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PERTURBATION OF 2-VARIABLE HYPONORMAL WEIGHTED SHIFTS

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Abstract : In this paper we consider a hyponormal 2-variable weighted shift T on $\ell^2(\mathbb{Z}_+^2)$ and investigate conditions under which its perturbation $\tilde{T}$ will still remain hyponormal. We show how hyponormality of the perturbed shift $\tilde{T}$ can be completely determined by identifying a set of positivity conditions. Finally we show that perturbation of a 2-variable hyponormal weighted shift is weakly hyponormal.

1. Introduction. A bounded linear operator T on a Hilbert space H is said to be hyponormal if $[T^*,T] := T^*T - TT^* \geq 0$ on H.

Let $\{e_n\}_{n=0}^{\infty}$ be the orthonormal basis of $\ell^2(\mathbb{Z}_+)$. For a positive bounded weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$, the unilateral weighted shift on $\ell^2(\mathbb{Z}_+)$ with weight sequence α is the operator W_α defined as follows: $W_\alpha e_n = \alpha_n e_{n+1}$ for all n.
It easily follows that W_α is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}$ for all n. Now suppose the i^{th} weight α_i is slightly perturbed and is replaced by x. If $W_{[i:x]}$ denote the perturbed shift, then $W_{[i:x]}$ will still remain hyponormal provided $\alpha_{i-1} \leq x \leq \alpha_{i+1}$.

Thus, if we originally have a strictly increasing weight sequence $\{\alpha_n\}$ then for each i it is possible to choose $\delta_i > 0$ such that $W_{[i:x]}$ is again hyponormal for $x \in (\alpha_i - \delta_i, \alpha_i + \delta_i)$. In other words, any slight perturbation of the i^{th} weight still keeps the perturbed shift hyponormal.

In this paper we consider a hyponormal 2-variable weighted shift on $\ell^2(\mathbb{Z}_+^2)$ and investigate conditions under which its perturbation will still remain hyponormal. For this we consider 2-variable positive weight sequences $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}_+^2}$ and $\beta = \{\beta_k\}_{k\in\mathbb{Z}_+^2}$ such that $\alpha_k < \alpha_{k+\varepsilon_1}$ and $\beta_k < \beta_{k+\varepsilon_2}$ for all $k \in \mathbb{Z}_+^2$ and $\varepsilon_1 = (1,0)$, $\varepsilon_2 = (0,1)$.

Let $\{e_k\}_{k\in\mathbb{Z}_+^2}$ denote the orthonormal basis for $\ell^2(\mathbb{Z}_+^2)$, and T_1, T_2 be operators on $\ell^2(\mathbb{Z}_+^2)$ defined as follows: $T_1 e_k = \alpha_k e_{k+\varepsilon_1}$ and $T_2 e_k = \beta_k e_{k+\varepsilon_2}$, for all $k \in \mathbb{Z}_+^2$.
It is assumed that T_1 and T_2 commute and hence $\alpha_k \beta_{k+\varepsilon_1} = \beta_k \alpha_{k+\varepsilon_2}$ for all $k \in \mathbb{Z}_+^2$.

We then consider the 2-variable weighted shift $T = (T_1, T_2)$ on $\ell^2(\mathbb{Z}_+^2)$. From (Curto, 1990) we have the following results:

THEOREM 1.1 *T is hyponormal if and only if*

$$\Delta_k := \begin{pmatrix} \alpha_{k+\varepsilon_1}^2 - \alpha_k^2 & \alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1} - \alpha_k\beta_k \\ \alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1} - \alpha_k\beta_k & \beta_{k+\varepsilon_2}^2 - \beta_k^2 \end{pmatrix} \geq 0 \ (\forall\, k \in \mathbb{Z}_+^2)$$

THEOREM 1.2 *T is weakly hyponormal if and only if*

$$\left\langle \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] \end{pmatrix} \begin{pmatrix} x \\ \lambda x \end{pmatrix}, \begin{pmatrix} x \\ \lambda x \end{pmatrix} \right\rangle \geq 0 \ (\forall\, x \in \ell^2(\mathbb{Z}_+^2) \text{ and } \lambda \in \mathbb{C}).$$

