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On Positively Quadratically Hyponormal Weighted Shifts

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Abstract

Consider the sequence of positive weights $\alpha(x,y) : \sqrt{x}, \sqrt{y}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \ldots$ with a Bergman tail. If $y = \frac{2}{3}$ then it was shown in [2] that for $0 < x \leq y$, the weighted shift operator $W_{\alpha(x,y)}$ is positively quadratically hyponormal. In this paper we show that there exists an interval (k_1, k_2) about $\frac{2}{3}$ such that if $y \in (k_1, k_2)$ then for $0 < x \leq y$, $W_{\alpha(x,y)}$ is positively quadratically hyponormal. In fact, using Mathematica graphs we show that the largest such interval is $[k_1, k_2)$ where $k_1 = \frac{29}{46} \approx 0.630435$ and $k_2 = 0.737144$.

Mathematics Subject Classification: 47B37, 47B20

Keywords: quadratic hyponormality, positive quadratic hyponormality

1 Introduction

Let H be a separable infinite dimensional complex Hilbert space and $B(H)$ denote the algebra of bounded linear operators on H. For $A, B \in B(H)$, let $[A, B] := AB - BA$. We say that an n-tuple $T = (T_1, \ldots, T_n)$ of operators on H is hyponormal if the operator matrix $([T_j^*, T_i])_{i,j=1}^n$ is positive on the direct sum of n copies of H. For $k \geq 1$ and $T \in B(H)$, T is k-hyponormal if $(I, T, \ldots, T^k)$ is hyponormal. Again, T is weakly k-hyponormal if $p(T)$ is hyponormal for every polynomial p of degree $\leq k$. It can be shown easily that k-hyponormality of T implies weak k-hyponormality of T.

For $k=2$, weak 2 hyponormality, often referred to as quadratic hyponormality (for which we write q.h.) was first considered in detail by Curto in [2]. The definition of positively quadratically hyponormal operators (for which we write p.q.h.) was introduced by Curto-Fialkow in [3]. Subsequently, it has been shown that $2-hyponormality \Longrightarrow p.q.h. \Longrightarrow q.h.$ though the reverse is not always true.

In this paper we consider the weighted shift $W_{\alpha(x,y)}$ with a positive weight sequence $\alpha(x,y) : \sqrt{x}, \sqrt{y}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \ldots$ having a Bergman tail. We determine an interval (k_1,k_2) such that for $y \in (k_1,k_2)$ and $0<x\leq y$, $W_{\alpha(x,y)}$ is p.q.h. We also show that the largest such interval is $[k_1,k_2)$ where $k_1=\frac{29}{46}\approx 0.630435$ and $k_2=0.737144$

2 Preliminaries and notation

Let $\{e_n\}_{n=0}^{\infty}$ be the canonical orthonormal basis for $\ell^2(\mathbb{Z}_+)$ and let $\alpha : \{\alpha_n\}_{n=0}^{\infty}$ be a bounded sequence of positive numbers. Let W_α be a unilateral weighted shift defined by $W_\alpha e_n = \alpha_n e_{n+1}$ for $n\geq 0$. Then W_α is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}\,\forall\, n\geq 0$.
We recall here some terminologies and notations from [3]. Also we note that an operator T is quadratically hyponormal if $T+sT^2$ is hyponormal for every $s\in\mathbb{C}$.
Let W_α be a hyponormal weighted shift. For $s\in\mathbb{C}$, we let $D(s) := [(W_\alpha + sW_\alpha^2)^*, (W_\alpha+sW_\alpha^2)]$ and let P_n be projection onto $\bigvee_{i=0}^{n}\{e_i\}$ and $D_n := D_n(s) =$

$$P_n[(W_\alpha+sW_\alpha^2)^*,(W_\alpha+sW_\alpha^2)]P_n = \begin{pmatrix} q_0 & \bar{r}_0 & 0 & \cdot & \cdot & 0 & 0 \\ r_0 & q_1 & \bar{r}_1 & \cdot & \cdot & 0 & 0 \\ 0 & r_1 & q_2 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & q_{n-1} & \bar{r}_{n-1} \\ 0 & 0 & 0 & \cdot & \cdot & r_{n-1} & q_n \end{pmatrix},$$

