubber bands

Let us look at the signs in

$$8\pi T^{11} = E_2^2 + E_3^3 - E_1^2 + B_2^2 + B_3^3 - B_1^2$$

The parallel and perp components come with opposite signs. The stress can be positive or negative, but the sign has nothing to do with the signs of E.

[5] ρ is a delta function on the world line of charge.

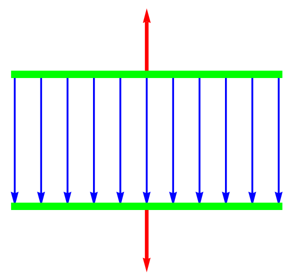

Figure 6.6: You need to apply a force to hold two oppositely charged capacitor plates apart (red arrows). If the arrow is in the z-direction, $T^{zz} < 0$. You get this sign if you think of the electric field lines as rubber band: As if the pressure is negative.

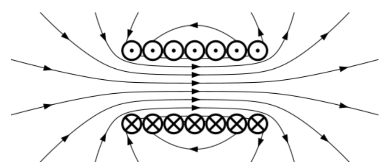

Figure 6.7: The magnetic field lines of a solenoid. The stress in the radial direction $T^{\rho\rho} > 0$ inside the solenoid. This is because $B_z \neq 0$ while $B_\rho \approx 0$. You get the right sign if you replaced the field lines by rubber bands: Stretched rubber bands along the z-axis, that fan out in the radial direction, will lead to a positive pressure in the radial direction and negative pressure in the axial direction.

6.4 Applications

6.4.1 Radiation pressure

The luminosity of the sun $L_\odot = 3.6 \times 10^{26}$ $[W]$, giving a stream of 10^{45} photons/sec. The radial component of Maxwell energy momentum tensor at a distance R from the sun is then

$$T_{0\hat{r}} = \frac{L_\odot}{4\pi R^2 c} \tag{6.16}$$

Consider a macroscopic (black) particle of radius r that perfectly absorbs radiation. The force on the particle at a distance R from the sun is then

$$F_{radiation} = \frac{r^2}{4R^2 c} L_\odot \tag{6.17}$$

The gravitational force on a particle with density ρ is

$$F_{gravity} = \frac{4\pi\rho r^3 M_\odot G}{3R^2} \tag{6.18}$$

Where $M_\odot = 2 \times 10^{33}$ $[gram]$ and Newton constant $G = 6.7 \times 10^{-8}$ $[cgs]$. The ratio of the two is then

$$\frac{F_{radiation}}{F_{gravity}} = \frac{3L_\odot}{16\pi r\rho c M_\odot G} \approx \frac{0.06\,[gr/cm^2]}{\rho r} \tag{6.19}$$

For water $\rho = 1$ $[gm/cm^3]$. For earth, $r = 6 \times 10^8$ $[cm]$, the ratio is minuscule: 10^{-10}. However, for very small grains, of radius less than 6×10^{-2} $[cm] \approx 600$ $[\mu]$ radiation dominates.

Radiation pressure cleans the solar neighborhood from fine dust. This could be a mechanism of transporting viruses from our solar system to distant parts of the universe[6].

Exercise 6.13 (Comet tails)**.** *Can you figure out the shape of a comet tail? Suppose the tail is associated with a planet in circular non-relativistic orbit.* Hint: Figure out the tail in the rotation frame.

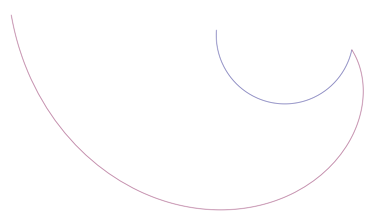

Figure 6.8: A planet encircling a star and the tail of dust it sprays (tail)

6.4.2 Solar sails

The computation above can be applied to solar sails. A sail of area A and width d can be used to sail away from the sun provided

$$F_{radiation} = \frac{A}{4\pi R^2 c} L_\odot > F_{gravity} = \frac{\rho A d M_\odot G}{R^2} \tag{6.20}$$

Cancelling the similar terms we get

$$\frac{1}{4\pi c} L_\odot > \rho d M_\odot G \tag{6.21}$$

Up to factors of order unity we get, the same estimate as above. You need very thin sails to build solar sails.

[6] The assumption that the particle is black is not reasonable when the radiation penetrates a distance comparable to the size.

6.4.3 Halbach array

Exercise 6.14 (Paradox)**.** *The Halbach array is shown in the fig 6.4.3. Make a qualitative plot of its field lines. You will find a proper plot here. Since the field is large and essentially parallel to the array on one side of the array, and small on the other. You may then wonder if the array is a Baron von Munchhausen: Properly oriented, it will float in gravitational field. Discuss the Maxwell stress tensor and resolve this apparent paradox.*

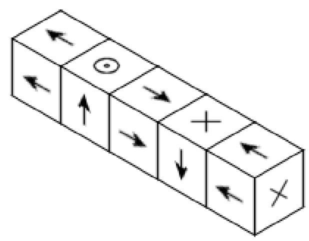

Figure 6.9: Halbach array gives a large magnetic field above the array and small one below it.

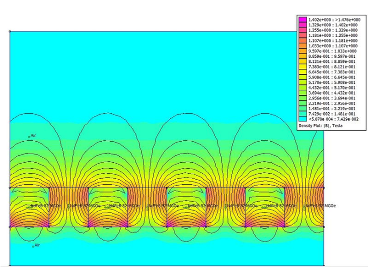

Figure 6.10: Halbach field (Wikipedia)

Chapter 7

Poisson equation, Cloaking

Vector fields in 3 dimensions are determined by their vorticity and sources. The basic techniques for solving such problems is described in the context of static electric and magnetic fields. Maxwell equation in dielectric media can be reinterpreted as Maxwell equations in curvilinear coordinates. This is the basis for cloaking.

7.1 Vector fields in 3D: Source and vorticity

The source ρ and vorticity ω of a vector field $\mathbf{V}$ in 3 dimensions are defined by

$$\nabla \cdot \mathbf{V} = 4\pi\rho, \quad \nabla \times \mathbf{V} = 4\pi\omega$$

The vorticity is always sourcesless

$$\nabla \cdot \omega = 0$$

Radial vector fields are vorticity free. $\mathbf{x}$ is vector field with uniform source, $\nabla \cdot \mathbf{x} = 3$.

The converse is also true: The sources ρ and ω with $\nabla \cdot \omega = 0$ determine the field $\mathbf{V}$. By linearity, we can decompose the problem in to two problems:

$$\mathbf{V} = \mathbf{E} + \mathbf{B}$$

where $\mathbf{E}$ is irrotational (conservative)

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad \nabla \times \mathbf{E} = 0$$

and $\mathbf{B}$ is sourceless

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = 4\pi\omega$$

As we shall see the equations for $\mathbf{E}$ and $\mathbf{b}$ are solved by the same technique.

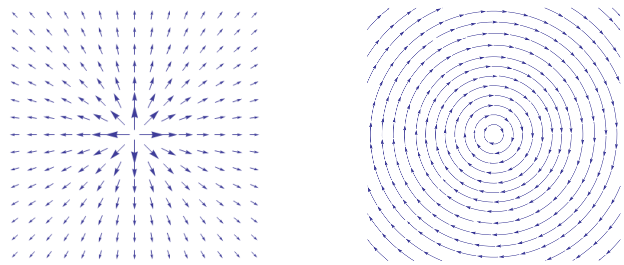

Figure 7.1: Left: An irrotational field with a source at the origin. A sourceless field with vorticity along the z-axis

7.1.1 Static electric fields: $\dot{\mathbf{B}} = 0$

The electric field $\mathbf{E}$ is determined by Gauss law and Faraday inductions law

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad \nabla \times \mathbf{E} + \dot{\mathbf{B}} = 0$$

and dot is a derivative with respect to $x^0 = ct$. When $\dot{\mathbf{B}} = 0$ the electric field is conservative and is the gradient of a potential that solves Poisson's equation

$$\Delta\phi = -4\pi\rho \tag{7.1}$$

Remark 7.1 (Time dependent ρ). *ρ may be time independent. However, time dependent ρ would normally entail $\dot{\mathbf{B}} \neq 0$*

7.1.2 Harmonic functions and Poisson's equation

Solutions of

$$\Delta\phi = 0 \tag{7.2}$$

are called Harmonic functions. A fundamental fact about Harmonic functions is:

Theorem 7.2 (Harmonic functions). *If ϕ is Harmonic, then $\phi(x)$ is the average value of ϕ on a sphere which has x as its center.*

The theorem is evident in one dimension, because a harmonic function is a linear function–clearly the average of equidistant neighbors. We shall postpone the proof in the general case after we assemble some more tools.

Corollary 7.3 (Boundary). *On a given set, harmonic functions assume their maxima and minima of its boundary.*

This is also known as Ernshaw's theorem: A charge can not be trapped by electrostatic fields alone.

Exercise 7.4. *Suppose that locally*

$$E_j(x) = E_j(0) + g_{jk}x^k + O(x^2)$$

Show that $\nabla \cdot \mathbf{E} = 0$ if $Tr\ g = 0$.

7.1.3 The case of two dimensions

A point in the plane can be described by a the complex number $z = x + iy$ (or, alternatively, by $\bar{z} = x - iy$). A general function is a function of both $f(z, \bar{z})$ while analytic functions are functions of z only. The real and imaginary parts of a complex valued function described a vector field.

Exercise 7.5. *Show that*

$$2\partial = 2\partial_z = \partial_x - i\partial_y, \quad 2\bar{\partial} = 2\partial_{\bar{z}} = \partial_x + i\partial_y,$$

Show that the Laplacian is

$$\Delta = 4\partial_z \bar{\partial}_z,$$

Using this show that $f(z) + g(\bar{z})$ for analytic f and g is Harmonic. Show that grad, div and curl are

$$\begin{aligned} \nabla f &\Longrightarrow\ 2\bar{\partial} f, \\ \nabla \cdot \mathbf{V} &\Longrightarrow \partial V + \bar{\partial} \bar{V} \\ \nabla \times \mathbf{V} &\Longrightarrow i\left(\partial V - \bar{\partial} \bar{V}\right) \end{aligned}$$

7.1.4 The Green function of the Laplacian

For unit point charge e in 3-dimensions the electric field, by symmetry must be radial and to satisfy Gauss law must be

$$\nabla \cdot \left(\frac{\mathbf{x}}{|\mathbf{x}|^3}\right) = 4\pi\delta(\mathbf{x}) \Longrightarrow \mathbf{E} = e\,\frac{\mathbf{x}}{|\mathbf{x}|^3}\,, \quad \phi(x) = \frac{e}{|\mathbf{x}|} \tag{7.3}$$

A useful identity

$$\Delta_x G(\mathbf{x} - \mathbf{y}) = \delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad G(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|} \tag{7.4}$$

This is aka Poisson kernel, aka the Green function. G is a radial function and

$$G'(r) = \frac{1}{4\pi r^2} \tag{7.5}$$

is the inverse area of the sphere of radius r.

Remark 7.6 (Operator interpretation)**.** *It is useful to interpret this equation as an operator identity: When interpreted as an integral operator, the delta function represents the identity. The Laplacian in a differential operator and G is an integral operator. From this point of view $G = \Delta^{-1}$: The inverse of the Laplacian.*

By linearity, the solution of Poisson equation for a source term $\rho(x)$ reduces to an integration:

$$\phi(\mathbf{x}) = -4\pi \int G(\mathbf{x} - \mathbf{y})\rho(\mathbf{y})d\mathbf{y}, \tag{7.6}$$

The solution is unique up to Harmonic function.

Exercise 7.7 (Generalization to arbitrary dimension)**.**

1. *Show that in dimension d*

$$G'(r) = \frac{1}{\omega_d\, r^{d-1}}, \qquad r = |\mathbf{x}|$$

where ω_d is the area of the d dimensional unit sphere.

2. *Compute ω_d using the Gaussian integral*

$$\int d^d x e^{-x^2} = \omega_d \int_0^\infty r^{d-1} dr e^{-r^2}$$

7.1.5 Proof of theorem 9.32

Let $G(x)$ be the Green function of the Laplacian in d-dimensions and ϕ Harmonic:

$$\begin{aligned}\Delta\left(G(x)\phi(x)\right) &= G\Delta\phi + 2\nabla\phi\cdot\nabla G + \phi\Delta\left(G\right)\\ &= 2\nabla\phi\cdot\nabla G + \phi(x)\,\delta(x)\end{aligned}$$

Integrate this identity on a ball at the origin. The last term (on the right) gives $\phi(0)$. Since G is a radial function, the middle term can be written as

$$\int_{|x|\le R} 2\nabla\phi\cdot\nabla G dV = \omega_d \int_0^R r^{d-1}\, dr\, G'(r) \underbrace{\int_{|x|=r} dS\cdot\nabla\phi}_{0\ by\ Gauss}$$

(ϕ is Harmonic, the flux through any closed surface of $\nabla\phi$ vanishes so the integral on the right vanishes for any $r > 0$.)

It remains to integrate the term on the left

$$\begin{aligned}\int_{|x|\le R} \Delta\left(\phi G\right)\, dV &= \int_{|x|=R} dS\cdot\nabla\left(\phi(\mathbf{x})G(r)\right)\\ &= \int_{|x|=R} dS\cdot\Big(\underbrace{(\nabla\phi)G(r)}_{0\ by\ Gauss} + \phi(\mathbf{x})\, G'(r)\hat{\mathbf{r}}\Big)\\ &= G'(R)\int_{|x|=R} dS\cdot\hat{r}\,\phi(\mathbf{x})\end{aligned}$$

which is precisely the average of ϕ over the sphere of radius R.

Exercise 7.8. *Analyze the stability of a dipole* $\mathbf{d}$ *in electrostatic fields:*

1. *Show that its is*

$$\mathcal{E} = -\mathbf{d}\cdot\mathbf{E}$$

2. *Suppose first that* $\mathbf{d}$ *is a fixed vector. Show that* $\mathcal{E}$ *is harmoic*

$$\Delta\mathcal{E} = (\mathbf{d}\cdot\nabla)\Delta\phi = 0$$

3. *Using the principle of virtual work show that the torque on the dipole is*

$$\mathbf{T} = \mathbf{d} \times \mathbf{E}$$

4. *Suppose that at every point* $\mathbf{x}$ *the dipole is co-oriented with the field*

$$\mathbf{d}(\mathbf{x}) = \pm\, d\, \hat{\mathbf{E}}(\mathbf{x})$$

and the dipole is placed initially at a point where $\mathbf{E}(0) = 0$ *(no sources near the origin). Show that one of* $\mathcal{E}_\pm = \pm d|\mathbf{E}(\mathbf{x})|$ *always has a minimizer at* $\mathbf{x} = 0$*.*

5. *What is the form of the minimizer?*

6. *Can you relate this to the stability of the Levitron?*

7.1.6 Uniformly moving charge

For a charge e at rest at the origin

$$E_j = e\frac{x_j}{r^3}, \quad B_j = 0 \tag{7.7}$$

Everything is time independent so we can compute this at any time we want.

Consider the Lorentz boost

$$\Lambda = \begin{pmatrix} C & S & 0 & 0 \\ S & C & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C = \cosh\phi,\ S = \sinh\phi \tag{7.8}$$

where ϕ is the rapidity connected with the usual γ and β by

$$\gamma = \cosh\phi, \quad \beta = \tanh\phi \tag{7.9}$$

The two Lorentz scalars are

$$E^2 - B^2 = \frac{e^2}{r^4}, \quad E \cdot B = 0 \tag{7.10}$$

The transformation rules are

$$F'_{\mu\nu}(x') = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta F_{\alpha\beta}(x') = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta F_{\alpha\beta}(\Lambda x) \tag{7.11}$$

In the frame where we see a moving charge, everything depends on time. So let us compute everything at $t' = 0$ when the charge is at the origin. We have

$$x = Cx', \quad y = y', \quad z = z' \tag{7.12}$$

In particular

$$\begin{aligned} r^2 &= C^2x'^2 + y'^2 + z'^2 = C^2x'^2 + (C^2 - S^2)(y'^2 + z'^2) \\ &= C^2r'^2 - S^2(y'^2 + z'^2) \end{aligned} \tag{7.13}$$

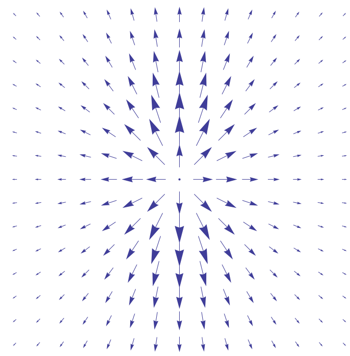

Figure 7.2: The vector field of a moving charge with rapidity $\phi = 1$. The field is manifestly radial but not spherically symmetric.

Let us turn to the fields. E_x does not change

$$E'_x = F'_{01} = \Lambda_0{}^\alpha \Lambda_1{}^\beta F_{\alpha\beta} = (\Lambda_0{}^0\Lambda_1{}^1 - \Lambda_0{}^1\Lambda_1{}^0)F_{01} = (C^2 - S^2)E_x = E_x \quad (7.14)$$

Hence, at $t' = 0$

$$E_x(x) = e\frac{x}{r^3} = e\gamma\frac{x'}{r^3(r')} = E'_x(x') \quad (7.15)$$

where $r(r')$ is the ugly expression Eq. (7.13).