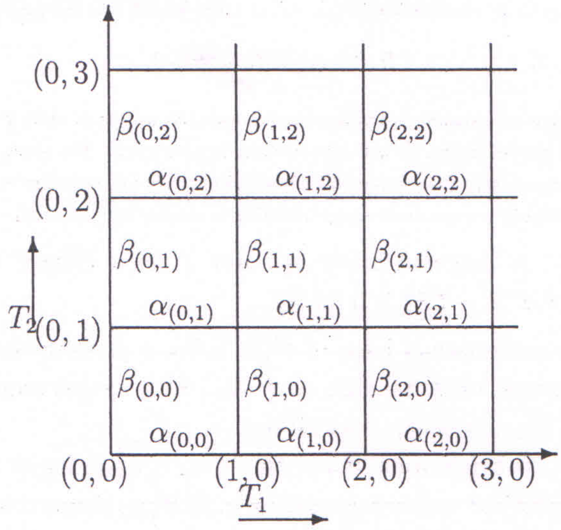

Figure 1

Here we begin with a hyponormal weighted shift $T = (T_1, T_2)$, as defined above, and having the following weight diagram.

Now suppose we perturb the weight α_k to x. If $\tilde{T} = (\tilde{T_1}, \tilde{T_2})$ denotes the perturbed shift then $\tilde{T_1}, \tilde{T_2}$ fail to be commutative. So to preserve commutativity it is absolutely necessary to perturb more weights in other blocks. Since it is a necessity we restrict the number of these subsequent perturbations to the bare minimum. The further perturbation of weights in adjacent blocks are as follows:

(1) β_k changes to $y = \dfrac{\beta_k x}{\alpha_k}$

(2) $\alpha_{k-\varepsilon_1}$ changes to $z = \dfrac{\alpha_{k-\varepsilon_1}\alpha_k}{x}$

(3) $\beta_{k-\varepsilon_2}$ changes to $t = \dfrac{\beta_{k-\varepsilon_2}\alpha_k}{x}$

with the understanding that if $k = (0, k_2)$ then we neglect (2), and if $k = (k_1, 0)$ we neglect (3).

$\tilde{T} = (\tilde{T}_1, \tilde{T}_2)$ is the perturbed shift with weight sequences $\{\tilde{\alpha}_\tau\}_{\tau \in \mathbb{Z}_+^2}$ and $\{\tilde{\beta}_\tau\}_{\tau \in \mathbb{Z}_+^2}$ given as follows:

$$\tilde{\alpha}_\tau = \begin{cases} x, & \text{if } \tau = k \\ z, & \text{if } \tau = k - \varepsilon_1 \\ \alpha_k, & \text{if } \tau \neq k, \tau \neq k - \varepsilon_1 \end{cases} \quad \text{and} \quad \tilde{\beta}_\tau = \begin{cases} y, & \text{if } \tau = k \\ t, & \text{if } \tau = k - \varepsilon_2 \\ \beta_k, & \text{if } \tau \neq k, \tau \neq k - \varepsilon_2 \end{cases}$$

Figure 2 gives the corresponding weight diagram:

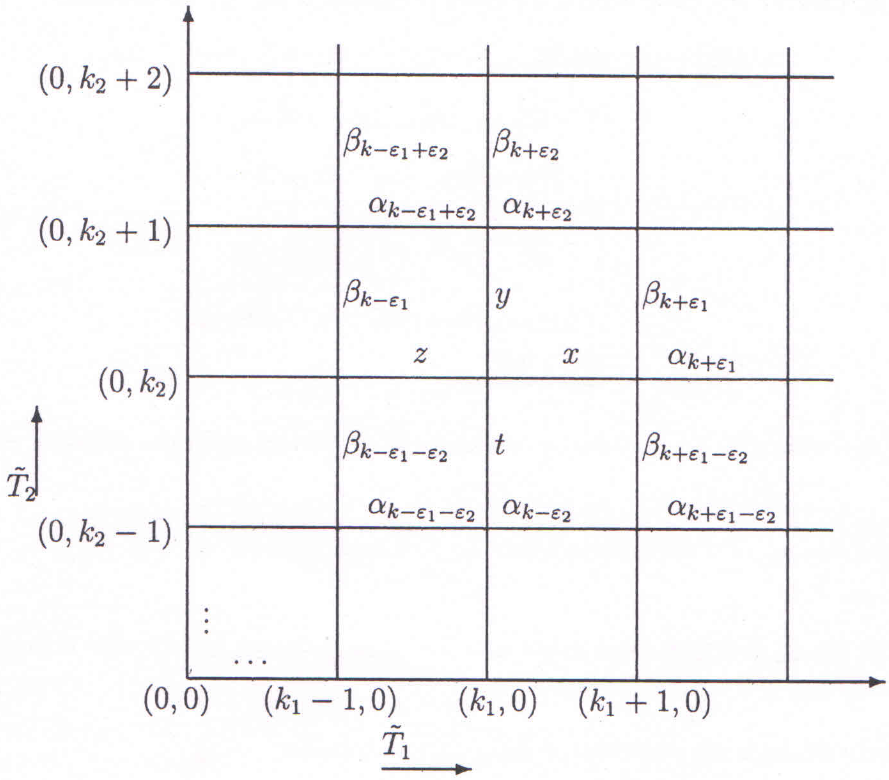

Figure 2

As $\alpha_k < \left(\frac{\beta_{k-\varepsilon_2}}{\beta_{k-2\varepsilon_2}}\right)\alpha_k$, so by keeping $x < \left(\frac{\beta_{k-\varepsilon_2}}{\beta_{k-2\varepsilon_2}}\right)\alpha_k$ we will preserve the condition $\beta_{k-2\varepsilon_2} < t$. Similarly, by keeping x suitably near α_k, we can preserve the conditions $\beta_{k-2\varepsilon_2} < t < y < \beta_{k+\varepsilon_2}$ and $\alpha_{k-2\varepsilon_1} < z < x < \alpha_{k+\varepsilon_1}$.

In this paper, we show how hyponormality of the perturbed shift $\tilde{T}$ can be completely determined by identifying the conditions of positivity of $\tilde{\Delta}_{k-2\varepsilon_2}, \tilde{\Delta}_{k-\varepsilon_1-\varepsilon_2}, \tilde{\Delta}_{k-2\varepsilon_1}, \tilde{\Delta}_{k-\varepsilon_1}, \tilde{\Delta}_{k-\varepsilon_2}$

and $\tilde{\Delta}_k$, where

$$\tilde{\Delta}_\tau := \begin{pmatrix} \tilde{\alpha}^2_{\tau+\varepsilon_1} - \tilde{\alpha}^2_\tau & \tilde{\alpha}_{\tau+\varepsilon_2}\tilde{\beta}_{\tau+\varepsilon_1} - \tilde{\alpha}_\tau\tilde{\beta}_\tau \\ \tilde{\alpha}_{\tau+\varepsilon_2}\tilde{\beta}_{\tau+\varepsilon_1} - \tilde{\alpha}_\tau\tilde{\beta}_\tau & \tilde{\beta}^2_{\tau+\varepsilon_2} - \tilde{\beta}^2_\tau \end{pmatrix} \quad (\forall\, \tau \in \mathbb{Z}^2_+)$$

Finally, we show that it is always possible to choose δ_k such that for all $x \in (\alpha_k - \delta_k, \alpha_k + \delta_k)$, $\tilde{T}$ will remain weakly hyponormal. In other words, perturbation of 2-variable hyponormal shift remains weakly hyponormal.