where $q_k := u_k + |s|^2 v_k$, $r_k := s\sqrt{w_k}$, $u_k := \alpha_k^2 - \alpha_{k-1}^2$, $v_k := \alpha_k^2\alpha_{k+1}^2 - \alpha_{k-1}^2\alpha_{k-2}^2$, $w_k := \alpha_k^2(\alpha_{k+1}^2-\alpha_{k-1}^2)^2$ for $k\geq 0$ and $\alpha_{-1}=\alpha_{-2}:=0$.
Clearly, W_α is q.h. if and only if $D_n(s)\geq 0$ for every $s\in\mathbb{C}$ and every $n\geq 0$. Let $d_n(\cdot) := det(D_n(s))$. Then it follows form [3] that $d_0=q_0$, $d_1=q_0q_1-|r_0|^2$ and $d_{n+2}=q_{n+2}d_{n+1}-|r_{n+1}|^2 d_n$ for $n\geq 0$, and that d_n is actually a polynomial in $t:=|s|^2$ of degree $n+1$, with Maclaurin expansion $d_n(t) := \sum_{i=0}^{n+1} c(n,i)t^i$. This gives that for $n\geq 0$ and $1\leq i\leq n+1$,

1. $c(n,0) = u_0u_1 \dots u_n \geq 0$
2. $c(n, n+1) = v_0v_1 \dots v_n \geq 0$
3. $c(1,1) = u_1v_0 - u_0v_1 - w_0 \geq 0$
4. $c(n,i) = u_nc(n-1,i) + v_nc(n-1,i-1) - w_{n-1}c(n-2,i-1)$ for $n \geq 2$
5. $c(n,1) = u_nc(n-1,1) + (v_nu_{n-1} - w_{n-1}u_0 \dots u_{n-1})$ for $n \geq 2$.

Also we recall from Definition 4.2 [3] that W_α is p.q.h. if $c(n,i) \geq 0$ for all $n, i \geq 0$ with $0 \leq i \leq n+1$.

3 Statement of problem

Problem : Consider the weighted shift $W_{\alpha(x,y)}$ with a positive weight sequence $\alpha(x,y) : \sqrt{x}, \sqrt{y}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \dots$ having a Bergman tail. In [2] it was shown that for $y = \frac{2}{3}$ and $0 < x \leq y$, the weighted shift $W_{\alpha(x,y)}$ is p.q.h. So, does there exist an interval (k_1, k_2) about $\frac{2}{3}$ such that for $y \in (k_1, k_2)$ and $0 < x \leq y$, the weighted shift $W_{\alpha(x,y)}$ is p.q.h. ?

Remark 3.1. *We must have $x \leq y \leq \frac{3}{4}$ because $W_{\alpha(x,y)}$ cannot be p.q.h. if it is not hyponormal in the first place.*

Remark 3.2. *If (k_1, k_2) exists then it must be contained in $[\delta_1, \delta_2]$.*

Remark 3.2 is in view of Theorem 2.2 [5] which states the following :
Let $\alpha(x) : \sqrt{x}, \sqrt{x}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \dots$ be a weight sequence with Bergman tail and let $QH(W_{\alpha(x)}) = \{x \in \mathbb{R}_+ : W_{\alpha(x)} \text{ is q.h.}\}$. Then $QH(W_{\alpha(x)}) = [\delta_1, \delta_2]$ where $\delta_1 \approx 0.1673$ and $\delta_2 \approx 0.7439$ with errors less than .001.
Now suppose (k_1, k_2) exists. Then for $y \in (k_1, k_2)$ and $0 < x \leq y$, $W_{\alpha(x,y)}$ is p.q.h. and hence q.h. In particular $W_{\alpha(y)}$ is q.h. and so $y \in QH(W_{\alpha(y)}) = [\delta_1, \delta_2]$.