For the transverse directions

$$E'_y = F'_{02} = \Lambda_0{}^\alpha \Lambda_2{}^\beta F_{\alpha\beta} = \Lambda_0{}^\alpha F_{\alpha 2} = CF_{02} = \gamma E_y \quad (7.16)$$

and so

$$E_y(x) = e\frac{y}{r^3}, \quad = E'_y(x') = \gamma e\frac{y}{r^3} = \gamma e\frac{y'}{r^3(r')} \quad (7.17)$$

The formula is the same but for different reasons. In one case γ came form the field transformation and in the other from the coordinates.

It now follows that in both frames the field is radial, because

$$\frac{x}{y} = \frac{E_x(x)}{E_y(x)} = \frac{E'_x(x)}{E'_y(x)} = \frac{x'}{y'} \quad (7.18)$$

Remark 7.9. *This is a bit surprising. One could have argued that since the field is radial in the rest frame, you may expect it to point in the direction of the particle at the retarded time, not now.*

The total strength of the field is

$$E'^2 = e^2\frac{\gamma^2}{r^4} = \gamma^2 E^2 \quad (7.19)$$

It is stronger in the frame where the charge is seen moving (computed for the same event).

7.2 Coulomb gauge

The Coulomb gauge is aka the radiation gauge aka the transverse gauge is:

Theorem 7.10 (Coulomb gauge)**.** *It is always possible to choose the vector potential $A_\mu = (-\phi, \mathbf{A})$ so that*

$$\nabla \cdot \mathbf{A} = 0, \quad \Delta\phi = -4\pi\rho$$

The theorem is a consequence what we have learned about Poisson equation. Suppose $\nabla \cdot \mathbf{A} \neq 0$. Let Λ be a solution of the Poisson's equation

$$\Delta\Lambda = \nabla \cdot \mathbf{A}$$

The gauge transformation

$$A'_\mu = A_\mu - \partial_\mu \Lambda$$

reproduces the the same field F with $\mathbf{A}'$ satisfying the Coulomb gauge condition

$$\nabla \cdot \mathbf{A}' = \nabla \cdot \mathbf{A} - \Delta\Lambda = 0$$

What about ϕ'? It is determined by

$$\mathbf{E} = -\nabla\phi' - \dot{\mathbf{A}}'$$

Taking the divergence of this we see that ϕ' as a solution of Poisson's equation:

$$-\Delta\phi' = \nabla \cdot \mathbf{E} = 4\pi\rho \tag{7.20}$$

Remark 7.11 (Causality)**.** *The Coulomb gauge is a-causal: The scalar potential ϕ is fixed by the* instantaneous *charge distribution. You move a charge here and the potential ϕ changes immediately everywhere. The fact that the potential changes faster than light has no use for transferring information because the fields are still causal.*

Remark 7.12 (Free space)**.** *When $\rho = 0$ we may take $\phi = 0$ together with $\nabla \cdot \mathbf{A} = 0$.*

7.3 Magnetic fields in the case $\dot{\mathbf{E}} = 0$

The magnetic field is determined by Ampere's law and "Gauss" law for the magnetic field

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} + \dot{\mathbf{E}} = \frac{4\pi}{c}\mathbf{J}$$

In the case $\dot{\mathbf{E}} = 0$ this reduces to

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \frac{4\pi}{c}\,\mathbf{J} \tag{7.21}$$

Note that $\dot{\mathbf{E}} = 0$ implies $\dot{\rho} = 0$ and then $\nabla \cdot \mathbf{J} = 0$, which is a consistency condition for the Ampere equation in this case.

Since $\mathbf{B}$ is sourceless it is given by a vector potential for which we are allowed to impose a Coulomb gauge condition:

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \nabla \cdot \mathbf{A} = 0$$

Using the identity

$$\nabla \times (\nabla \times \mathbf{A}) = -\Delta \mathbf{A} + \nabla(\nabla \cdot \mathbf{A})$$

we find that $\mathbf{A}$ is a solution of (the vector valued) Poisson's equation

$$\Delta \mathbf{A} = -\frac{4\pi}{c}\mathbf{J}$$

Remark 7.13 (Consistency)**.** *The equation is consistent with the gauge condition* $\nabla \cdot \mathbf{A} = 0$ *since* $\nabla \cdot \mathbf{J} = 0$.

7.3.1 Biot-Savart law

We have seen that in the case that $\dot{\mathbf{E}} = 0$ the magnetic field is determined by a Poisson equation for the vector potential. It follows that

$$\mathbf{A}(\mathbf{x}) = \frac{1}{c}\int \frac{\mathbf{J}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\, d\mathbf{y} \tag{7.22}$$

This gives gives $\mathbf{B}$ as an explicit line integral over the current:

$$\begin{aligned}\mathbf{B}(\mathbf{x}) &= \nabla \times \mathbf{A}(\mathbf{x}) \\ &= \frac{1}{c}\int \underbrace{\nabla_x \left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)}_{Coulomb} \times \mathbf{J}(y)\, dy \\ &= \frac{1}{c}\int \frac{\mathbf{J}(\mathbf{y}) \times (\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\, d\mathbf{y}\end{aligned}$$

This is the Biot-Savart law.

Exercise 7.14 (A straight line of current)**.** *Consider a cylindrically symmetric tube carrying constant current I along the z-axis. Using the cylindrical symmetry of the problem and the integral version of Ampere's law show that*

$$\mathbf{B} = 2I\,\frac{\hat{\mathbf{z}} \times \mathbf{x}}{|\mathbf{x}|^2}$$

Exercise 7.15 (Constant magnetic fields)**.** *The vector potential of a constant magnetic field is a linear vector values function and so of the form*

$$\mathbf{A} = \mathbf{a} \times \mathbf{x} + (\mathbf{b} \cdot \mathbf{x})\mathbf{c}$$

Show that

$$\mathbf{B} = 2\mathbf{a} + \mathbf{b} \times \mathbf{c}, \quad \nabla \cdot \mathbf{A} = \mathbf{c} \cdot \mathbf{b}$$

7.3.2 Magnetic dipole

The current associated with thin loop of radius a in the $x-y$ plane carrying current I is

$$\begin{aligned} 2\mathbf{J} &= (-y, x, 0)\, I\delta(x^2 + y^2 - a^2)\delta(z) \\ &= \tfrac{1}{2}(I\hat{\mathbf{z}} \times \nabla)\theta(a^2 - x^2 - y^2)\,\delta(z) \end{aligned}$$

and $\theta(x) = 1$ for $x > 0$ and 0 otherwise–the standard step function. Now consider the limit $a \to 0$ and $I \to \infty$ so that that Ia^2 is fixed.

Exercise 7.16 (Delta function)**.** *Show that*

$$\lim_{a\to 0} \frac{\theta(a^2 - x^2 - y^2)}{\pi a^2}\delta(z) = \delta(\mathbf{x})$$

The $a \to 0$ limit represents a point dipole, characterized by a vector

$$\mathbf{m} = \left(\frac{\pi^2 I a^2}{c}\right)\hat{\mathbf{z}}$$

Ampere equation takes the form

$$(\nabla \times \mathbf{B})(\mathbf{x}) = 4\pi(\mathbf{m} \times \nabla)\delta(\mathbf{x})$$

To find $\mathbf{B}$ we could plug the source into Biot-Savart. However, this is not much simpler then retracing the derivation. For $\mathbf{A}$ we find

$$\begin{aligned} \mathbf{A}(\mathbf{x}) &= \int d\mathbf{y}\, \frac{(\mathbf{m} \times \nabla)_y \delta(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \\ &= \int d\mathbf{y} \left(\underbrace{(\mathbf{m} \times \nabla)_y \left(\frac{\delta(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|}\right)}_{bdry\ term} - (\mathbf{m} \times \nabla)_x \frac{\delta(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \right) \\ &= -(\mathbf{m} \times \nabla)_x \int d\mathbf{y}\, \frac{\delta(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \\ &= -(\mathbf{m} \times \nabla)\, \frac{1}{|\mathbf{x}|} \\ &= \frac{\mathbf{m} \times \mathbf{x}}{|\mathbf{x}|^3} \end{aligned}$$

To compute $\mathbf{B}$ we need a version of the vector identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

where $\mathbf{a} = \nabla$ is a differential operator,$\mathbf{b} = \mathbf{m}$ a fixed vector and $\mathbf{c} = \mathbf{x}r^{-3}$ a vector valued function. Reflection shows that the right form is

$$\nabla \times (\mathbf{m} \times \mathbf{c}) = \mathbf{m}(\nabla \cdot \mathbf{c}) - (\mathbf{m} \cdot \nabla)\mathbf{c}$$

From the solution of the Coulomb problem we know that

$$\nabla \cdot (\mathbf{x} r^{-3}) = 4\pi\delta(\mathbf{x})$$

Hence

$$\mathbf{B}(\mathbf{x}) = 4\pi\mathbf{m}\delta(\mathbf{x}) - (\mathbf{m}\cdot\nabla)\left(\frac{\mathbf{x}}{r^3}\right)$$

It follows that the magnetic field of a dipole is

$$\mathbf{B} = \frac{-\mathbf{m} + 3(\mathbf{m}\cdot\hat{\mathbf{x}})\hat{\mathbf{x}}}{|\mathbf{x}|^3} + 4\pi\mathbf{m}\delta(\mathbf{x})$$

Exercise 7.17. *Verify all steps.*

Remark 7.18 (Singularity)**.** *The magnetic field has a bad (non-integrable) singularity at the origin. One way to see this is to consider the total flux through the origin. Take the plane oriented with* $\mathbf{m}$ *through the dipole. The flux through such a plane is*

$$\begin{aligned}\mathbf{B}\cdot\mathbf{m} &= -\frac{(\mathbf{m}\cdot\mathbf{m})(\mathbf{x}\cdot\mathbf{x}) - 3(\overbrace{(\mathbf{m}\cdot\mathbf{x})}^{=0})^2}{|\mathbf{x}|^{\mathbf{5}}} + 4\pi\mathbf{m}^2\delta(\mathbf{x}) \\ &= -\frac{(\mathbf{m}\times\mathbf{x})^2}{|\mathbf{x}|^{\mathbf{5}}} + 4\pi\mathbf{m}^2\delta(\mathbf{x})\end{aligned}$$

The first term has an non-integrable singularity at the origin. At the same time, we know that the total flux through any surface must be zero

Exercise 7.19 (Vanishing flux)**.** *Show that the total flux through any such plane at distance* ε *from the origin vanishes.*

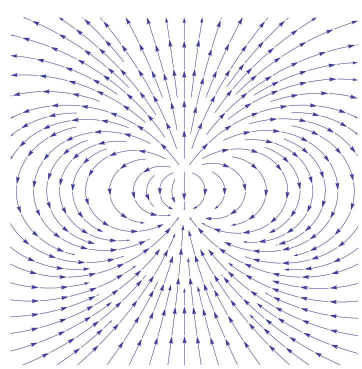

Figure 7.3: Dipole field

7.3.3 Dirac monopole

Dirac monopoles were invented, by Dirac of course, to explain the quantization of charge: The charge of the proton is exactly minus that of the electron, in contrast with, say, their their mass ratio which does not look like a simple fraction. Dirac realized that if there was even a single monopole of magnetic charge e_m anywhere in the universe, say, behind Andromeda, then charged quantization will be a consequence of quantum mechanics: The electric charge of any quantum particle e will be constrained by

$$\frac{2e_m e}{\hbar c} \in \mathbb{Z}$$

This is Dirac charge quantization. This Dirac quantization can be viewed as a consequence of the Aharonov-Bohm effect.

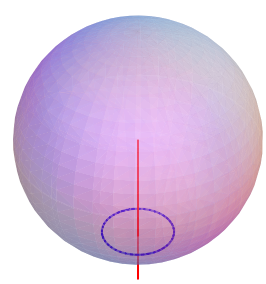

Figure 7.4: Dirac string and monopole

We look for a vector potential whose flux is the monopole charge e_m, and with nice Coulombic field

$$\mathbf{B} = e_m \frac{\mathbf{x}}{|\mathbf{x}|^3}$$

There is no smooth $\mathbf{A}$ that does it. So imagine we allow $\mathbf{A}$ with a singularity along the negative z axis. (But $\mathbf{B}$ is still smoth.) Consider the sphere minus the south pole. Then, by Stokes,

$$4\pi e_m = \int \nabla \times \mathbf{A}\, dS = \int_\ell \mathbf{A} \cdot d\ell$$

Suppose we $\mathbf{A}$ is azimuthal, i.e. has only one covariant component A_ϕ. Then

$$4\pi e_m = A_\phi(\theta = \pi) 2\pi \Longrightarrow A_\phi(\theta = \pi) = 2e_m$$

You will see the singularity in the normalized coordinate

$$A_{\hat{\phi}}(\theta = \pi) = \frac{A_\phi}{\sqrt{g_{\phi\phi}}} = \frac{2e_m}{r\sin\theta}$$

Exercise 7.20. *Show this.* (Recall that $A_\phi \mathbf{e}^\phi = A_{\hat{\phi}}\hat{\phi}$, with $\mathbf{e}^\phi \cdot \mathbf{e}^\phi = g_{\phi\phi}$)

How do we extend this to any θ? Note that on the negative z-axis $2 = 1 - \cos\theta$. This regularizes the potential on the positive real axis and maintains the total flux through the sphere. Hence

$$\mathbf{A} = e_m \frac{1-\cos\theta}{r\sin\theta}\,\hat{\phi}$$

is nicely behaved along the positive z axis. This singularity is called the Dirac string. A computation you are asked to do in the next exercise shows that $\mathbf{A}$ is the vector potential of a monopole

$$\nabla \times \mathbf{A} = e_m \frac{\mathbf{x}}{|\mathbf{x}|^3}$$

Exercise 7.21 (Monopole)**.** *Show that the covariant component of the vector potential is $A_\phi = g(1-\cos\theta)$. Show that*

$$B_r = B^r = B_{\hat{r}} = \frac{e_m}{r^2}$$

You may think of the string along the negative z-axis as a flux tube that brings in the magnetic flux that emanates from the monopole. By the Aharonov-Bohm effect, such a string is invisible if the flux satisfies Dirac quantization rule.

7.3.4 Application to geometry: Linking number

Suppose you have two loops γ_1 and γ_2 in space and you want to know if they link. Imagine that the loop γ_1 carries a unit current. Then, if the loop γ_2 links n times the loop γ_1 we have

$$\int_{\gamma_2} \mathbf{B}(\mathbf{x_2}) \cdot d\mathbf{x}_2 = \frac{4\pi n}{c}$$

Now plug $\mathbf{B}$ from the solution of Poisson's equation to get

$$\int_{\gamma_2}\int_{\gamma_1} \frac{d\mathbf{x}_2 \cdot (\mathbf{x_2} - \mathbf{x_1}) \times d\mathbf{x_1}}{|\mathbf{x_2} - \mathbf{x_1}|^3} = 4\pi n$$

If $n \neq 0$ the loops link. The converse is, however, not always true.

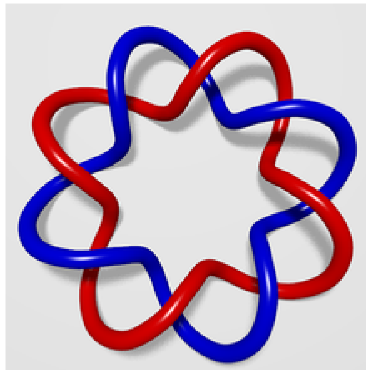

Figure 7.5: Linking curves with linking number 4, (Wikipedia)

7.4 Cloaking

7.4.1 Maxwell equation in a dielectric medium

In a dielectric medium, the homogeneous equations, Faraday's law and no-monoples, are the same as in free space[1]

$$\nabla \cdot \mathbf{B} = 0, \quad \dot{\mathbf{B}} + \nabla \times \mathbf{E} = 0, \qquad (homogenous)$$

In the absence of external sources, the inhomogeneous Maxwell equations, Gauss and Ampere laws, are modified to

$$\nabla \cdot \mathbf{D} = 0, \quad \dot{\mathbf{D}} - \nabla \times \mathbf{H} = 0 \qquad (inhomogeneous)$$

Dot stands for derivative with respect to x^0 and D and H are defined through the constitutive relations[2]

$$\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H}$$

where ε and μ are tensors. In a fixed, narrow, band of frequencies, these tensors can be viewed as functions of the spatial coordinates alone. This will be assumed from now on. In free (Euclidean) space

$$\varepsilon = \mu = g$$

where g is the (Euclidean) metric. (Recall that $g^i{}_j = \delta^i_j$.)

[1] **E** and **B** then represent averages over a macroscopically small, but microscopically large, ball.

[2] Hopefully no confusion will arise between ε as dielectric constant and ε as Levi-Civita tensor.