2. Hyponormality conditions. To check positivity of $\tilde{\Delta}_{k-2\varepsilon_2}$ we consider

$$\begin{aligned} f_1(x) : &= \det \tilde{\Delta}_{k-2\varepsilon_2} \\ &= (t^2 - \beta^2_{k-2\varepsilon_2})(\alpha^2_{k+\varepsilon_1-2\varepsilon_2} - \alpha^2_{k-2\varepsilon_2}) \\ &\quad -(\alpha_{k-\varepsilon_2}\beta_{k+\varepsilon_1-2\varepsilon_2} - \alpha_{k-2\varepsilon_2}\beta_{k-2\varepsilon_2})^2 \\ &= \left(\frac{\beta^2_{k-\varepsilon_2}\alpha^2_k}{x^2} - \beta^2_{k-2\varepsilon_2}\right)(\alpha^2_{k+\varepsilon_1-2\varepsilon_2} - \alpha^2_{k-2\varepsilon_2}) \\ &\quad -(\alpha_{k-\varepsilon_2}\beta_{k+\varepsilon_1-2\varepsilon_2} - \alpha_{k-2\varepsilon_2}\beta_{k-2\varepsilon_2})^2 \\ \therefore\ f_1'(x) &= -\frac{2\beta^2_{k-\varepsilon_2}\alpha^2_k}{x^3}(\alpha^2_{k+\varepsilon_1-2\varepsilon_2} - \alpha^2_{k-2\varepsilon_2}) < 0. \end{aligned}$$

Now, $f_1(\alpha_k) = \det \Delta_{k-2\varepsilon_2} \geq 0$. So by continuity of f_1 we can make the following conclusion C1:

(1) If $\det \Delta_{k-2\varepsilon_2} > 0$ then there exists $\delta_k > 0$ such that for all $x \in (\alpha_k - \delta_k, \alpha_k + \delta_k)$ $\tilde{\Delta}_{k-2\varepsilon_2} \geq 0$.

(2) If $\det \Delta_{k-2\varepsilon_2} = 0$ then there exists $\delta_k > 0$ such that $\tilde{\Delta}_{k-2\varepsilon_2} \geq 0$ for all $x \in (\alpha_k - \delta_k, \alpha_k)$, and $\tilde{\Delta}_{k-2\varepsilon_2} \ngeq 0$ for $x \in (\alpha_k, \alpha_k + \delta_k)$.

Similarly to check the positivity of $\tilde{\Delta}_{k-\varepsilon_1-\varepsilon_2}$, we consider

$$\begin{aligned} f_2(x) : &= \det \tilde{\Delta}_{k-\varepsilon_1-\varepsilon_2} \\ &= (\beta^2_{k-\varepsilon_1} - \beta^2_{k-\varepsilon_1-\varepsilon_2})(\alpha^2_{k-\varepsilon_1} - \alpha^2_{k-\varepsilon_1-\varepsilon_2}) \\ &\quad -(zt - \alpha_{k-\varepsilon_1-\varepsilon_2}\beta_{k-\varepsilon_1-\varepsilon_2})^2 \\ &= (\beta^2_{k-\varepsilon_1} - \beta^2_{k-\varepsilon_1-\varepsilon_2})(\alpha^2_{k-\varepsilon_1} - \alpha^2_{k-\varepsilon_1-\varepsilon_2}) \\ &\quad -\left(\frac{\alpha_{k-\varepsilon_1}\alpha^2_k\beta_{k-\varepsilon_2}}{x^2} - \alpha_{k-\varepsilon_1-\varepsilon_2}\beta_{k-\varepsilon_1-\varepsilon_2}\right)^2 \end{aligned}$$

$$f_2'(x) = 4\frac{\alpha_{k-\varepsilon_1}\alpha_k^2\beta_{k-\varepsilon_2}}{x^3}\left(\frac{\alpha_{k-\varepsilon_1}\alpha_k^2\beta_{k-\varepsilon_2}}{x^2} - \alpha_{k-\varepsilon_1-\varepsilon_2}\beta_{k-\varepsilon_1-\varepsilon_2}\right)$$