Remark 3.3. *If (k_1, k_2) exists then it must be contained in $(0.625, \delta_2]$.*

Remark 3.3 is in view of Theorem 3.7 [6] where it was shown that for $y = \frac{5}{8} = 0.625$, $W_{\alpha(x,y)}$ is not p.q.h.

In the next section we shall show that (k_1, k_2) exists and also determine the biggest such interval. Before that we record a few definitions and results from [1] which are to be used in solving our problem.

Definition 3.4. *[1] Let $\alpha : \alpha_0, \alpha_1, \dots$ be a weight sequence.*
(1) A weighted shift W_α has property $B(k)$ if $u_{n+1}v_n \geq w_n$, $(n \geq k)$
(2) A weighted shift W_α has property $C(k)$ if $v_{n+1}u_n \geq w_n$, $(n \geq k)$

Corollary 3.5. *[1] Let W_α be a weighted shift with property $C(2)$. Then W_α is p.q.h. if and only if $c(n+1,n) \geq 0 \,\forall\, n \in \mathbb{N}$*

Lemma 3.6. *[1] If W_α has property $B(n+1)$ for some $n \geq 1$, then W_α has property $C(n)$.*

Theorem 3.7. *[1] If W_α be a weighted shift with property $B(k)$ for some $k \geq 2$, then W_α is p.q.h. if and only if $c(n+i-1,i) \geq 0$ for $n = 1, 2, \ldots, k$*

4 Determination of k_1 and k_2

In view of Remark 3.3, we shall consider $y \in (0.625, \frac{2}{3}]$ for determining k_1, and we consider $y \in [\frac{2}{3}, 0.7439]$ for determining k_2.

CASE I : Determining k_1

Choose $y \in (0.625, \frac{2}{3}]$, $0 < x \leq y$ and denote the sequence $\alpha(x,y)$ as $\alpha_0, \alpha_1, \alpha_2, \ldots$. Then we have $\alpha_0 = \sqrt{x}$, $\alpha_1 = \sqrt{y}$ and $\alpha_n = \sqrt{\frac{n+1}{n+2}}$ for $n \geq 2$. Using the expressions of u_n, v_n and w_n as given in section2, we see that $u_{n+1}v_n - w_n = \frac{1}{40}(\frac{2}{3} - y) \geq 0$ for $n = 3$, and for $n \geq 4$ we have $u_{n+1} = \frac{1}{(n+2)(n+3)}, v_n = \frac{4}{(n+1)(n+3)}, w_n = \frac{4}{(n+1)(n+2)(n+3)^2}$ and so $u_{n+1}v_n = w_n$. Therefore, we have $u_{n+1}v_n - w_n \geq 0$ for $n \geq 3$ and so by Definition 3.4, $W_{\alpha(x,y)}$ has property $B(3)$.

Since $W_{\alpha(x,y)}$ has property $B(3)$ so by Theorem 3.7, $W_{\alpha(x,y)}$ is p.q.h. if and only if $c(n+i-1,i) \geq 0$ for $n = 1, 2$ and $i = 1, 2, 3$.

Again, since $W_{\alpha(x,y)}$ has property $B(3)$, so by Lemma 3.6, $W_{\alpha(x,y)}$ has property $C(2)$ and hence by Corollary 3.5, $W_{\alpha(x,y)}$ is p.q.h. if and only if $c(n+1,n) \geq 0$ for all $n \in \mathbb{N}$.

Combining the above two results we get that $W_{\alpha(x,y)}$ is p.q.h. if and only if $c(2,1), c(3,2)$ and $c(4,3)$ are ≥ 0. Using the expressions of $c(n,i)$ from Section2 and simplifying we get,

$c(2,1) = \frac{3}{4}\, x\, (\frac{4}{5} - y)(y - x)$

$c(3,2) = \frac{1}{80}\, x\, [(5y - 4y^2 - 3y^3) - x(32 - 112y + 138y^2 - 60y^3)]$

$c(4,3) = \frac{1}{2800}\, x\, [(41y - 79y^2 + 37y^3) - x(128 - 420y + 475y^2 - 184y^3)]$

Clearly, for $y \in (0.625, \frac{2}{3}]$ and $0 < x \leq y$ we get $c(2,1) \geq 0$

Regarding $c(3,2)$, if we define $f(y) := \frac{5y - 4y^2 - 3y^3}{32 - 112y + 138y^2 + 60y^3}$ then it is seen from the Mathematica graph and also by rigorous calculation that for $y \in (0.625, \frac{2}{3}]$

and $0 < x \leq y$, $f(y) \geq y$ and so $c(3,2) \geq 0$.