7.4.2 Maxwell in curvilinear coordinates

Maxwell equations are made from div and curl. These are geometric and have a relatively simple expression in curvilinear coordinates:

$$\nabla \cdot \mathbf{E} = \frac{1}{\sqrt{g}}\partial_j(\sqrt{g}E^j), \qquad (\nabla \times E)^i = \frac{\varepsilon^{ijk}}{\sqrt{g}}\partial_j E_k$$

The homogeneous Maxwell equations in curvilinear coordinates[3], are

$$\partial_j(\sqrt{g}B^j) = \partial_j(\sqrt{g}\,\underbrace{\mu^{jk} H_k}_{H_k \neq B_k}) = 0,$$
$$\sqrt{g}\dot{B}^j + \varepsilon^{ijk}\partial_j E_k = \sqrt{g}\mu^{jk}\dot{H}_k + \varepsilon^{ijk}\partial_j E_k = 0 \quad (Faraday)$$

We write the equations in terms of E_j and H_j both with lower indexes. In the case of vacuum $\mu = g$.

The inhomogeneous equations (without sources) are

$$\partial_j(\sqrt{g}D^j) = \partial_j(\sqrt{g}\,\underbrace{\varepsilon^{jm} E_m}_{E_m \neq D_m}) = 0, \quad (Gauss)$$
$$\sqrt{g}\dot{D}^j - \varepsilon^{ijk}\partial_j H_k = \sqrt{g}\,\underbrace{\varepsilon^{jm}\dot{E}_m}_{D_m \neq E_m} - \varepsilon^{ijk}\partial_j H_j = 0 \quad (Ampere)$$

This shows that in a dielectric material where the dielectric and permeability tensors are the same (and time independent) hen we can reinterpret the constitutive relations as metric:

$$\varepsilon^{jm} = \mu^{km} \Longleftrightarrow g^{jm}$$

Now in most materials $\mu \approx 1$ while $\varepsilon \neq 1$, so this is not a general prescription. You need to engineer the material to make $\varepsilon = \mu$.

7.4.3 Metric and dielectric medium

The geometric description of the constitutive relation is due to Ulf Leonhardt (Now at Weizmann). This of what we have foudn like this: You have a dielectric with $\varepsilon, = \mu$ in a space with metric is $g \neq \varepsilon$. Maxwell equations are precisely the same as those in a (different) space, with metric g', but no dielectric, i.e $g' = \varepsilon' = \mu'$, provided

$$\sqrt{g}\,\mu^{ij} = \sqrt{g}\,\varepsilon^{ij} = \sqrt{g'}\,(g')^{ij},$$

as an identity between *functions*. This means:

$$\big(\sqrt{g}\,\mu^{ij}\big)(x) = \big(\sqrt{g}\,\varepsilon^{ij}\big)(x) = \big(\sqrt{g'}\,(g')^{ij}\big)\,\underbrace{(x)}_{not\ x'} \tag{7.23}$$

[3]In section Tensors for a discussion of div and curl in curvilinear coordinates.

So far, this is just a reinterpretation of the symbols. It says that whenever $\mu = \varepsilon$ you can reinterpret dielectrics media in one space as the geometry of an empty, but in general *different* space. This is not cloaking: You got a new space, possibly curved, and the curvature now replaces the dielectric.

You get cloaking provided the two spaces are geometrically the same, and the change in g reflects a coordinate change. In this case

$$(g')^{ij} = g^{\alpha\beta} \frac{\partial (x')^i}{\partial x^\alpha} \frac{\partial (x')^j}{\partial x^\beta} \tag{7.24}$$

This translates a dielectric into empty space in curved coordinates. This is the basis of cloaking.

Exercise 7.22. *Consider the scaling $(x')^j = \alpha x^j$. What is the value of ε and μ that behaves like vacuum? (Note that time has* not *been scaled.).*

7.4.4 Dirichlet to Neuman: Calderon problem

Cloaking means that you can not tell what is inside an inaccessible region by manipulating and measuring in its exterior. Cloaking means that you can not do CT, MRI, and can't explore for oil.

Here is a prototype of cloaking. You are given a medium with unknown ε, and are allowed to manipulate and measure only electrostatic potential and normal field on the surface. Can you determine $\varepsilon(x)$ inside?

Exercise 7.23. *Show that if $\varepsilon(x)$ is known, then the field on the surface, $E_n(x)$ is a linear functional of $\phi(x)$ on the surface.*

The equation for the potential is

$$\partial_j \left(\sqrt{g} \varepsilon^{jm} \partial_m \phi \right) = 0$$

(The metric g is known.) This is a tensorial equation, it retains its form under coordinate change. Consider a coordinate transformation that only affects the interior of the body. This will not affect anything you can do or measure, , but it will scamble ε and ϕ inside the body. It follows that you can not determine ε from (static) boundary data.

7.4.5 An invisible dielectric ball

Take g to be the usual spherical coordinate metric with diagonal covariant components

$$g_{r,\theta,\phi} = (1, r^2, r^2 \sin^2 \theta), \quad \sqrt{g} = r^2 \sin \theta$$

Consider change of the radial coordinate, r, alone

$$h(r') = \begin{cases} r & r' > 1 \\ h(r') & r' < 1 \end{cases}, \quad h(1) = 1, \quad h(0) = 0, \quad h'(r') > 0$$

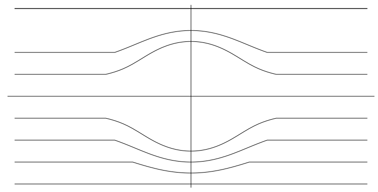

Figure 7.6: Straight lines, geodesics in the Euclidean metric, look like deformed curves when plotted in the coordinates x' associated with the deformed metric g'. In cloaking you use this fact, and Fermat idea, to force the light avoid the hiding place.

$h(r')$ is a nice, monotonic, 1-1. This introduces a dielectric medium inside the unit ball. We want to find the dielectric media.

Using the covariant version of Eq. 7.24:

$$(g')_{r,\theta,\phi}(x') = \left(\left(\frac{1}{h'(r')}\right)^2, h^2(r'), h^2(r')\sin^2\theta\right), \quad \left(\sqrt{g'}\right)(x') = \frac{h^2(r')}{h'(r')}\sin\theta$$

The right hand side of Eq. (7.23) is the diagonal matrix

$$\frac{\sin\theta}{h'(r')}\left((hh'(r'))^2, 1, \sin^{-2}\theta\right)$$

viewed as a function of (r', θ). Replace the argument $r' \to r$ gives the function

$$\frac{\sin\theta}{h'(r)}\left((hh'(r))^2, 1, \sin^{-2}\theta\right)$$

Plugging in Eq. 7.23 gives for the contravariant components of the (diagonal) tensors

$$\mu^{r,\theta,\phi} = \varepsilon^{r,\theta,\phi} = \frac{1}{h'(r)}\left(\left(\frac{h(r)h'(r)}{r}\right)^2, \frac{1}{r^2}, \frac{1}{r^2\sin^2\theta}\right)$$

The dielectric tensor looks nicer, and may have clearer physical meaning, in normalized components

$$\mu_{\hat{r},\hat{\theta}\hat{\phi}} = \varepsilon_{\hat{r},\hat{\theta}\hat{\phi}} = \frac{1}{h'(r)}\left(\left(\frac{h(r)h'(r)}{r}\right)^2, 1, 1\right)$$

$h(r) = r$ for $r > 1$ one gets $\mu = \varepsilon = 1$ outside, as one must. However, inside the ball we have a non-trivial dielectric medium. Such a medium is invisible.

Exercise 7.24. *Explain why the tensor represents and isotropic medium.*

7.4.6 Cloaking

In cloaking you want more than an invisible ball. You want to use its interior to hide something. This can be achieved with an h that creates a protected cavity inside ball of radius 1/2.

$$r' = \frac{1+r}{2}\theta(1-r) + r\theta(r-1)$$

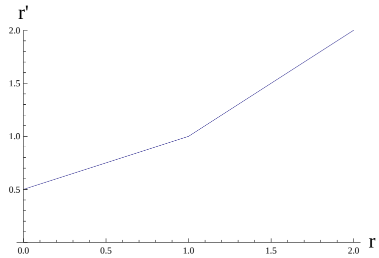

Figure 7.7: Coordinate change which creates a hiding cache in a ball of radius 1/2. The origin The surface of the sphere $r' = \frac{1}{2}$ is mapped to the origin $r = 0$. The interior of the sphere is mapped into a different world $r < 0$. The real world with a hole $r' > \frac{1}{2}$ is mapped into a fictitious world without a hole which looks empty.

h is a piecewise linear function which inside the unit ball is simply

$$r = h(r') = 2r' - 1$$

Plugging in the equation for the dielectric functions we get in the coordinates of physical space (with the standard spherical coordinates)

$$\mu_{\hat{r},\hat{\theta},\hat{\phi}} = \varepsilon_{\hat{r},\hat{\theta},\hat{\phi}} = \frac{1}{2}\left(\left(\frac{2(2r-1)}{r}\right)^2, 1, 1\right), \quad 1/2 < r < 1$$

The radial component of $\mu = \varepsilon$ vanishes at the boundary of the protected acvity: $r = 1/2$. Not the discontinuity on the boundary: The limit from inside the ball gives $\varepsilon = \frac{1}{2}(4, 1, 1)$ while the limit from outside is $(1, 1, 1)$.

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Chapter 8

Electromagnetic waves

Maxwell equations in vacuum are reduced to the wave equation for the potentials. We discuss the notions of polarization, Green's function, retarded and advanced solutions.

8.1 Electromagnetic waves

In the absence of sources, the inhomogeneous Maxwell equations are:

$$0 = \partial^\mu F_{\mu\nu} = \partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{8.1}$$

In the absence of charges the Coulomb gauge allows us to choose $\phi = A_0 = 0$ and $\nabla \cdot \mathbf{A} = 0$. In particular, it says that[1]

$$\partial_\mu A^\mu = 0$$

This is known as the Lorenz gauge condition (see section 9.1.2). Here we derived it as a special case of the Coulomb gauge in the absence of charges. In fact, one can *always* impose the Lorentz gauge condition (even when there are charges), but the proof that one can do that shall only be given later. It is manifestly gauge invariant.

In the Lorentz gauge the potentials satisfy the wave equation

$$\Box A^\mu = 0, \quad \Box = \partial^\mu \partial_\mu = -\frac{1}{c^2}\partial_{tt} + \Delta$$

Remark 8.1 (Lorentz invariance)**.** *Since the Dalambertian, $\Box$ and the Lorenz gauge conditions are manifestly a Lorentz invariants, Lorentz transformations of electromagnetic waves are electromagnetic waves.*

[1] Assuming $\det |g| = 1$

8.1.1 Electric and Magnetic fields

The electric and magnetic fields in vacuum satisfy the wave equations. This follows from Faraday and Ampere laws:

$$\dot{\mathbf{E}} + \nabla \times \mathbf{B} = 0, \quad \dot{\mathbf{B}} - \nabla \times \mathbf{E} = 0$$

Substituting one into the other gives

$$\ddot{\mathbf{E}} + \nabla \times (\nabla \times \mathbf{E}) = 0, \quad \ddot{\mathbf{B}} + \nabla \times (\nabla \times \mathbf{B}) = 0$$

The constraints are Gauss laws

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0 \tag{8.2}$$

Combining we get that the electric and magnetic fields satisfy the wave equation

$$\ddot{\mathbf{E}} - \Delta\mathbf{E} = 0, \quad \ddot{\mathbf{B}} - \Delta\mathbf{B} = 0 \tag{8.3}$$

Waves that satisfy the wave equation, Eq. (8.3), and the divergence-less constraint, Eq. (8.2) are called transverse waves.

8.2 Plane waves

Plane waves are (the real part of)

$$A_\mu(x) = a_\mu e^{ik\cdot x}, \quad k \cdot x = k_\mu x^\mu \tag{8.4}$$

with a_μ a 4-vector of fixed amplitudes and $k_\mu = (-\omega, \mathbf{k})$ a fixed 4-wave vector. It follows

$$F_{\mu\nu} = i(k_\mu a_\nu - k_\nu a_\mu)e^{ik\cdot x}$$

Exercise 8.2. *Show that F is invariant under*

$$a_\mu \to a_\mu + \lambda k_\mu \tag{8.5}$$

Explain why this is a gauge transformation.

Maxwell equations, Eq. (8.1), reduce to an algebraic equation for k and a linear equation for a_ν

$$(k \cdot k)\ a_\nu - k_\nu(k \cdot a) = 0 \tag{8.6}$$

This, together with the Lorentz gauge condition, Eq. (8.5), allows us to impose that the two terms in Eq.(8.6) separately vanish

$$k \cdot k = 0, \quad k \cdot a = 0$$

This says that k is a light-like vector and fixes the dispersion relation

$$\omega = \pm c|\mathbf{k}|$$

The second condition, $k \cdot a = 0$, the Lorenz gauge condition, is a transversality condition on the amplitude.

We are still free to orient the Euclidean frame so that the wave propagates in the z-direction. The light-like vector k_μ and the amplitude a_μ are then

$$k_\mu = \omega(-1,0,0,1), \quad a_\mu = \underbrace{(a_0, a_1, a_2, -a_0)}_{Lorentz\ gauge}$$

This is how a plane wave looks in the Lorenz gauge.

There is a remnant gauge freedom that allows us to choose a_0. Using the fact that k is light-like, $k_0 \neq 0$ hence the gauge freedom, Eq. (8.5), allows us to set $a_0 = 0$. This reduces the Lorenz gauge to the Coulomb gauge:

$$k_\mu = \omega(-1,0,0,1), \quad a_\mu = \underbrace{(0, a_1, a_2, 0)}_{Coulomb\ gauge} \tag{8.7}$$

In the Coulomb gauge, the amplitudes are orthogonal (in Euclidean space) to the direction of propagation.

8.2.1 Electric and magnetic fields

For plane waves

$$\mathbf{E} = -i\frac{\omega}{c}\mathbf{A}, \quad \mathbf{B} = -i\mathbf{k} \times \mathbf{A}$$

Since $\mathbf{k} \cdot \mathbf{A} = 0$ this implies that $\mathbf{E}$ and $\mathbf{B}$ are orthogonal and have equal magnitudes. $\mathbf{E}, \mathbf{B}$ and $\mathbf{k}$ form an orthogonal triad.

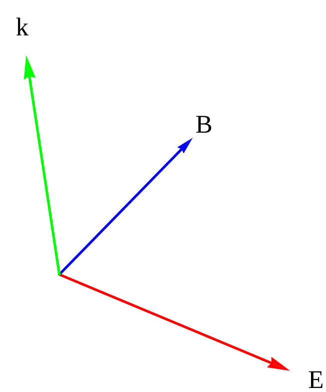


Figure 8.1: The triad of $\mathbf{E}, \mathbf{B}, \mathbf{k}$ for a plane wave. The wave propagates in the $\hat{\mathbf{k}} = \hat{\mathbf{E}} \times \hat{\mathbf{B}}$ direction

8.2.2 Doppler

Since $k \cdot x$ is a Lorentz scalar

$$k \cdot x = k' \cdot x'$$

Lorentz transformation of a plane wave is a still a plane wave. The wave has, in general, different wave vectors and amplitudes in different frames:

$$k'_\mu = \Lambda_\mu{}^\nu k_\nu, \quad a'_\mu = \Lambda_\mu{}^\nu a_\nu$$

Longitudinal Doppler

Consider a plane wave propagating in the z-direction, Eq. (8.7). Boosting the wave with rapidity ϕ in the *same* direction is the same as viewing the wave from an inertial frame boosted in the *opposite* direction. The associated Lorentz transformation is

$$\Lambda_0{}^3 = \Lambda_3{}^0 = -\sinh\phi, \quad \Lambda_0{}^0 = \Lambda_3{}^3 = \cosh\phi, \quad \Lambda_1{}^1 = \Lambda_2{}^2 = 1 \tag{8.8}$$

The Lorentz transformation of the light-like vector $k_\mu = \omega(1,0,0,1)$ gives $k'_\mu = \omega'(1,0,0,1)$ where

$$\omega' = \omega(\cosh\phi + \sinh\phi) = \omega e^\phi = \omega\sqrt{\frac{1+\beta}{1-\beta}} \tag{8.9}$$

This is linear in the velocities for small speeds. $a_{1,2}$ are not affected by the boost.

Transverse Doppler

Consider, as before, a wave propagating in the z-direction, but a boost in the x-direction so that

$$\Lambda_0{}^1 = \Lambda_1{}^0 = -\sinh\phi, \quad \Lambda_0{}^0 = \Lambda_1{}^1 = \cosh\phi, \quad \Lambda_2{}^2 = \Lambda_3{}^3 = 1 \tag{8.10}$$

The wave vector $k_\mu = \omega(1,0,0,1)$ is transformed to a light like wave vector $k'_\mu = \omega(\cosh\phi, -\sinh\phi, 0, 1)$ in the $x - z$ plane. The new frequency is

$$\omega' = \omega\cosh\phi = \omega\,\gamma$$

This is quadratic in the velocities for small speeds.

8.2.3 Particle production: GZK limit

The cosmic microwave background (CMB) provides a shield that screens ultra high energy cosmic rays: The GZK limit says that protons with energies above 5×10^{13} MeV are screened by the $3^\circ K$ thermal photons of the CMB. As an exercise let us compute the threshold for pion production.

Let us compute the threshold for particle production. The total energy-momentum of a proton with rapidity ϕ and counter-propagating photon in the plane is

$$p_\mu = m_P(\cosh\phi, \sinh\phi) + \hbar\omega(1, -1) = (p_0, p_1)$$

The energy in the center of mass frame, E_{cm}, is the scalar

$$E_{cm} = \sqrt{p_0^2 - p_1^1} = \sqrt{m_P^2 + 2m_P\hbar\omega e^\phi} = m_P + m_\pi$$

and the equality on the right expresses The threshold for pion production.

$$m_\pi^2 + 2m_P m_\pi = 2m_P\,\hbar\omega\, e^\phi$$

Since $m_\pi \ll m_P$ one finds a simple formula for the rapidity

$$e^\phi \approx \frac{m_\pi}{\hbar\omega}$$

The corresponding energy threshold is

$$m_p\cosh\phi \approx \tfrac{1}{2}m_p e^\phi \approx \frac{m_p m_\pi}{2\times 3k_B} \approx 2.5\times 10^{14}\ Mev$$

which is factor 5 too large from the estimate given above.

Exercise 8.3. *Can you figure out why the estimate is too big?*

8.2.4 Laser cooling and optical molasses

Laser cooling is a cool application of the Doppler effect to slow down atoms. Think of the atom as a two level system with energy gap E. Suppose you point a laser beam with frequency $\hbar\omega < E$ at the atom. Atoms that move towards the light source will see bluer light and if they move fast enough, they will be able to absorb the light. This will slow the atom down. At the same time, slow atoms will be transparent to the light.

The remnant velocity is the one whose Doppler shift is comparable to the natural line width Γ of the atomic a level. This gives the velocity $c\Gamma/E$. The corresponding temperature is of the order

$$k_b T \approx \tfrac{1}{2}Mc^2\left(\frac{\Gamma}{E}\right)^2 \tag{8.11}$$

where M is the mass of the atom. This leads to low temperatures wheneresvr the energy level has long life time so Γ is small. Indeed,

$$\left(\frac{\Gamma}{E}\right)^2 = O(\alpha^6) = O(10^{-12})$$

8.3 Polarization

8.3.1 Amplitude and phase

Scalar plane waves are simply characterized by their frequency ω, wave vector $\mathbf{k}$, amplitude and phase. Electromagnetic waves, being vector valued, are more complicated. In addition to the amplitude and phase they are also characterized by their polarization.

The electric field of an electromagnetic wave propagating is the real part of

$$\mathbf{E}_0\, e^{i\phi}, \quad \phi = \mathbf{k}\cdot\mathbf{x} - \omega t \tag{8.12}$$

The amplitude, $\mathbf{E}_0$, is a *complex* vector in the plane perpendicular to direction of propagation, $\mathbf{k}\cdot\mathbf{E}_0 = 0$. $\mathbf{E}_0$ has 4 real amplitudes. What is the physical interpretation of these four amplitudes?

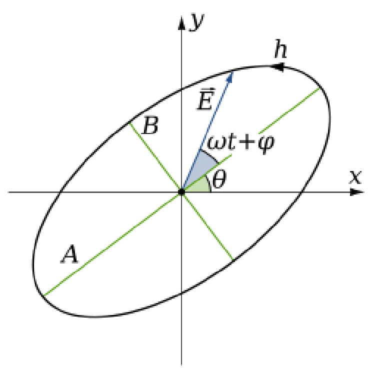


Figure 8.2: Four Stokes parameters describe elliptically polarized light. Three numbers identify the size of the ellipse, its tilt to the axes, its eccentricity. A fourth number gives the purity (the coherence) of the light. The plane of the ellipse is perpendicular to the direction of propagation $\mathbf{k}$.

8.3.2 Polarization

Let $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ denote orthogonal unit vectors in the plane perpendicular to $\mathbf{k}$. Write

$$\mathbf{E}_0 = E_+\mathbf{z}_+ + E_-\mathbf{z}_-, \quad \sqrt{2}\mathbf{z}_\pm = \hat{\mathbf{x}} \pm i\hat{\mathbf{y}}$$

We may formally identify $E_\pm$ with the components of a an (un-normalized) spin 1/2 namely, $|\psi\rangle \iff (E_+, E_-)^t$. Quantum mechanics provides us with a canonical procedure for factoring out the normalization and the overall phase in of a quantum state: The density matrix:

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{|E_+|^2 + |E_-|^2}\begin{pmatrix} |E_+|^2 & E_+E_-^* \\ E_+^*E_- & |E_-|^2 \end{pmatrix} \tag{8.13}$$

ρ does not care about the overall amplitude, $\sqrt{|E_+|^2 + |E_-|^2}$, and overall phase of the wave. Since $Tr\rho = 1$ while $\det\rho = 0$ the two eigenvalues of ρ are 1 and 0: ρ is a projection $\rho^2 = \rho$

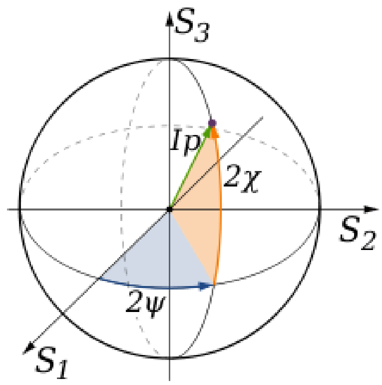


Figure 8.3: The Poincare sphere associates with every point on the sphere a polarization. The north and south poles represent right and left circularly polarized light and the equator with linearly polarized light.

8.3.3 Poincare sphere

Any 2×2 hermitian matrix of unit trace can be written as

$$\rho = \tfrac{1}{2}\left(1 + \mathbf{s}\cdot\sigma\right) = \frac{1}{2}\begin{pmatrix} 1+s_3 & s_1+is_2 \\ s_1-is_2 & 1-s_3 \end{pmatrix} \tag{8.14}$$

where $\mathbf{s} = (s_1, s_2, s_3) \in \mathbb{R}^3$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of Pauli matrices:

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = -\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{8.15}$$

From Eq.(8.14)

$$4\det\rho = 1 - \mathbf{s}\cdot\mathbf{s} \tag{8.16}$$

so ρ is a projection if $\mathbf{s}$ is a unit vector $\hat{\mathbf{s}}$. This allows us to identify state of polarization with points on the unit sphere.

Exercise 8.4. *Show that antipodal points on the Poincare sphere represent orthogonal polarization in the sense that $\rho_{\hat{s}} \cdot \rho_{-\hat{s}} = 0$.*

8.3.4 Stokes parameters

Since we are oblivious to the overall phase, we may write $(E_+, E_-) = (\cos\chi, e^{i\psi}\sin\chi)$, with $\cos\chi \geq 0$, i.e. $0 \leq \chi \leq \pi/2$. The corresponding points on the unit sphere are

$$s_3 = \cos^2\chi - \sin^2\chi = \cos 2\chi,$$
$$s_1 + is_2 = 2e^{-i\psi}\cos\chi\sin\chi = e^{-i\psi}\sin 2\chi$$

This makes 2χ and ψ the standard spherical coordinates.

Circular polarization

The north poles correspond to $\cos\chi = 1$ so that $\mathbf{E_0} \Longrightarrow \mathbf{z}_+$. The electric field is–up to an overall amplitude and phase–

$$\sqrt{2}\mathbf{E} = \hat{\mathbf{x}}\cos\phi - \hat{\mathbf{y}}\sin\phi, \quad \phi = \mathbf{k}\cdot\mathbf{z} - \omega t \tag{8.17}$$

As ϕ increases from 0 to 2π the vector $\mathbf{E}$ describes a circle in the x-y plane which is turning clockwise. At a fixed z and as a function of time the field rotates counter clockwise while propagating: It behaves like a left handed screw and so is called left circularly polarized.

Exercise 8.5 (South pole)**.** *Show that the south pole represent right circular polarization.*

Linear polarization

The equator is $s_3 = 0 \Longrightarrow \cos^2\chi = \sin^2\chi$. As the overall phase does not play a role we may take for the amplitude $(e^{-i\psi/2}, e^{i\psi/2})/\sqrt{2}$. The corresponding $\mathbf{E}_0$ is:

$$\begin{aligned}\mathbf{E}_0 &= \tfrac{1}{2}(\hat{\mathbf{x}} + i\hat{\mathbf{y}})e^{-i\psi/2} + \tfrac{1}{2}(\hat{\mathbf{x}} - i\hat{\mathbf{y}})e^{i\psi/2}\\ &= \hat{\mathbf{x}}\cos(\psi/2) + \hat{\mathbf{y}}\sin(\psi/2)\end{aligned}$$

As ϕ increases from 0 to 2π this describes a line element in the x-y plane at angle $\psi/2$ to the x axis. $\psi = 0$ corresponds to $\hat{\mathbf{x}}$ polarized wave and $\psi = \pi$ to $\hat{\mathbf{y}}$ polarized wave.

8.3.5 Partially polarized light

The discussion so far addressed ideal plane waves where k is sharply defined and the amplitude $\mathbf{E}_0$ is an honest constant. Such a wave is coherent. Many light sources are incoherent. One way to model incoherence is as a statistical average

$$\mathbf{E}_0 = \sum p_j \mathbf{E}_j, \quad p_j \geq 0,$$

where $\mathbf{E_j}$ represent independent (normalized) light sources, namely

$$\langle(\mathbf{E}_j)_a(\mathbf{E_k})_b\rangle = 0 \quad j \neq k, \forall a, b$$

The polarization of such a mixture is naturally defined as the mixture of polarizations

$$\rho = \sum p_j \rho_j \tag{8.18}$$

Exercise 8.6. *Show that*

$$\rho = \frac{1}{\langle |E_+|^2\rangle + \langle |E_-|^2\rangle} \begin{pmatrix} \langle |E_+|^2\rangle & \langle E_+E_-^*\rangle \\ \langle E_+^*E_-\rangle & \langle |E_-|^2\rangle \end{pmatrix} \tag{8.19}$$

It is still true that $Tr\rho = 1$. But now ρ is not a projection, because $\det\rho$ need not vanish:

$$\det\rho\big(\langle |E_+|^2\rangle + \langle |E_-|^2\rangle\big)^2 = \langle |E_+|^2\rangle\langle |E_-|^2\rangle - \langle E_+E_-^*\rangle\langle E_+^*E_-\rangle \geq 0 \tag{8.20}$$

(Schwartz inequality was used here). By 8.16 this implies that the vector $|\mathbf{s}| \leq 1$; the vector lies in the unit ball. The light we get from the sun is completely unpolarized. It is associated with $\mathbf{s} = 0$, the center of the Poincare ball.

Remark 8.7 (Combing a tennis ball)**.** *One amusing, essentially topological, property of the transverse nature of electromagnetic waves is that it is not possible to have a fully spherically symmetric electromagnetic wave. The point is that a spherical wave, with* **k** *pointing radially, has* **E** *tangent to the sphere. It is a basic fact in topology that any vector field on the sphere must vanish at (at least) two points. The field can not be "the same" everywhere.*

8.3.6 3D glasses

When you view a 3D movie, the 3D glasses transmit a picture with right circular polarization to, say, the right eye and left circular polarization to the left eye.

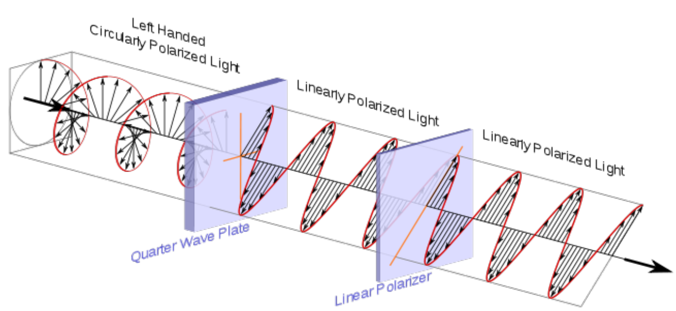


Figure 8.4: An arrangement that transmits right circular polarization.

Exercise 8.8. *Can you give an (ergonomic) argument why spectators would prefer circular to linear polarization?*

Exercise 8.9. *Define quarter wave plate as the rotation of the Poincare sphere that turns circular polarization to linear. Show that it is represented by Hadamard gate H*

$$\sqrt{2}H = \sigma_3 + \sigma_1$$

Exercise 8.10. *Explain why the filtering associated by linear polarizers can be described by the projections*

$$P_{H,V} = \frac{\mathbb{1} \pm \sigma_1}{2}$$

It follows from the two exercises above that the right and left glasses can be represented by 2×2 matrices

$$g_1 = P_H H, \quad g_2 = P_V H$$

Exercise 8.11. *Give a physical interpretation of the identity*

$$P_V \sigma_3 = \sigma_3 P_H$$

in terms of rotation of the glasses.

Exercise 8.12. *Explain why holding the glasses backwards is represented by transposition:*

$$g_j \Longleftrightarrow g_j^t$$

If you place glass 1 rotated by $\pi/2$ behind glass 2 inverted the joint system is represented by represented by the matrix product $\sigma_3 g_1\, g_2^t$. A computation gives 0 which means that no light passes through.

Exercise 8.13. *Show that there are 64 ways of arranging the pair of glasses. How many of these let no light through.*

8.4 The wave equation

So far, we have discussed plane wave solutions of the electromagnetic wave equation. It is instructive to study some basic properties of the wave equation in general.

8.4.1 The wave equation in one dimension

The one dimensional wave equation, for a scalar field ϕ, in light-cone coordinates $u = x^1 - x^0$, $v = x^1 + x^0$ takes the form

$$\Box\phi = 4\partial_{uv}\phi = 0$$

The general solution of which is

$$\phi(u, v) = f(u) + g(v)$$

with (essentially) arbitrary f and g. f describes a wave rigidly propagating to the right at speed c and g a wave rigidly propagating to the left at speed c.

The functions f and g are determined by the initial (Cauchy) data

$$\phi_0(x^1, x^0 = 0) = f(x) + g(x), \quad \dot{\phi}_0(x^1, x^0 = 0) = -\big(f'(x) - g'(x)\big)$$

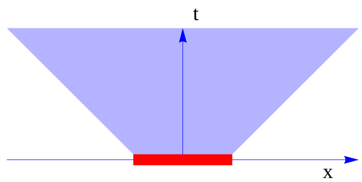


Figure 8.5: If the initial data, ϕ_0 and $\dot{\phi}_0$ are localized in the red interval, the solution at later times lives in the cone. This is called the domain of influence of the initial (red) data.

This allows for reconstructing f and g from the initial data by integration. As we shall discuss in more detail below, the localization of ϕ_0 and $\dot{\phi}_0$ does not imply that f and g are localized, see Fig. 8.6.

Any solution can be thought of a a linear combination of the solution with vanishing initial data for $\dot{\phi} = 0$ and the complementary case, where the initial data for ϕ vanish. We are, of course interested in the case where the initial data are localized bump functions.

Figure 8.6: The initial data is $\phi_0 = 0$ and bump function $\dot{\phi}_0 = 2\cosh^{-2} x$ shown on the left. Also shown are $f(x) = \tanh(x)$ and $g(x) = \tanh(-x)$. On the right you see the initial data $\dot{\phi}(0)$ again, and the wave $\phi(x)$ at time 2. This is supposed to illustrate that wave lingers near the origin forever and the failure of Huygens principle in one dimension.

Consider first the initial data $\dot{\phi}_0 = 0$

$$\dot{\phi}_0 = 0 \Longrightarrow f - g = const \Longrightarrow \phi_0 = f + \tfrac{1}{2}const = g - \tfrac{1}{2}const$$

By assumption, ϕ_0 is a (localized) bump function. It follows that so are $2f + const$ and $2g - const$. One propagates to the right and the other to the left. The wave lives inside the light cone and after a while there is no remnant of the wave in the interval of the initial data. The wave propagated to infinity.

In contrast, when $\phi_0(x, 0) = 0$

$$\phi_0 = 0 \Longrightarrow f = -g \Longrightarrow 2g' = \dot{\phi}_0$$

Now, the localized initial data $\dot{\phi}_0$ *do not* guarantee that f and g are localized (only that f' and g' are bump function). See fig. 8.6. As a consequence *Huygens principle fails in one dimension.*

Exercise 8.14. *The energy density of a scalar wave is*

$$\dot{\phi}^2 + (\nabla\phi)^2$$

1. *Express the energy density in terms of f and g.*

2. *Explain why the failure of Huygens principle in one dimension does not lead to conflict with energy conservation.*

8.4.2 Huygens principle

The solution of the wave equation in d dimensions is better behaved than the solution in one dimension: Huygens principle hold. This means that the value of the wave at a space time point is fully determined by the intersection of the backward light cone with the initial data.

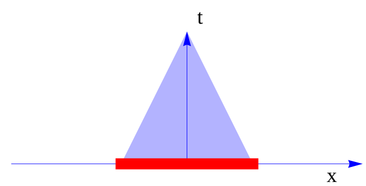


Figure 8.7: The domain of dependence of the initial red data. When Huygens principle holds, the value of the wave at the space time point is determined by the intersection of the backward light-cone with the initial data. In general, it is determined by the intersection of the *interior* of the cone with the initial data.

Explicit integral representation for the initial value problem of the wave equation in three dimensions, is given e.g. Wikipedia see also Pinchover and Rubinstein.