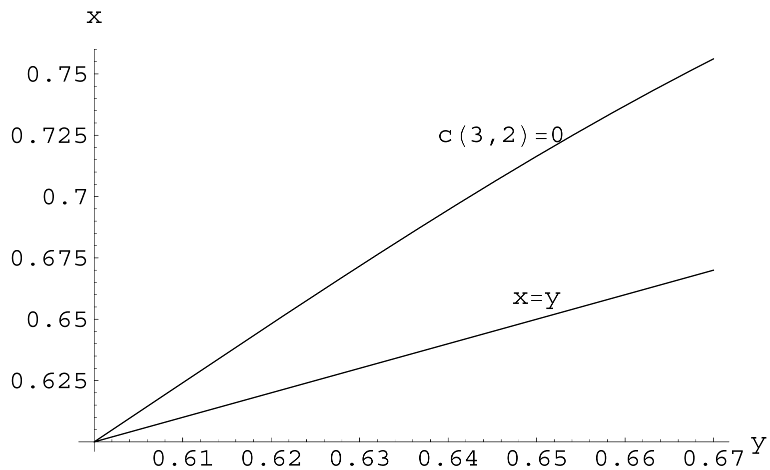

Figure 1

To check whether $c(4,3) \geq 0$, we define $f(y) := \frac{41y-79y^2+37y^3}{128-420y+475y^2-184y^3}$. Then
(i) for $y \in (0.625, \frac{29}{46})$, $f(y) < y$ and so for $f(y) < x \leq y$ we have $c(4,3) < 0$
(ii) for $y \in [\frac{29}{46}, \frac{2}{3}]$, $f(y) \geq y$ and so for $0 < x \leq y$ we have $c(4,3) \geq 0$

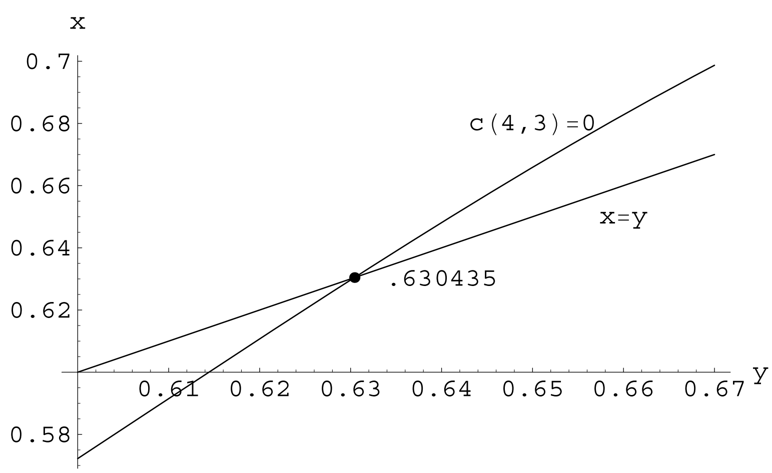

Figure 2

Hence we conclude that $W_{\alpha(x,y)}$ is p.q.h. for $0 < x \leq y$ if and only if $y \in [\frac{29}{46}, \frac{2}{3}]$. Thus, $k_1 = \frac{29}{46} \approx 0.630435$

CASE II : Determining k_2

Choosing $y \in [\frac{2}{3}, 0.7439]$ and $0 < x \leq y$ and proceeding as in Case-I we see that $W_{\alpha(x,y)}$ has property $B(4)$. So by Theorem 3.7, $W_{\alpha(x,y)}$ is p.q.h. if and only if $c(n+i-1,i) \geq 0$ for $n = 1,2,3$ and $i = 1,2,3,4$. That is, if and only if $c(1,1), c(2,1), c(2,2), c(3,1), c(3,2), c(3,3), c(4,2), c(4,3), c(4,4), c(5,3), c(5,4), c(6,4)$ are all ≥ 0.