8.4.3 Covariant superposition

The wave equation is an algebraic equation in Fourier space. A solution can be written as

$$\phi(x) = \frac{1}{(2\pi)^2} \int d^4k\, \delta(k\cdot k)\, \tilde{\phi}(k)\, e^{ik\cdot x} \tag{8.21}$$

with $\tilde{\phi}(k)$ an arbitrary function of the 4-vector k. Of course, because of the δ function only the values that the function takes on the light-cone are relevant. This expression for a scalar wave ϕ is manifestly Lorentz invariant. The generalization to vector waves A_μ and tensor waves is clear.

It is instructive to split the solution to the forward and backward light cone we have

$$\phi(k) = \theta(-t)\phi_<(\mathbf{k}) + \theta(t)\phi_>(\mathbf{k}) \tag{8.22}$$

We can carry out the time integration to get:

$$\phi_{>}(t,\mathbf{x}) = \frac{1}{(2\pi)^2}\int \frac{d\mathbf{k}}{2|\mathbf{k}|}\,\tilde{\phi}(|\mathbf{k}|,\mathbf{k})\, e^{i(|\mathbf{k}|t-\mathbf{k}\cdot\mathbf{x})} \tag{8.23}$$

The forward light cone is associated with out going (retarded) waves. Similarly

$$\phi_{<}(t,\mathbf{x}) = \frac{1}{(2\pi)^2}\int \frac{d\mathbf{k}}{2|\mathbf{k}|}\,\tilde{\phi}(-|\mathbf{k}|,\mathbf{k})\, e^{-i(|\mathbf{k}|t+\mathbf{k}\cdot\mathbf{x})} \tag{8.24}$$

the backward light-cone can be associated with incoming (advanced) waves. Note that in both cases, Lorentz invariance induces a weight on the three dimensional $\mathbf{k}$ space.

8.4.4 Waves with Gaussian waists

Exercise 8.15 (Light cone)**.** *Show that the wave equation in 3+1 dimensions can be written as*

$$(\partial_{uv} + \bar{\partial}\partial)\phi(u,v,z,\bar{z}) = 0, \quad z = x^1 + ix^2, \quad \bar{z} = x^1 - ix^2$$

where U, v are light cone coordinates, $v = x^3 - x^0$, $u = x^3 + x^0$

One is often interested in narrow pencils of light. Here is an example of a monochromatic wave with a Gaussian waist

Exercise 8.16 (Gaussian waists)**.** *Show that*

$$\phi = c(u)e^{-c(u)z\bar{z}}e^{iv/\lambda}$$

solves the wave equation provides

$$\lambda c^2(u) + ic'(u) = 0$$

whose solution is

$$c(u) = \frac{1}{\ell^2 + i\lambda u}$$

8.4.5 Monochromatic waves

Monochromatic waves are solutions of the wave equation whose time dependence is $e^{i\omega t}$. Hence

$$\Delta\phi = -{k_0}^2\phi \tag{8.25}$$

In Fourier space the solution is supported on the sphere

$$\mathbf{k}\cdot\mathbf{k} = k_0^2 \tag{8.26}$$

The smallest wave length that such a wave can accommodate is $2\pi/k_0$: The frequency limis the spatial resolution.

8.4.6 Evanescent waves

Near a planar boundary between two media one can can sometimes arrange for monochromatic, evanescent plane wave solution to the wave equation: Plane waves in the half-space which decay in x and propagate in the z-direction. Namly

$$e^{-\kappa x}e^{ikz}, \qquad k^2 = k_0^2 + \kappa^2$$

The noteworthy fact about this waves is that the frequency k_0, does not limit anymore the spatial resolution $1/k$. Near $x = 0$ one can find waves with $k \gg k_0$.

Example 8.17 (Transversal waves)**.** *The notion of transversality for evanescent waves is different from ordinary plane waves. For the wave*

$$\mathbf{E} = \mathbf{E}_0 e^{-\kappa x}e^{ikz}$$

Gauss law $\nabla \cdot \mathbf{E} = 0$ *reduces to*

$$\kappa(E_0)_1 + ik(E_0)_3 = 0$$

which allows $(E_0)_3 \neq 0$ *for a wave propagating the in z-direction.*

8.4.7 The eikonal equation: Geometric optics

8.5 Waves in dielectric media: Birefringence:

In the absence of external sources, the time evolution of the fields in a dielectric is dictated by Faraday and Ampere laws

$$\dot{\mathbf{E}} + \nabla \times \mathbf{B} = 0, \quad \dot{\mathbf{H}} - \nabla \times \mathbf{D} = 0$$

subject to the constraints

$$\nabla \cdot \mathbf{D} = 0, \quad \nabla \cdot \mathbf{B} = 0 \tag{8.27}$$

In Fourier space $(\mathbf{x}, t) \leftrightarrow (\omega, \mathbf{k})$ the differential equations reduce to algebraic equations

$$\omega\varepsilon^{-1}\,\mathbf{D} + \mathbf{k} \times \mu\mathbf{H} = 0, \quad \omega\mathbf{H} - \mathbf{k} \times \mathbf{D} = 0, \quad \mathbf{k} \cdot \mathbf{D} = 0, \quad \mathbf{k} \cdot \mu\mathbf{H} = 0$$

where μ and ε are the constitutive relations. We assume that μ, ε are positive matrices. (Possibly functions of ω.) Substitution gives for $\mathbf{D}$

$$\omega^2\varepsilon^{-1}\,\mathbf{D} + \mathbf{k} \times (\mu\mathbf{k} \times \mathbf{D}) = 0 \tag{8.28}$$

which can be written as a (generalized) eigenvalue problem for the 3×3 symmetric matrix

$$M_{jk}(\omega, \mathbf{k}) = \omega^2(\varepsilon^{-1})_{jk} + \varepsilon_{jmn}\varepsilon_{abk}\mu_{na}k_m k_b$$

Given $\mathbf{k}$ a non-trivial solution for the eigenvector $\mathbf{D}$ exists provided $\det M = 0$. This fixes the dispersion relation $\omega_j^2(\mathbf{k})$ with $j = 0, 1, 2$, the three eigenvalues of Eq. (8.28). One eigenvalue is always trivial since the matrix always 0 as an eigenvalue, corresponding to the eigenvector $\mathbf{k}$. There are, therefore, in general, only two non-trivial eigenvalues

Exercise 8.18. *Show that that if ε and μ are symmetric, so is M.*

In the frame where ε and μ are diagonal

$$M_{jk}(\omega, \mathbf{k}) = \omega^2(\varepsilon^{-1})_j \delta_{jk} + \mu_n \varepsilon_{njm} \varepsilon_{nbk} k_m k_b$$

In the special case that μ is a scalar

$$M_{jk}(\omega, \mathbf{k}) = \big(\omega^2(\varepsilon^{-1})_j - \mu \mathbf{k} \cdot \mathbf{k}\big)\delta_{jk} + \mu k_j k_k$$

For $\mathbf{k}$ in the principal direction j we get

$$\omega^2 = \varepsilon_j \mu \, k^2$$

The wave propagates at different speeds $\sqrt{e_j \mu}$ along the principal directions of ε. This is birefringence.

8.6 Green's function for the wave equation

The Green function, aka fundamental solution, is the solution to the wave equation with a point source term

$$\Box G = 4\pi \delta^{(4)}(x) \tag{8.29}$$

This equation merits some discussion regarding the existence and the uniqueness of the solution. The issue of existence has to do with the fact that the source terms is singular: A delta function (a distribution). What regularity properties shall we require of the solutions? When we studied Poisson's equation–an elliptic equation–a singular source had a Greens function that was a pretty regular function (the Coulomb singularity is integrable). This is a feature of elliptic equations. We will not have this luxury in the case of the wave equation which is hyperbolic: We shall have to allow for solutions G that are distributions if we insist on causality.

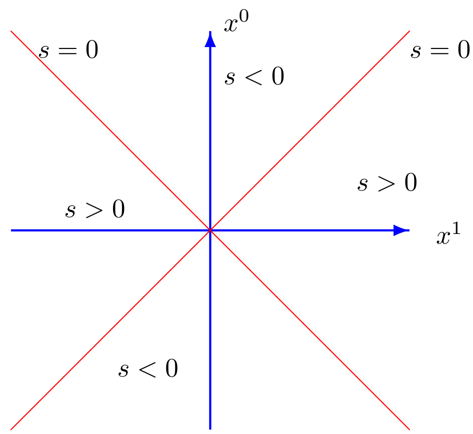
The second issue is uniqueness. As usual, we have the freedom of adding solutions of the free wave equation. We fix the solution by imposing causality: We shall denote by $G_>$ the solutions which is created by the source. This is also called a retarded solution. Formally, the wave equation with a source also admits advanced solutions where the wave is fully absorbed by the source (and solutions of mixed type). We select the retarded solutions because it is causal.

As we shall show below, the retarded Green function in 3+1 dimensions is

$$G_>^{3+1}(x) = 2\theta(t)\delta(x \cdot x) = \frac{\delta(|\mathbf{x}| - ct)}{|\mathbf{x}|}, \quad x = (ct, \mathbf{x}), \tag{8.30}$$

$G_>$ lives *on* the light-cone–rather than in its interior. This is the Huygens principle.

Exercise 8.19. *Show the second identity (on the right).*



The case of $d+1$ dimensions

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Although our main interest is in 3+1 dimensions, it is instructive to look at the general case of $d+1$. Causality means that $G_>^{d+1}$ lives in the forward light cone. Since the source terms is isotropic let us seek a solution of the form

$$G_>^{d+1}(x) = \theta(t) f(s), \quad s = x_\mu x^\mu \tag{8.31}$$

We need to ascertain that $G_>$ solves the homogeneous wave equation for $t > 0$. To do so, let us compute the two derivatives in the Dalambertian one at a time:

$$\partial_\mu f(s) = (\partial_\mu s)\, f'(s) = 2x_\mu f'(s) \tag{8.32}$$

It follows

$$\begin{aligned}
\tfrac{1}{2}\partial^\mu \partial_\mu f(s) &= \partial^\mu \left(x_\mu f'(s) \right) \\
&= (\partial^\mu x_\mu) f'(s) + 2(x_\mu x^\mu)\, f''(s) \\
&= (d+1) f'(s) + 2s\, f''(s) \\
&= (d-1) f' + 2(s\, f')'
\end{aligned}$$

Hence

$$\partial^\mu \partial_\mu f = 2(d-1) f' + 4(s\, f')' = 2(d-3) f' + 4(sf)''$$

This is a simple, second order ordinary differential equation which can be integrated directly to give

$$(d-1)f + 2(s\,f') = c_1$$

To compute the constant consider the left hand side outside of the forward light cone. Since we insist on causal solutions both f and f' must vanish outside the light cone.

Exercise 8.20. *Why?*

This says that $c_1 = 0$. We are left with a first order equation

$$(d-1)f + 2s\,f' = (d-3)f + 2(sf)' = 0 \tag{8.33}$$

This simple differential equation can be integrated in any dimension d.

Exercise 8.21 (Regular solutions)**.** *Show that the regular solutions of the differential equation 8.33 are (the real and imaginary parts of)*

$$f(s) = c_2(-s)^{\frac{1-d}{2}} \quad d \neq 1,$$

The solution found in the exercise distinguished d even and d odd. For even d it looks ok for the (real part) lives in the forward light cone. However, for odd d the solution is not causal: Causality forces $c_2 = 0$ and we are left empty handed. What have we missed? We missed solutions that are distributions.

Example 8.22. *In 2+1 dimensions*

$$G_{>}^{2+1} = \lambda_2 \theta(t)\,\mathrm{Re}\left(\frac{1}{\sqrt{-s}}\right)$$

is causal but does not satisfy the Huygens principle.

Back to $3+1$

Let us now focus on the case $d = 3$. The differential equation 8.33 reduces to

$$(sf)' = 0$$

which is easily integrated to

$$sf(s) = c_2$$

with c_2 the integration constant above. We use causality to conclude that $c_1 2 = 0$ and we are left with

$$sf(s) = 0$$

If you treat this as an equation for a *function* f then $f = 0$. Have we done something wrong?

The subtle point here is that f need not be a function. After all the source was a distribution so we should allow also distributional solutions. Now we can find a non-trivial solution, namely

$$s\delta(s) = 0$$

We conclude that

$$G_>^{3+1}(x) = \lambda\theta(t)\delta(s)$$

and Huygens principle holds.

Exercise 8.23 (Solution in $1+1$)**.** *Using the similar arguments show that*

$$G_>^{1+1}(x) = \lambda_1\theta(t)\theta(-s)$$

The solution lives in *the forward light-cone rather than its boundary. It does not satisfy Huygens principle.*

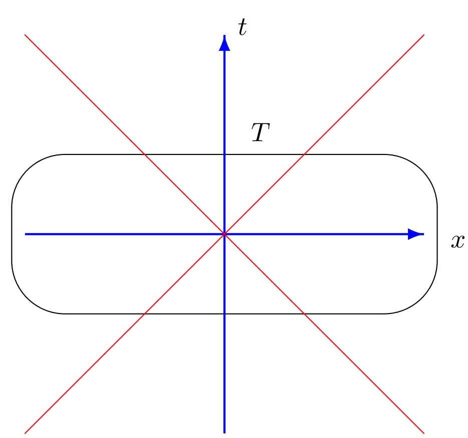
It remains to determine λ_3 in 3+1 dimensions. We do that by Gauss law. We integrate the wave equation with a source at the origin of a space-time box Ω [2]as in fig. 8.6

$$\int_\Omega \Box G_> d\Omega = 4\pi \int_\Omega \delta^{(4)}(x) d\Omega = 4\pi \tag{8.34}$$

The left-hand side can be converted to an integral on the boundary of the box

$$\int_\Omega \Box G_> \, d\Omega = \int_{\partial\Omega} (\partial_\mu G_>)\, dS^\mu$$

Exercise 8.24 (Signs-Sigh)**.** *On the early and late face of the box, determine the sign in* $dS^0 = \pm dV$*.*



Inspecting the figure one realizes that only the intersection of the top face of the box with the forward light cone contributes. Everywhere else the retarded solution vanishes. Using $dS^0 = dV$ and Eq. 8.32

$$\int_{\partial\Omega} (\partial_\mu G_>)\, dS^\mu = \int_{t=T} dV\, \partial_0 G_> = \lambda_3 \int_{t=T} dV\, (-2cT)\, \delta'(s)$$

[2]Minkowski cartesian coordinates with $|g| = 1$ are assumed.

With $s = r^2 - c^2T^2$

$$\begin{aligned}
\int_{t=T} dV\, \delta'(s) &= 4\pi \int_0^\infty dr\, r^2\, \frac{d\delta}{dr}\frac{dr}{ds} \\
&= 2\pi \int_0^\infty dr\, r\, \frac{d\delta}{dr} \\
&= 2\pi \int_0^\infty dr\, \left(\frac{d(r\delta)}{dr} - \delta\right) \\
&= -2\pi \int dr \delta(r^2 - c^2T^2) \\
&= -\frac{\pi}{cT}
\end{aligned}$$

This and Eq.8.34 fixes λ_3

$$\lambda_3(2cT)\frac{\pi}{cT} = 2\lambda_3\pi = 4\pi \Longrightarrow \lambda_3 = 2$$

Exercise 8.25. *Compute the normalization constants λ_{1+1} and λ_{2+1}.*

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Chapter 9

Radiation

Retarded solution of the wave equation. Retarded potentials, retarded electromagnetic fields. Lienard Wiechert potentials and fields; Dipole radiation.

9.1 Wave equation with a source term

We have seen that the retarded Green function for the wave equation in 3+1 dimensions is

$$G_>(x) = 2\,\theta(x^0)\delta\big(x\cdot x\big) = \frac{\delta(ct-r)}{r}, \quad r = |\mathbf{x}| \geq 0 \tag{9.1}$$

By the homogeneity of Minkowski space, for source term located at the space-time point y rather than the origin, the Green function is $G_>(x-y)$

9.1.1 Retarded waves for an arbitrary source

By the linearity of the wave equation, and the homogeneity of Minkowski space-time, the retarded solution of the wave equation, generated by an arbitrary[1] source ρ

$$\Box\phi = -4\pi\rho(x) \tag{9.2}$$

is

$$\phi_\rho(x) = \int d^4y\, G_>(x-y)\rho(y) \tag{9.3}$$

9.1.2 Application: Lorenz gauge

Let us show that one can always impose the Lorenz[2] gauge condition

$$\partial^\mu A_\mu = 0 \tag{9.4}$$

[1]Some condition on the localization of the sources should be imposed. This is related to Olber's paradox: If you assume constant density of stars, and that intensity of radiation falls like r^{-2} the night sky shoudl be as bright as the sun.

[2]This is the Danish Ludvig Lorenz, a contemporary of the Dutch, Nobel laureate Hendrik Lorentz of the Lorentz transformation.

Suppose $\partial^\mu A_\mu \neq 0$. Let Λ be the (retarded) solution of the wave equation with a source term

$$\Box\Lambda = -\partial^\mu A_\mu \tag{9.5}$$

Then, the gauge transformation,

$$A'_\mu = A_\mu + \partial_\mu \Lambda \tag{9.6}$$

satisfies the Lorenz gauge condition:

$$0 = \partial^\mu A'_\mu = \partial^\mu A_\mu + \partial^\mu \partial_\mu \Lambda \tag{9.7}$$

9.1.3 Scalar wave generated by a moving point source

As a preparation for studying the radiation of electromagnetic waves, consider the simpler problem of radiation of scalar waves generated by a point source moving on a world line $z = \big(ct, \mathbf{z}(ct)\big)$, $-\infty < t < \infty$. The motion is assumed to be that of a real particle so the velocity is time-like. The source density in space time is

$$\rho(\mathbf{y}, t) = \delta^{(3)}\big(\mathbf{y} - \mathbf{z}(t)\big) \tag{9.8}$$

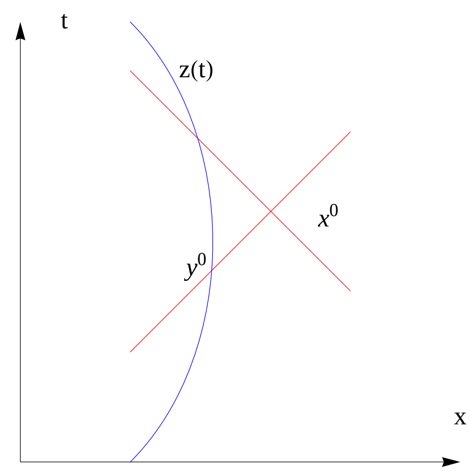

Figure 9.1: The world like $\mathbf{z}(t)$ of the point charge is the blue line. The wave at the point x is determined by the intersection of the backward light cone with the orbit at time y^0. There is one such point since the velocity is time like.

The retarded wave that the source generates is ($y^0 = ct$):

$$\begin{aligned} \phi_\rho(x) &= 2\int d^4y \, \underbrace{\delta^{(3)}\big(\mathbf{y} - \mathbf{z}(y^0)\big)}_{source} \, \underbrace{\delta\big((x-y)\cdot(x-y)\big)\,\theta(x^0 - y^0)}_{Green} \\ &= 2\int dy^0 \; \delta\big(R\cdot R\big)\theta(x^0 - y^0), \quad R = (x^0 - y^0, \mathbf{x} - \mathbf{z}(y^0)) \end{aligned}$$

The θ function guarantees causality: The past influences the present. The delta function says that signals propagate with the speed of light–Huygens principle holds. If the orbit $\mathbf{z}(z^0)$ is that of a physical particle, a single point contributes: a single particle has a single image (in the absence of mirrors) .

To compute the remaining integral use

$$\int \delta\big(f(y)\big)dy = \frac{1}{|f'(y_0)|}, \quad f(y_0) = 0. \tag{9.9}$$

Here $f = R \cdot R$ whose derivative is related to the the 4-velocity of the source $u = -\dot{R}$:

$$\frac{df}{dy^0} = \left(\frac{df}{d\tau}\right)\left(\frac{d\tau}{dy^0}\right) = 2\frac{R\cdot\dot{R}}{\gamma} = -2\,\frac{R\cdot u}{\gamma}$$

We obtain a simple looking formula for the wave $\phi(x)$ at the observing point x:

$$\phi(x) = \frac{\gamma(y^0)}{|R\cdot u(y^0)|}, \quad R = x - y, \quad R\cdot R = 0 \tag{9.10}$$

which is manifestly causal and satisfies Huygens principle.

The formula looks simple but at the price of being *implicit*. The right hand side *is not* an explicit function of the argument x. To compute $\gamma(y^0)$, $u(y^0)$ and $R = x - y$ on the right hand side you need first to evaluated at the *earlier time* y^0 (see Fig. 9.1.3), which is not explicitly given. This time is determined as the solution of the equation

$$(x^0 - y^0)^2 = \big(\mathbf{x} - \mathbf{z}(y^0)\big)^2$$

which may be arbitrarily complicated if the orbit $\mathbf{z}(z^0)$ is complicated.

As expected, the amplitude of *scalar waves* decays like $1/R$. However, you may argue if you want to call the general solution we have found a wave in all cases. For example you would probably not call the solution for a source at rest *a wave*.

9.2 Maxwell equation in the Lorenz gauge

The inhomogeneous Maxwell equations are

$$\partial^\mu F_{\mu\nu} = -\frac{4\pi}{c}\,j_\nu \tag{9.11}$$

Expressed in terms of the potentials, (this guarantees the homogeneous equations) one gets a system of second order PDE

$$\partial^\mu{}_\mu A_\nu - \partial^\mu{}_\nu A_\mu = -\frac{4\pi}{c}\,j_\nu \tag{9.12}$$

In the Lorenz gauge, $\partial^\mu A_\mu = 0$, Maxwell equations reduce to 4 *decoupled* wave equations

$$\Box A_\nu = \partial^\mu{}_\mu A_\nu = -\frac{4\pi}{c}\,j_\nu \tag{9.13}$$

The equations are coupled through the Lorenz gauge condition. If the current j^μ is not conserved then the derivation is inconsistent with the Lorentz gauge. Conversely, if current is conserved, then Lorentz gauge condition follows for all times provided the initial data for A_μ and $\dot{A}_\mu$ satisfy it.

Exercise 9.1. *Show that if one imposes the Lorentz gague condition as initial data for the wave equation, then the Lorenz gauge condition holds for all times provided current is conserved.*

9.3 Lienard-Wiechert: Retarded potentials

Our aim here is to compute the potential A_μ generated by a point charge moving with a given orbit. The Maxwell equations, Eqs. (9.13), can be viewed as 4 independent copies of the scalar wave equation with given source terms. We can therefore transcribe the solution from the previous section to this case. The source is now

$$j_\nu(y) = e\delta^{(3)}\big(\mathbf{y}-\mathbf{z}(y^0)\big)\underbrace{v_\nu(y^0)}_{velocity} = e\frac{\delta^{(3)}\big(\mathbf{y}-\mathbf{z}(y^0)\big)}{\gamma}\underbrace{u_\nu(y^0)}_{4-velocity} \tag{9.14}$$

Comparing with Eq. (10.6) for scalar waves we see that the retarded potentials are:

$$A_\nu(x) = e\frac{u_\nu}{|R\cdot u|} = -e\frac{u_\nu(y^0)}{R\cdot u(y^0)} \tag{9.15}$$

where removed the absolute value by taking into account that R is forward *light-like* and u forward *time-like* so $R\cdot u < 0$.

The result admits the following interpretation: The vector potential, being a 4-vector, must be of the form

$$(scalar)(vector)_\mu$$

We have (at least) two 4-vectors at our disposal: R and u. Between these we can form 3 scalars: Two uninteresting $u\cdot u = -c^2$ and $R\cdot R = 0$and one interesting $R\cdot u$. This, plus dimension analysis and the limit case of a charge at rest determines Eq. (9.1).

The result can also be viewed as the covariant form of Coulomb law:

$$A_\nu(x) = \frac{e}{|\mathbf{x}|}(1,0,0,0) = -\frac{e}{c|\mathbf{x}|}\underbrace{(-c,0,0,0)}_{u_\nu} \Longrightarrow -e\,\frac{u_\nu}{R\cdot u}$$

9.3.1 The Lorenz Gauge condition

We still need to verify the Lorenz gauge condition.

A clever argument: Since the condition is Lorentz invariant, it is sufficient to verify it in some Lorentz frame. So let us do that in the frame where the charge is instantaneously at rest at the early time. Then,

$$\partial^\mu A_\mu = \partial^0 A_0$$

with

$$A_0 = -\frac{e}{|\mathbf{x}|}$$

when you change t of the event x the distance $\mathbf{x}$ does not change because the particle is at rest. Hence

$$\partial^0 A_0 = 0$$

A sneaky argument: Suppose in the distant past the source was a rest. The coulomb solution then satisfies the Lorenz gauge in the past. By conservation of charge, the Lorenz gauge is preserved by the evolution.

An honest Computation: The reason for doing also an honest computation is that this will force us to derive the identity that describes how the retarded time depends upon variation of the observation event x:

$$\partial_\mu \tau = \frac{R_\mu}{R \cdot u} \tag{9.16}$$

This follows by differentiating $R \cdot R = 0$

$$0 = \tfrac{1}{2}\partial_\mu (R \cdot R) = R_\alpha \, \partial_\mu (x^\alpha - z^\alpha) = R_\mu - R \cdot u \, (\partial_\mu \tau)$$

Back to the honest verification of the Lorenz gauge condition:

$$0 = \partial_\mu A^\mu = -e \partial_\mu \left(\frac{u^\mu}{R \cdot u} \right)$$

This will hold provided

$$(R \cdot u)^2 \partial_\mu \left(\frac{u^\mu}{R \cdot u} \right) = (R \cdot u) \, \partial_\mu u^\mu - u^\mu \partial_\mu (R \cdot u) \overbrace{= 0}^{?} \tag{9.17}$$

To verify that this is indeed so, let us prepare

$$\partial_\mu u^\alpha = \dot{u}^\alpha (\partial_\mu \tau), \quad \partial_\mu R^\alpha = \delta^\alpha_\mu - u^\alpha (\partial_\mu \tau)$$

Substituting this in Eq. (9.17) and using Eq. (9.16) we find

$$\begin{aligned}
(R \cdot u) \, \partial_\mu u^\mu - u^\mu \partial_\mu (R \cdot u) &= (R \cdot u) \, \dot{u}^\mu (\partial_\mu \tau) - u^\mu u_\alpha \partial_\mu R^\alpha - u^\mu R_\alpha \dot{u}^\alpha (\partial_\mu \tau) \\
&= (R \cdot u) \, \frac{\dot{u} \cdot R}{R \cdot u} - u^\mu u_\alpha \partial_\mu R^\alpha - u^\mu R_\alpha \dot{u}^\alpha \frac{R_\mu}{R \cdot u} \\
&= -u^\mu u_\alpha \partial_\mu R^\alpha \\
&= -u \cdot u + u^\mu \partial_\mu \tau \\
&= -u \cdot u + u \dot{} u = 0
\end{aligned}$$

We have verified that the solution indeed satisfies the Lorenz condition.

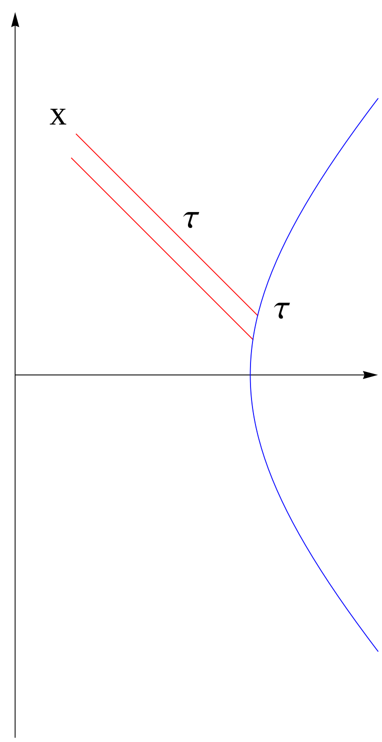

Figure 9.2: The self time τ parametrizes the blue orbit. It can be extended to a function on space time by pushing the value of τ to the forward light-cone. The figure illustrates how τ changes when the point of observation x changes. The red lines are light-like.

9.4 Retarded Fields

In the Lorenz gauge the potentials we found are causal. This implies the causality of the fields. We want now to derive direct formulas for the fields.

To find the fields we need to differentiate the potentials with respect to the space-time coordinates x^μ. Formally,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_{[\mu} A_{\nu]} \tag{9.18}$$

and the right hand side is a convenient notation. The word formal above refers to the fact that in taking the partial derivatives we need to remember that τ, the retarded time, is a function of the point of observation x. If we want to treat x and τ as independent variables, ∂_μ needs to be interpreted as

$$\partial_\mu \Longrightarrow \frac{\partial}{\partial x^\mu} + \left(\frac{\partial \tau}{\partial x^\mu}\right)\frac{\partial}{\partial \tau} = \frac{\partial}{\partial x^\mu} + \left(\frac{R_\mu}{R \cdot u}\right)\frac{\partial}{\partial \tau} \tag{9.19}$$

The x^μ differentiation sees only the first term in R, the location of the observer, while the τ differentiation only sees the second term, the location of the charge.

Using the explicit form of the potential:

$$-\,\partial_\mu A_\nu = e\,\partial_\mu\left(\frac{u_\nu}{R\cdot u}\right) = -e\,\frac{u_\mu u_\nu}{(R\cdot u)^2} + e\left(\frac{R_\mu}{R\cdot u}\right)\partial_\tau\left(\frac{u_\nu}{R\cdot u}\right) \tag{9.20}$$

Because F is anti-symmetric, the first term in Eq. (9.20) drops upon anti-symmetrization and only the second term contributes

The field F depend on the location, velocity u and the acceleration $\dot u$ of the charge at the early time. It does not depend on any higher derivatives, e.g. the jerk $\ddot u$. Now compute:

$$\begin{aligned}\partial_\tau\left(\frac{u_\nu}{R\cdot u}\right) &= \frac{\dot u_\nu}{R\cdot u} - \frac{u_\nu}{(R\cdot u)^2}\,\partial_\tau(R\cdot u) \\ &= \frac{\dot u_\nu}{R\cdot u} + \frac{u_\nu}{(R\cdot u)^2}\,u\cdot u - \frac{u_\nu}{(R\cdot u)^2}\,R\cdot\dot u \\ &= \frac{\dot u_\nu}{R\cdot u} - c^2\,\frac{u_\nu}{(R\cdot u)^2} - \frac{u_\nu}{(R\cdot u)^2}\,R\cdot\dot u\end{aligned} \tag{9.21}$$

Consequently

$$\left(\frac{R_\mu}{R\cdot u}\right)\partial_\tau\left(\frac{u_\nu}{R\cdot u}\right) = \frac{R_\mu \dot u_\nu}{(R\cdot u)^2} - c^2\,\frac{R_\mu u_\nu}{(R\cdot u)^3} - \frac{R_\mu u_\nu}{(R\cdot u)^3}\,R\cdot\dot u \tag{9.22}$$

We get F by anti-symmetrizing:

$$F_{\mu\nu} = -e\underbrace{\left(\frac{R_{[\mu}\dot u_{\nu]}}{(R\cdot u)^2} - \frac{R_{[\mu}u_{\nu]}}{(R\cdot u)^3}\,R\cdot\dot u\right)}_{radiation} + e\underbrace{c^2\,\frac{R_{[\mu}u_{\nu]}}{(R\cdot u)^3}}_{\text{"}Coulomb\text{"}} \tag{9.23}$$

Since u is normalized to c the last term, is order $O(c^0)$. It decays with distance like R^{-2}. This is, essentially, the Coulomb term. The first two terms are proportional to the acceleration and so formally of order $O(c^{-2})$. They decay more slowly at large distance, like R^{-1}. These are the radiating terms.

9.4.1 Interpretation

The formula is complicated and at first also opaque. It may be useful to view it from general principles.

We have three vectors in the problem: R, the light-like vector connecting the point of observation and the source, u the particle 4-velocity and $\dot u$ its 4-acceleration. From these we can make three interesting scalars

$$R\cdot u,\quad R\cdot\dot u,\quad \dot u\cdot\dot u$$

The remaining scalars are not interesting

$$R\cdot R = 0,\quad u\cdot u = -c^2,\quad u\cdot\dot u = 0 \tag{9.24}$$

F must be linear in the acceleration. This is because the potential did not depend on the acceleration at all. As a consequence, that the scalar $\dot{u}\cdot\dot{u}$ should not appear and the scalar $R\cdot\dot{u}$ can only appear in the numerator. Given this we can reconstruct all the three terms in F, up to sign, just by the fact that F is a tensor and dimension analysis. From the tensorial properties of F it must be of the form

$$(tensor)_{\mu\nu} = (scalar)\,(vector)_\mu (vector)_\nu$$

Since F has dimension of $[charge][length^{-2}]$ and u has the dimension of c, one possible term is

$$ec^2 \frac{R_{[\mu}\, u_{\nu]}}{(R\cdot u)^3}$$

which gives the last term in Eq. (10.5). You can even get the numerical factor (and the sign) by looking at the limiting case of the Coulomb field of a particle at rest where $u_\mu = (-c, 0, 0, 0)$.

Looking at the terms proportional to the acceleration $\dot{u}$ one possibility is

$$e\frac{R_{[\mu}\, \dot{u}_{\nu]}}{(R\cdot u)^2}$$

which gives the first term up to sign and numerical factors. The middle term is obtained similarly.

9.5 Particle instantaneously at rest

The formula for F simplifies for a particle instantaneously at rest at the origin (at the early time), i.e. $y^0 = 0$: $u^\mu = (c, 0, 0, 0)$ and $R\cdot u = -c|\mathbf{x}|$. This determines F on the forward light-cone $(|\mathbf{x}|, \mathbf{x})$.