Using the expressions of $c(n,i)$ from Section2 and simplifying we get,
$c(1,1) = \frac{1}{4}\, xy\,(3-4x)$
$c(2,1) = \frac{3}{20}(4-5y)(y-x)$
$c(2,2) = \frac{2}{20}\, xy\,[(3-5y^2) - x(4-5y)]$
$c(3,1) = \frac{1}{60}\, x\,(3-4y)(y-x)$
$c(3,2) = \frac{1}{80}\, x\,[(-5y+4y^2+3y^3) - x(-32+112y-138y^2+60y^3)]$
$c(3,3) = \frac{1}{40}\, xy\,[(-12+27y-16y^2) - x(-16+38y-24y^2)]$
$c(4,2) = \frac{1}{16800}\, x\,[(75y-86y^2-21y^3) - x(264-842y+966y^2-420y^3)]$
$c(4,3) = \frac{1}{2800}\, x\,[(41y-79y^2+37y^3) - x(128-420y+475y^2-184y^3)]$
$c(4,4) = \frac{1}{5600}\, xy\,[(192-390y+193y^2) - x(256-608y+454y^2-105y^3)]$
$c(5,3) = \frac{1}{201600}\, x\,[(111y-194y^2+63y^3) - 2x(132-397y+417y^2-162y^3)]$
$c(5,4) = \frac{1}{33600}\, x\,[(41y-73y^2+28y^3) - x(128-420y+475y^2-174y^3-15y^4)]$
$c(6,4) = \frac{1}{50803200}\, x\,[(1776y-2942y^2+765y^3) - x(4224-12704y+13344y^2-4914y^3-405y^4)]$

Now $c(1,1), c(2,1)$ and $c(3,1)$ are obviously ≥ 0 for $0 < x \leq y \leq \frac{3}{4}$

Thus we only need to check $c(2,2), c(3,2), c(3,3), c(4,2), c(4,3), c(4,4), c(5,3), c(5,4)$ and $c(6,4)$. Of these we find that other than $c(4,2), c(4,4), c(5,4)$ and $c(6,4)$, all the rest are ≥ 0 for $y \in [\frac{2}{3}, 0.7439]$ and $0 < x \leq y$. This is clear from the following figure which shows that the graphs of $c(2,2), c(3,2), c(3,3), c(4,3)$ and $c(5,3)$ are all above the $x = y$ line in the region $y \in [\frac{2}{3}, 0.7439]$.

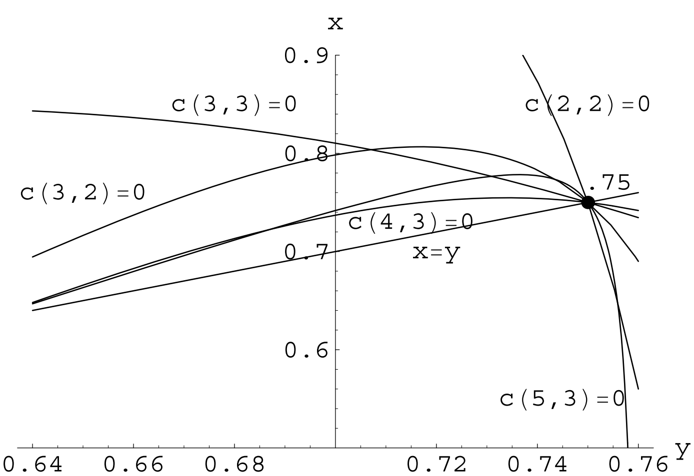

Figure 3

To check whether $c(4,2) \geq 0$, we define $f(y) := \frac{75y-86y^2-21y^3}{264-842y+966y^2-420y^3}$. Then

(i) for $y \in (0.737144, 0.7439)$, $f(y) < y$ and so for $f(y) < x \le y$ we have $c(4,2) < 0$
(ii) for $y \in [\frac{2}{3}, 0.737144)$, $f(y) \ge y$ and so for $0 < x \le y$ we have $c(4,2) \ge 0$

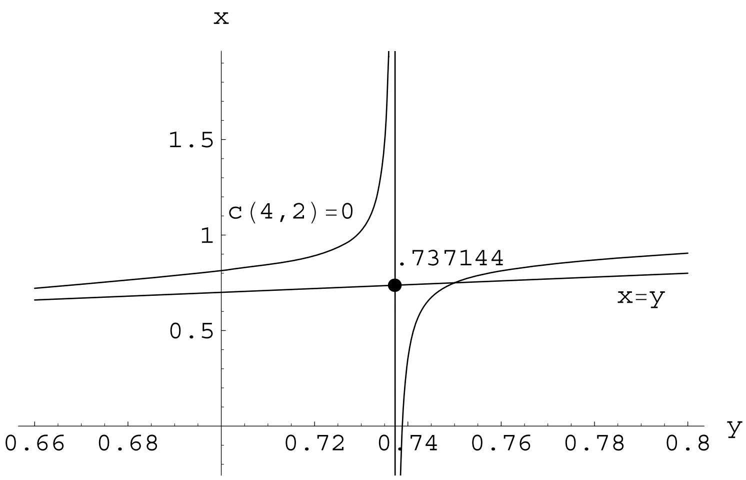

Figure 4

To check whether $c(5,4) \ge 0$, we define $f(y) := \frac{41y-73y^2+28y^3}{128-420y+475y^2-174y^3-15y^4}$. Then
(i) for $y \in (0.742207, 0.7439]$, $f(y) < y$ and so for $f(y) < x \le y$ we have $c(5,4) < 0$
(ii) for $y \in [\frac{2}{3}, 0.742207]$, $f(y) \ge y$ and so for $0 < x \le y$ we have $c(5,4) \ge 0$

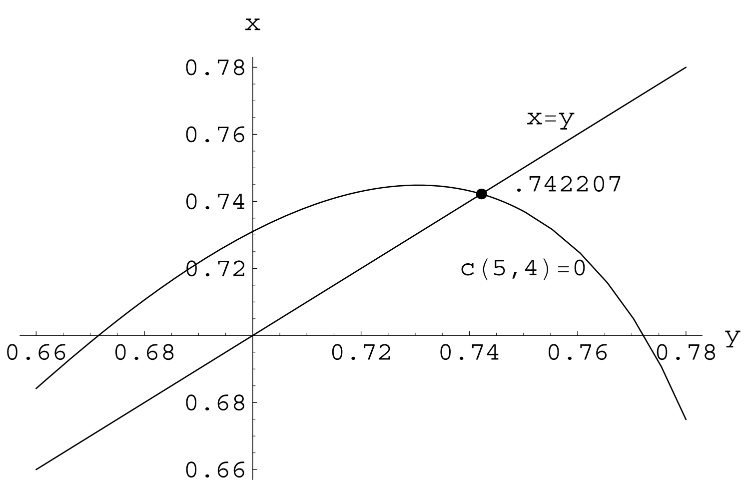

Figure 5

To check whether $c(6,4) \ge 0$, we define $f(y) := \frac{1776y-2942y^2+765y^3}{4224-12704y+13344y^2-4914y^3-405y^4}$. Then
(i) for $y \in (0.742654, 0.7439]$, $f(y) < y$ and so for $f(y) < x \le y$ we have

$c(6,4) < 0$
(ii) for $y \in [\frac{2}{3}, 0.742654]$, $f(y) \geq y$ and so for $0 < x \leq y$ we have $c(6,4) \geq 0$