9.5.1 The Magnetic field:

Consider first the magnetic field. Since the spatial components of u vanish, we have (in Cartesian coordinates)

$$\begin{aligned}
F_{ij}(|\mathbf{x}|,\mathbf{x}) &= \varepsilon_{ijk}B^k(|\mathbf{x}|,\mathbf{x}) \\
&= -e\left(\frac{R_{[i}\dot{u}_{j]}}{(R\cdot u)^2} - \frac{\overbrace{R_{[i}u_{j]}}^{=0}}{(R\cdot u)^3}\,R\cdot\dot{u}\right) + ec^2\frac{\overbrace{R_{[i}u_{j]}}^{=0}}{(R\cdot u)^3} \\
&= -\frac{e}{c^2}\frac{R_{[i}\dot{u}_{j]}}{|\mathbf{x}|^2} \\
&= \frac{e}{c^2}\frac{(\mathbf{a}(0)\times\mathbf{x})_k}{|\mathbf{x}|^2} \qquad (9.25)
\end{aligned}$$

where $\mathbf{a}$, the 3-vector of acceleration $\dot{u}_\mu = (0, \mathbf{a})$, is orthogonal to $u = (c, 0)$.

It follows that, 3 vector of magnetic field on the light-cone emanating from teh origin, $(|\mathbf{x}|, \mathbf{x})$ is

$$\mathbf{B}(|\mathbf{x}|, \mathbf{x}) = \frac{e}{c^2} \frac{\mathbf{a}(0) \times \mathbf{x}}{|\mathbf{x}|^2} \tag{9.26}$$

The main conclusions we draw from this are

1. The field decays like the inverse distance
2. The field is perpendicular to both the line of sight and the acceleration vector: there is no radiation in the direction of acceleration.
3. You can use the homogeneity of space-time to write a slightly more complicated formula for the case where the particle is at rest at an arbitrary point.

9.5.2 The electric field

Recall that $F_{0j} = -E_j$. For a particle at rest at the origin at lab time $y^0 = 0$ $R \cdot u = -c|\mathbf{x}|$ and $R_0 = -|\mathbf{x}|$. Hence, on the light-cone emanating from the origin

$$\begin{aligned} E_j = -F_{0j} &= e \left(\frac{R_{[0}\dot{u}_{j]}}{(R \cdot u)^2} - \frac{R_{[0}u_{j]}}{(R \cdot u)^3} R \cdot \dot{u} \right) - ec^2 \frac{R_{[0}u_{j]}}{(R \cdot u)^3} \\ &= \frac{e}{c^2|\mathbf{x}|^3} \left(|\mathbf{x}|\, R_{[0}\dot{u}_{j]} - \frac{1}{c} R_{[0}u_{j]}\, R \cdot \dot{u} \right) + \frac{e}{c|\mathbf{x}|^3} R_{[0}u_{j]} \\ &= \frac{e}{c^2|\mathbf{x}|^3} \left(-(\mathbf{x} \cdot \mathbf{x})\, \mathbf{a}_j + (\mathbf{x} \cdot \mathbf{a})\, \mathbf{x}_j \right) + \frac{e}{r^2} \mathbf{x}_j \end{aligned}$$

Since

$$(\mathbf{x} \cdot \mathbf{x})\mathbf{a} - (\mathbf{x} \cdot \mathbf{a})\, \mathbf{x} = -\big(\mathbf{x} \times (\mathbf{x} \times \mathbf{a})\big)$$

We can collect the above to a vector identity

$$\begin{aligned} \mathbf{E}(|\mathbf{x}|, \mathbf{x}) &= \frac{e}{c^2|\mathbf{x}|} \big(\hat{\mathbf{x}} \times (\hat{\mathbf{x}} \times \mathbf{a})\big) + \frac{e}{|\mathbf{x}|^2} \hat{\mathbf{x}} \\ &= \mathbf{B}(|\mathbf{x}|, \mathbf{x}) \times \hat{\mathbf{x}} + \frac{e}{|\mathbf{x}|^2} \hat{\mathbf{x}} \end{aligned} \tag{9.27}$$

The longitudinal part is Coulomb and the transversal part is radiation. **E** and **B** are mutually orthogonal. The two Lorentz scalars are

$$\mathbf{E} \cdot \mathbf{B} = 0, \quad \mathbf{E}^2 - \mathbf{B}^2 = -\frac{e^2}{|\mathbf{x}|^4}$$

Exercise 9.2. *Show that the Poynting vector is*

$$\frac{c}{4\pi} \mathbf{E} \times \mathbf{B} = \frac{e^2}{4\pi c^3 |\mathbf{x}|^2}\, \hat{\mathbf{x}}$$

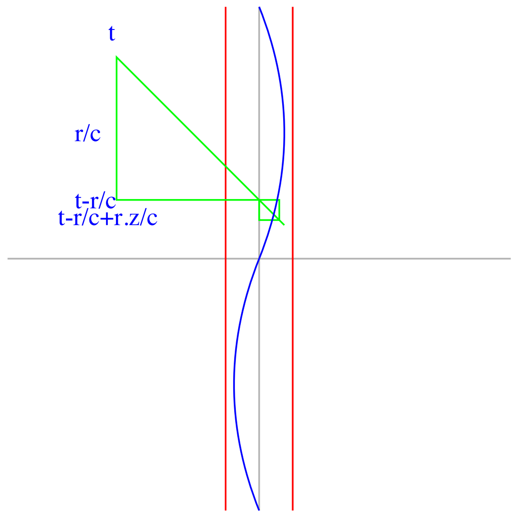

Figure 9.3: A space time diagram showing the geometric meaning of the successive approximations leading to Eq. (9.36). The world-line of the moving charge is the blue path. It is confined to a narrow corridor between the two red lines a distance 2ℓ apart. The retardation time is the intersection of the back light cone (green line) with the blue orbit. The first approximation $t - r/c$ is the intersection of the back light cone with the origin (gray vertical line). The next correction in Eq. (9.36) is the corner of the small green square.

9.5.3 Slow particles

We found the electric and magnetic field of a particle at rest at the origin. If the particle is at the origin but not at rest, we can get the fields by making a Lorentz transformation. In particular, if the particle is slow the field is

$$\mathbf{B}(x) = \frac{e}{c^2}\frac{\mathbf{a}(x^0 - |\mathbf{x}|) \times \mathbf{x}}{|\mathbf{x}|^2}\left(1 + O\left(\frac{v}{c}\right)\right) \tag{9.28}$$

(We have use the homogeneity of Minkowski space time to shift $y^0 = 0$ to any other time.) and similarly for $\mathbf{E}$.

9.6 Retardation from a distant source

To compute the $F_{\mu\nu}$ we need to compute the retardation: $x^0 - y^0$. This time is determined as the solution of the equation

$$x^0 - y^0 = |\mathbf{x} - \mathbf{z}(y^0)|$$

where $\mathbf{z}(y^0)$ is the obit of the charge. What we seek is an explicit, approximate, formula for $y^0(x)$.

In general, there is little one can say for arbitrary x^μ and general orbits $\mathbf{z}(y^0)$. In practice, one often is interested in the field far from the source: The light from a star; from radiating atom. In these cases the observer is far from the source:$|\mathbf{x}| \gg |\mathbf{z}|$. Assume the orbit $\mathbf{z}$ is confined to a ball of radius $\ell \ll |\mathbf{x}|$. Under this condition we can write

$$|\mathbf{x}-\mathbf{z}|^2 = |\mathbf{x}|^2 + |\mathbf{z}|^2 - 2\mathbf{x}\cdot\mathbf{z} = |\mathbf{x}|^2\left(1 - 2\frac{\mathbf{x}\cdot\mathbf{z}}{|\mathbf{x}|^2} + O\left(\frac{\ell}{|\mathbf{x}|}\right)^2\right)$$

Hence the implicit equation for the retardation for a distant source reduces to:

$$\underbrace{x^0 - y^0}_{large\ O(x)} = \underbrace{|\mathbf{x}|}_{O(x)} - \underbrace{\hat{\mathbf{x}}\cdot\mathbf{z}(y^0)}_{O(\ell)} + \underbrace{O\left(\frac{\ell^2}{|\mathbf{x}|}\right)}_{negligible} \tag{9.29}$$

Although this is still an implicit equation, which is in general, as difficult as the original one, it is good starting point for getting explicit approximate solution when one has additional dimensionless parameters.

How accurately do we actually need to compute y^0? The answer depends on the characteristic frequency of the source. The accuracy we need is such that it allows to locate the source to better than the characteristic wave length. To see this consider an oscillating charge

$$\mathbf{z}(y^0) = \ell\,\hat{\mathbf{n}}\cos(ky^0)$$

So, if

$$ky^0 = \underbrace{123456789}_{irrelevant}.{\color{blue}987}654321 \times (2\pi)$$

and we want an accuracy of 1%, then the all we care are the three digits in blue, just after the decimal point. In other words, cos is a periodic function, we need only the fractional part of ky^0. In particular we need to compute the fractional part of the retardation. We can now write Eq. (9.29) in dimensionless form

$$\underbrace{k(x^0 - z^0)}_{\gg 1} = \underbrace{k|\mathbf{x}|}_{\gg 1} - \underbrace{k\hat{\mathbf{x}}\cdot\mathbf{z}(z^0)}_{O(k\ell)} + \underbrace{O\left(\frac{k\ell^2}{|\mathbf{x}|}\right)}_{negligible} \tag{9.30}$$

Whether the term $O(k\ell)$ can be neglected or not does not depend anymore on how far the source is but rather on how big $k\ell$ is.

9.6.1 The dipole approximation: $k\ell \ll 1$

The dipole, approximation is concerned with the case $k\ell \ll 1$: the wave length of the emitted radiation is much larger than the size of the source. This is, for example, the case for the light emitted by a single atom where the characteristic wave length is thousands of times larger than the atom. In this case the adequate approximation it to keep only the big terms in Eq. (9.30) to get the an explicit, approximate, expression for the early-time y^0:

$$y^0(x) = x^0 - |\mathbf{x}| \tag{9.31}$$

The dipole approximation describes small antennas.

Harmonic motion

As an application consider a charge undergoing non-relativistic harmonic motion with acceleration

$$\mathbf{a}(y^0) = \mathbf{a}_0\, e^{iky^0} \tag{9.32}$$

$\mathbf{a}$ is a vector with complex amplitudes. Thus, for example $\mathbf{a}_o = \ell(1, i, 0)/\sqrt{2}$ describes circular motion with radius ℓ in the plane. Non-relativistic means $\omega\ell \ll c \Leftrightarrow k\ell \ll 1$ which is the condition for the dipole approximation to apply. This sets

$$ky^0 = k(x^0 - r), \quad r = |\mathbf{x}|$$

Using Eq. (9.28) for the magnetic field of a charge moving non-relativistically we get

$$\mathbf{B}(x) = \left(\frac{e\ \mathbf{a}_0 \times \hat{\mathbf{x}}}{c^2}\right) \underbrace{\frac{e^{ik(x^0-r)}}{r}}_{spehrical\ wave} \tag{9.33}$$

From this point of view, an outgoing spherical wave is a consequence of retardation.

Many particles

The radiation fields from many particles with prescribed orbits is, by linearity of the Maxwell equation, the sum of the radiation of the individuals ones. In general, each particle will have its own retarded time, and the formulas remain implicit. A simplification occurs in the dipole approximation for “ small antenna” where all the charges share the same retardation.

The dipole moment of a large collection of charges is:

$$\mathbf{d}(t) = \sum e_j \mathbf{z}_j\,(t) \tag{9.34}$$

and we assume that all the orbits $\mathbf{z}_j$ are such that the dipole approximation applies. In this case all the charges have the same retardation and the magnetic field is simple

$$\mathbf{B}(\mathbf{r}, t) = \frac{\ddot{\mathbf{d}}(t - r/c) \times \hat{\mathbf{x}}}{c^2|\mathbf{x}|} \tag{9.35}$$

9.6.2 Beyond dipole: Slow motion

The dipole approximation applies provided $k\ell \ll 1$. In the case that the charges move slowly, one can go beyond the dipole approximation and get explicit approximation for the retardation that apply when $k\ell = O(1)$ This is often practically important: Your cell phone works at $\omega \approx 2\ [GHz]$ associated with wavelength $15\ [cm]$ which is comparable to the cell phone antenna. When $k\ell = O(1)$ one needs to take the $O(k\ell)$ term in Eq. (9.30) into account. In general, this gives, once again an intractable implicit equation which depends on the details of the motion $\mathbf{z}(y^0)$. We can get an explicit simple formula in the special case that the charge moves slowly $|\dot{\mathbf{z}}| = v \ll c$. This is illustrated in the Fig. 9.3. Since the world-line is almost vertical, we may solve Eq. (9.30) iteratively,

$$y^0(x) = x^0 - |\mathbf{x}| + \hat{\mathbf{x}} \cdot \mathbf{z}(x^0 - |\mathbf{x}|) \tag{9.36}$$

(Eq. (9.31) has been substituted in Eq. 9.29). It should be clear that the error we make is proportional to $v/c \ll 1$.

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9.7 Power

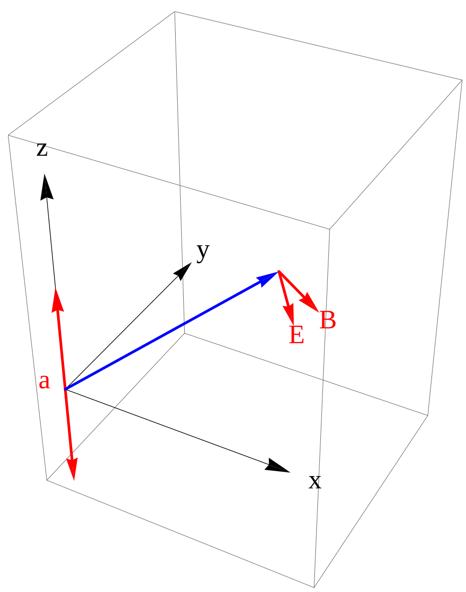


Figure 9.4: Dipole oriented along the z-axis and the associated fields. ϕ is the angle between the z-axis and the blue arrrow.

The power emitted by dipole can be computed from Poynting

$$\mathbf{P} = c\frac{\mathbf{E} \times \mathbf{B}}{4\pi} \tag{9.37}$$

Both **E** and **B** lie in the plane perpendicular to the line of sight so P is parallel to $\hat{r}$. Suppose that $\mathbf{a} = a\hat{\mathbf{z}}$. The magnitude of P in the direction ϕ relative to the z-axis is

$$P(\phi) = c\frac{E^2}{4\pi} = \frac{e^2a^2}{4\pi c^3}\frac{\sin^2\phi}{r^2} \tag{9.38}$$

The power through a spherical shell of radius r is then

$$P_T = 2\pi r^2 \int d\phi\, \sin\phi\, P(\phi) \tag{9.39}$$

Evidently

$$\int d\phi\, \sin\phi\, \sin^2\phi = -\int d(\cos\phi)\,(1-\cos^2\phi) = 2\left(1-\tfrac{1}{3}\right) = \frac{4}{3} \tag{9.40}$$

Hence, the total power

$$P = \frac{2}{3}\frac{e^2\mathbf{a}^2}{c^3} = \frac{2}{3}\frac{\ddot{\mathbf{d}}^2}{c^3} \tag{9.41}$$

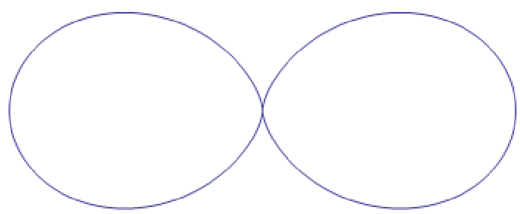


Figure 9.5: Polar plot of the power radiated by a dipole antenna as function of the angle. The maximal power is radiated in the plane perpendicular to the dipole.

A dipole antenna is not isotropic: It does not radiate at all in the directions of the dipole.

Remark 9.3. *You can not make an isotropic antenna. No matter how complicated an antenna you make the Poynting vector must vanish it at east two directions. This is a consequence of topology: The vector B is tangent to the sphere. It is a fact that every vector field tangent to the sphere must vanish at two points, at least. This is sometimes expressed as* you can not comb a tennis ball. *Hence P must vanish at two points at least.*

9.8 Classical instability of atoms

I now want to explain a puzzle in classical electrodynamics that turned out to be a window that opened the way to quantum mechanics. In classical physics atoms are unstable, and should collapse in no time by radiation.

Consider a charge e in Keplerian orbit around a nucleus of charge e in a circular orbit. The energy (non-relativistic) of the system is

$$E = -\frac{1}{2}\frac{e^2}{|\mathbf{x}|}$$

while the acceleration is

$$a = \frac{e^2}{m|\mathbf{x}|^2}$$

The rate of loss of energy by radiation is

$$-\dot{E} = \frac{2}{3}\frac{e^2}{c^3}a^2$$

We can now eliminate a and obtain a differential equation for the energy

$$\dot{E} = -kE^4, \quad k = \frac{2^5}{3(me)^2c^3}$$

Integrating the differential equation we find a blow up at finite time:

$$E(t) = E_0\,(1-\gamma t)^{-1/3}, \quad \gamma = -3E_0^3 k > 0$$

In other words, the charge would collapse on the nucleus in finite time $1/\gamma$.

The ground state energy of hydrogen-like atom is,

$$2E_0 = -mc^2\left(\frac{e^2}{\hbar c}\right)^2 = -mc^2\alpha^2$$

and its period is $2\pi\hbar/E_0$. Therefore, the decay time, counted in periods, is

$$\frac{E_0}{2\pi\hbar\gamma} = \frac{1}{128\pi}\times\alpha^{-3} = 6400$$

This means that the electron in hydrogen would fall on the on the proton in 2×10^{-12} seconds. The classical world is unprotected against collapse to the nucleus.

The apparent instability of the atoms in classical physics is one of the reasons for quantum mechanics.

9.9 Orienting an antenna

9.10 Reciprocity

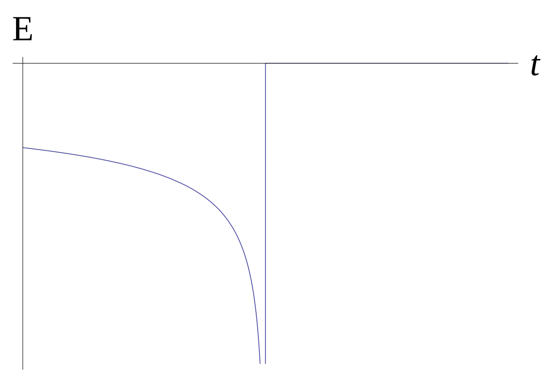


Figure 9.6: Blowup at finite time: The energy of a charged particle encircling the nucleus goes to $-\infty$ in finite time.

Chapter 10

Radiation reaction

Electrodynamics is a practical theory that proved itself in huge number of application. It is a consistent theory when the sources are continuous distributions. However, it is not a self-consistent fundamental theory of *point particles*.

10.1 Interacting systems

In studying interacting systems one needs to solve Newton equations for the motion of charged particles in a given electromagnetic field and simultaneously solve Maxwell equations for a given motion of the sources. Newton equations are non-linear and Maxwell equations are partial differential equations. One therefore ends up with a set of non-linear partial differential equation. This is a hard problem with few general methods other than simulations.

10.1.1 Non-relativistic interacting particles

The theory of interacting point charges simplifies in the limit of slowly moving point charges. This limit is perfectly consistent and practically useful. Let us discuss this first. Slow particles interact weakly with magnetic fields, and the forces is dominated by Coulomb forces:

$$m\dot{u}_\mu + \frac{e}{c}F_{\mu\nu}u^\nu = 0$$

If the particles move sufficiently slowly, $u^\mu \approx -c\delta^\mu_0$, and the charges are affected mostly by the electric field given by

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\dot{\mathbf{A}}$$

In the Coulomb gauge, the scalar potential Φ is the instantaneous solution of Poisson equation while $\mathbf{A}$, (with $\nabla\cdot\mathbf{A} = 0$), solves the wave equation with a source term

$$-\Delta\mathbf{A} - \frac{1}{c^2}\ddot{\mathbf{A}} = \underbrace{\frac{4\pi}{c}\mathbf{J} - \frac{1}{c}\nabla\dot{\Phi}}_{negligible}$$

which is small when the particles move slowly. The problem of interacting charges is then encapsulated by the Hamiltonian

$$H = \sum \frac{1}{2m_j}\mathbf{v}_j^2 + \frac{1}{2}\sum \frac{e_j e_k}{|\mathbf{x_j} - \mathbf{x_k}|}$$

The electromagnetic field has been integrated out of the problem. This is the starting point of musch of atomic physics.

10.1.2 Plasma and Magnetohydrodynamics

In plasma physics and magnetohydrodynamics Newton equations take the form of the partial differential equations similar to those of fluid mechanics[1]. Combined with Maxwell equations, one gets a theory of coupled, non-linear partial differential equations. The theory is technically hard, but conceptually is fine.

As a consequence there are relatively simple phenomena in plasma physics, that are only poorly understoon. For example the current understanding of the spontaneous generation of the magnetic field of the earth–driven by the earth rotation–is largely based on computer simulations.

10.1.3 Infinities

Electrodynamics of point-like charges is not a fully consistent theory. The source of trouble can already be seen by looking at the the field energy of a point charge:

$$\frac{1}{8\pi}\int \frac{e^2}{|\mathbf{x}|^4} d\mathbf{x} = \infty$$

The divergence comes from the singularity at $\mathbf{x} = 0$. (The integral at infinity is perfectly convergent in 3 spatial dimensions). Modern physics tinkers with the dimension of space time, and allows for elementary objects that are strings and branes, these affect the field energy. It may be that this divergence is the window for a future (string) theory free from such divergences.

10.2 Radiation reaction

Consider a charged particle moving non-relativistically in a circle due to the action of a central force. We allow for non-electromagnetic forces that make sure that the particle moves in a stationary orbit. The accelerating charge radiates and we want to compute the force that acts back on the charge because of this process. We have a formula for the power radiated by the charge and we want to convert the information in the power to information about the force.

For circular the acceleration is perpendicular to the velocity and we have

$$\mathbf{a}^2 = (\mathbf{a} \dot{\cdot} \mathbf{v}) - \dot{\mathbf{a}} \cdot \mathbf{v} = -\dot{\mathbf{a}} \cdot \mathbf{v}$$

[1] Thermodynamics needs to be thrown in to close the equations.

The radiated power can now be written as

$$P = \frac{2}{3}\frac{e^2}{c^3}\,\mathbf{a}^2 = -\frac{2}{3}\frac{e^2}{c^3}\,\dot{\mathbf{a}}\cdot\mathbf{v} = -\mathbf{F}_{AL}\cdot\mathbf{v}$$

If the particle is moving at constant speed, an opposite (non-electromagnetic) force must be applied to feed back the radiated energy. This means that the radiation acts back on the charge and produces a force

$$\mathbf{F}_{AL} = \frac{2}{3}\frac{e^2}{c^3}\,\dot{\mathbf{a}}$$

The jerk $\dot{\mathbf{a}}$ is opposite to the velocity, Fig 10.1. The force acts like a friction force. $-\mathbf{F}_{AL}$ is then the non-electromagnetic force that we need to apply to keep the orbit stationary. It is known as the Abraham-Lorentz force. It is a somewhat unusual force, in two ways. Unlike the usual friction it is not proportional to the velocity but rather to the jerk. Second, it is ultimately a force that a particle applies on itself. Something Newton would not approve of.

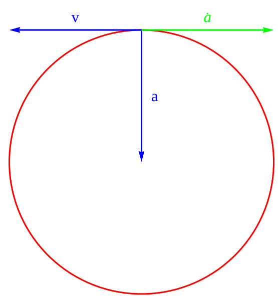


Figure 10.1: For a cicrular motion every derivative is a rotation by $\pi/2$

Exercise 10.1 (Covariant form of Abraham Lorentz force)**.** *Using the fact that Newton law*

$$ma_\mu = f_\mu$$

is consistent with $u\cdot u = -c^2$ *provided* $f\cdot u = 0$ *and* $a\cdot a + u\cdot\dot{a} = 0$*, show that the covariant form of Abaraham-Lorentz force is*

$$ma_\mu = \frac{2e^2}{3c^3}\left(\dot{a}_\mu - \frac{a\cdot a}{c^2}u_\mu\right)$$

10.2.1 When is radiation reaction important?

In MKS units

$$\underbrace{\frac{e^2}{c^3}}_{Gaussian} \iff \underbrace{\frac{e^2}{4\pi\varepsilon_0 c^3}}_{MKS} = 8\times 10^{-54}\ [Kg\ s]$$

a small number. You'd need huge jerk to get tiny forces.

If ω is the characteristic frequency of the problem, the radiation reaction force becomes comparable to the inertia at frequencies:

$$\frac{mc^3}{e^2} \gg \omega$$

If you take for m the mass and charge of the electron, then the left hand side is of the order of 10^{23} $[Hz]$. For most practical purposes, radiation reaction is negligible.

10.2.2 Friction

Small forces can still do something if they act for long time. Consider the equation of motion in a force field f with radiation reaction like friction:

$$m\mathbf{a} = \mathbf{f}(\mathbf{x}) + k\dot{\mathbf{a}}, \quad k = \frac{2}{3}\frac{e^2}{c^3} \tag{10.1}$$

From a conceptual point of view, this is a major modification of Newton law, since the order of the equation changed: It is not enough to specify initial conditions in the form of position and velocity, one also needs to specify the initial acceleration. This is contrary to much evidence we have.

However, since k is so small, a reasonable strategy for interpreting the equation is look at it iteratively. Namely, replace the equation by

$$m\mathbf{a} = \mathbf{f}(\mathbf{x}) + \frac{k}{m}\dot{\mathbf{f}} = \mathbf{f}(\mathbf{x}) + \frac{k}{m}(\mathbf{v}\cdot\nabla)\mathbf{f}$$

In this form the equation of motion is still second order and the radiation reaction looks like a friction term.

As an example consider the harmonic oscillator where $f(x) = -\kappa x$. The equation of motion is now

$$ma = -\kappa x - \frac{\kappa k}{m} v$$

which is solved by $x(t) = x(0)e^{\pm i\omega_\pm t}$ where ω is a solution of the quadratic equation

$$\omega_\pm^2 - \frac{\kappa}{m} \mp i\omega_\pm \frac{\kappa k}{m^2} = 0$$

Since k is very small, this is solved by

$$\omega_\pm = \sqrt{\frac{\kappa}{m}}\left(1 \pm i\, k\frac{\kappa}{m}\right)$$

This describes reasonable solutions of slowly decaying oscillations.

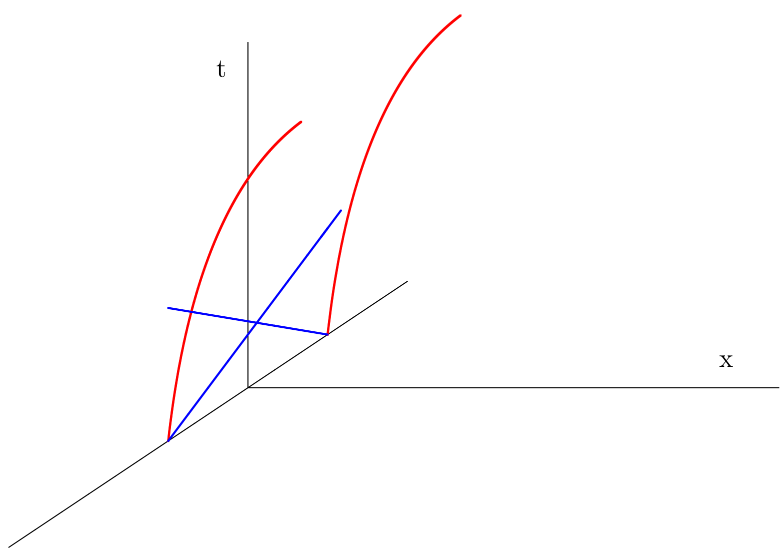
Exercise 10.2. *Write the equations of motion for the Kepler problem with radiation reaction friction term.*

10.2.3 The Dumbbell

A more careful derivation of the radiation reaction force starts with the forces an ex extended object of size ε. In this cased different parts of the object apply mutual forces that add up to the radiation reaction force. In the limit $\varepsilon \to 0$ one recovers Abaraham Lorentz with an extra bonus: The mass is related to the energy of the field.

We shall re-derive the radiation reaction force thinking of a point charge as the $\varepsilon \to 0$ of an object of size ε.

We want to compute the forces that the dumbbell applies on itself when it moves in a prescribed way. The world-lines of the dumbbell are shown in the figure (red) and the light-cone is drawn blue. We shall show that Newton third law is violated, so there is a net force acting on an extended body.



The dumbbell is made of two point charges separated by a rod of length ε. The dumbbell moves along the x-axis and is aligned with the y-axis. So the world line of the two charges is

$$x_\pm(t) = (ct, q(t), \pm\varepsilon/2, 0)$$

The two charges communicate when $x_\pm(t+\tau) - x_\mp(t)$ is light like. The time delay $\tau = O(\varepsilon)$ since the dumbbell is small.

$R_\pm(\tau) = x_\pm(\tau) - x_\mp(0)$ is a light like vector. We choose a Lorentz frame so that the dumbbell is at rest at time zero, i.e. $q(0) = \dot{q}(0) = 0$ and so

$$R_\pm(\tau) = (c\tau, q(\tau), \pm\varepsilon, 0) \tag{10.2}$$

We want to find the mutual forces on the dumbbell at time τ and express them in terms of the acceleration and jerk *at time* τ, *not at time* 0. This gives a funny situation where we specify the position and velocity at time 0 but specify the acceleration at time τ. (We take the jerk to be constant.)

Let a and $\dot a$ be the acceleration and jerk at time τ. Clearly

$$a(\tau) = a = a(0) + \dot a\tau$$

From this it follows that

$$\dot q(\tau) = a(0)\tau + \tfrac{1}{2}\dot a\tau^2 = a\tau - \tfrac{1}{2}\dot a\tau^2 \tag{10.3}$$

Integrating that gives the position

$$q(\tau) = \tfrac{1}{2}a(0)\tau^2 + \tfrac{1}{6}\dot a\tau^3 = \tfrac{1}{2}a\tau^2 + \left(\tfrac{1}{6} - \tfrac{1}{2}\right)\dot a\tau^3 = \tfrac{1}{2}a\tau^2 - \tfrac{1}{3}\dot a\tau^3 \tag{10.4}$$

This fixes the function $q(\tau)$ in $R_\pm$.

τ and ε are related by the condition that R is light like:

$$(c\tau)^2 = \varepsilon^2 + q^2(\tau)$$

which is a polynomial equation for τ of order 6. However, as $q(\tau)$ is quadratic in τ to leading order

$$c\tau \approx \varepsilon$$

The retarded field at time τ is determined by the velocity and acceleration of the particle at time 0:

$$F_{\mu\nu} = -e\left(\frac{R_{[\mu}a_{\nu]}}{(R\cdot u)^2} - \frac{R_{[\mu}u_{\nu]}}{(R\cdot u)^3}\,R\cdot a - c^2\frac{R_{[\mu}u_{\nu]}}{(R\cdot u)^3}\right) \tag{10.5}$$

The four velocity is $u_\mu(0) = (-c, 0, 0, 0)$, and the four acceleration is $a_\mu(0) = \big(0, a(0), 0, 0\big)$. We therefore have

$$R\cdot u = -c^2\tau$$

The electric field in the direction of motion on one of the charges due to the other at time τ is then

$$-E_x = F_{01} = -e\left(\frac{R_0 a_1}{(c^2\tau)^2} + \underbrace{\frac{R_1 u_0}{(-c^2\tau)^3}\,R\cdot a}_{O(\tau)} + c^2\frac{R_1 u_0}{(-c^2\tau)^3}\right)$$

We are interested in the terms that do not vanish when $c\tau = \varepsilon \to 0$. Since $R_1 = q(\tau) = O(\tau^2)$ the middle terms in the formula above tends to zero with τ. We drop it. The remaining terms are

$$\begin{aligned} E_x = F_{10} &= \frac{e}{c^3}\left(-\frac{a(0)}{\tau} + \frac{q(\tau)}{\tau^3}\right) \\ &= \frac{e}{c^3}\left(-\frac{a - \dot a\tau}{\tau} + \frac{\frac{1}{2}a - \frac{1}{3}\dot a\tau}{\tau}\right) \\ &= -\frac{e}{c^3}\left(\frac{a}{2\tau} - \frac{2}{3}\,\dot a\right) \end{aligned} \tag{10.6}$$

and we dropped terms that scale to zero with τ

The retarded force that one charge applies on the other in the x direction is

$$F = eE = -\underbrace{\frac{e^2}{2\tau c^3}}_{e.m.\ mass} a + \frac{2e^2}{3c^3}\,\dot{a}$$

Disregarding sel-forces, we can now write Newton law assuming bare dumbbell mass m_b, in an external force F_{ex} and radiation self-force as

$$m_{ef}\, a = \left(m_b + \frac{e^2}{2\tau c^3}\right) a = \left(\frac{2e^2}{3c^3}\right)\,\dot{a} + F_{ex}$$

The effective mass gets a contribution from the electromagnetic interaction. This is very nice except that as we send $\tau \to 0$ we need to take the bare mass large and *negative*.

Remark 10.3. *One can get the self radiation reaction using the following trick Amos Ori taught me. Let $f(e)$ be the self force. Then, in the limit $\tau \to 0$ the total force on the dumbbell is*

$$F_t = 2f(e) + \frac{4\varepsilon^2}{3c^3}\,\dot{a} = f(2e) = 4f(e)$$

This says that the radiation reaction force is $f(e) = \frac{2e^2}{3c^3}\,\dot{a}$ as we have seen before.

When is it important? Small forces are important in two cases. The first is when they are all that there is and the second if they operate for very long time.

10.2.4 Conceptual difficulties

Radiation reaction is conceptually problematic for several reasons. First, it changes the order of Newton equations of motion from second order to third, see Eq. (10.1). This means that fixing the initial position and velocity is not sufficient to determine the orbit. This is in contrast with common experience.

Another problem is that the equations of motion , Eq. (10.1), admit non-physical solutions. For example, with $\mathbf{f} =$ the equation is

$$ma = k\dot{a}\,, \quad k = \frac{2e^2}{3c^3}$$

It admit the solution

$$a(t) = a(0)e^{kt/m} \Longrightarrow v(t) = v(0) + \frac{a(0)m}{k}\,e^{kt/m}$$

A particle, initially at rest, self accelerate to large velocities. This can only be avoided if you tune $a(0) = 0$ precisely. In practice, we only tune $x(0)$ and $v(0)$, so how come we do not see these self-accelerations? The theory of self-force for point particles, must be fundamentally flawed.