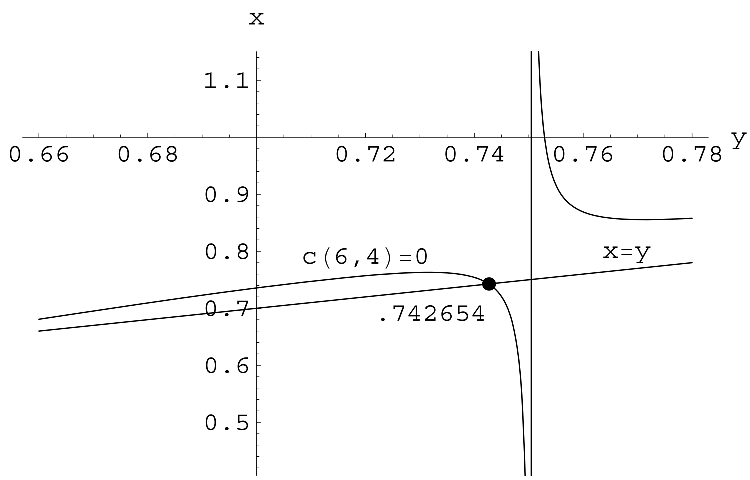

Figure 6

To check whether $c(4,4) \geq 0$, we define $f(y) := \frac{192-390y+193y^2}{256-608y+454y^2-105y^3}$. Then
(i) for $y \in (0.742847, 0.7439]$, $f(y) < y$ and so for $f(y) < x \leq y$ we have $c(4,4) < 0$
(ii) for $y \in [\frac{2}{3}, 0.742847]$, $f(y) \geq y$ and so for $0 < x \leq y$ we have $c(4,4) \geq 0$

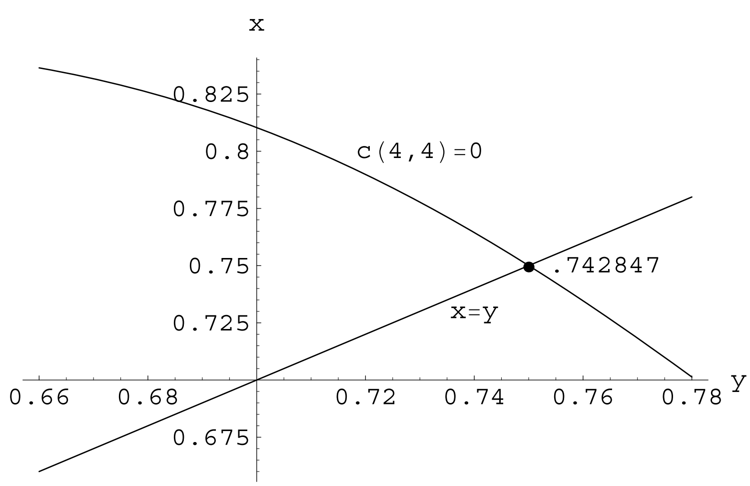

Figure 7

Hence we conclude that $W_{\alpha(x,y)}$ is p.q.h. for $0 < x \leq y$ if and only if $y \in [\frac{2}{3}, 0.737144)$. Thus, $k_2 = 0.737144$

5 Conclusion

For $0 < x \leq y \leq \frac{3}{4}$, let $\alpha(x, y)$ denote the sequence with Bergman tail given by $\alpha(x, y) : \sqrt{x}\,, \sqrt{y}\,, \sqrt{\frac{3}{4}}\,, \sqrt{\frac{4}{5}}\,, \ldots$. Then,

(A) There exists an interval (k_1, k_2) about the point $\frac{2}{3}$ such that for every $y \in (k_1, k_2)$ and $0 < x \leq y$, the weighted shift operator $W_{\alpha(x,y)}$ is p.q.h.

(B) For $y \leq \frac{29}{46} \approx 0.630435$ there exists $0 < x \leq y$ such that $W_{\alpha(x,y)}$ is not p.q.h.

(C) For $y > 0.737144$ there exists $0 < x \leq y$ such that $W_{\alpha(x,y)}$ is not p.q.h.

(D) $W_{\alpha(x,y)}$ is p.q.h. for $0 < x \leq y$ if and only if $y \in [k_1, k_2)$, where $k_1 = \frac{29}{46} \approx 0.630435$ and $k_2 = 0.737144$

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