$$\therefore\ f_2'(\alpha_k) = 4\frac{\alpha_{k-\varepsilon_1}\beta_{k-\varepsilon_2}}{\alpha_k}\left(\alpha_{k-\varepsilon_1}\beta_{k-\varepsilon_2} - \alpha_{k-\varepsilon_1-\varepsilon_2}\beta_{k-\varepsilon_1-\varepsilon_2}\right)$$

$$\begin{cases} > 0, & \text{if}\, \alpha_{k-\varepsilon_1}\beta_{k-\varepsilon_2} > \alpha_{k-\varepsilon_1-\varepsilon_2}\beta_{k-\varepsilon_1-\varepsilon_2} \\ < 0, & \text{if}\, \alpha_{k-\varepsilon_1}\beta_{k-\varepsilon_2} < \alpha_{k-\varepsilon_1-\varepsilon_2}\beta_{k-\varepsilon_1-\varepsilon_2} \\ = 0, & \text{if}\, \alpha_{k-\varepsilon_1}\beta_{k-\varepsilon_2} = \alpha_{k-\varepsilon_1-\varepsilon_2}\beta_{k-\varepsilon_1-\varepsilon_2} \end{cases}$$

From the continuity of f_2 we can make the following conclusion C2:

(1) If $\det \Delta_{k-\varepsilon_1-\varepsilon_2} > 0$ then there exists $\delta_k > 0$ such that $\tilde{\Delta}_{k-\varepsilon_1-\varepsilon_2} \geq 0$ for all $x \in (\alpha_k - \delta_k, \alpha_k + \delta_k)$

(2) If $\det \Delta_{k-\varepsilon_1-\varepsilon_2} = 0$ then there exists $\delta_k > 0$ such that

(i) If $\alpha_{k-\varepsilon_1}\beta_{k-\varepsilon_2} > \alpha_{k-\varepsilon_1-\varepsilon_2}\beta_{k-\varepsilon_1-\varepsilon_2}$, then $\tilde{\Delta}_{k-\varepsilon_1-\varepsilon_2} \geq 0$ for all $x \in (\alpha_k, \alpha_k + \delta_k)$, and $\tilde{\Delta}_{k-\varepsilon_1-\varepsilon_2} \ngeq 0$ for $x \in (\alpha_k - \delta_k, \alpha_k)$,

(ii) If $\alpha_{k-\varepsilon_1}\beta_{k-\varepsilon_2} < \alpha_{k-\varepsilon_1-\varepsilon_2}\beta_{k-\varepsilon_1-\varepsilon_2}$, then $\tilde{\Delta}_{k-\varepsilon_1-\varepsilon_2} \geq 0$ for all $x \in (\alpha_k - \delta_k, \alpha_k)$, and $\tilde{\Delta}_{k-\varepsilon_1-\varepsilon_2} \ngeq 0$ for $x \in (\alpha_k, \alpha_k + \delta_k)$,

(iii) If $\alpha_{k-\varepsilon_1}\beta_{k-\varepsilon_2} = \alpha_{k-\varepsilon_1-\varepsilon_2}\beta_{k-\varepsilon_1-\varepsilon_2}$, then $\tilde{\Delta}_{k-\varepsilon_1-\varepsilon_2} \geq 0$ for all $x \in (\alpha_k + \delta_k, \alpha_k - \delta_k)$.

For positivity of $\tilde{\Delta}_{k-2\varepsilon_1}$, we consider

$$\begin{aligned} f_3(x) :&= \det \tilde{\Delta}_{k-2\varepsilon_1} \\ &= (z^2 - \alpha_{k-2\varepsilon_1}^2)(\beta_{k-2\varepsilon_1+\varepsilon_2}^2 - \beta_{k-2\varepsilon_1}^2) \\ &\quad -(\alpha_{k-2\varepsilon_1+\varepsilon_2}\beta_{k-\varepsilon_1} - \alpha_{k-2\varepsilon_1}\beta_{k-2\varepsilon_1})^2 \\ &= \left(\frac{\alpha_{k-\varepsilon_1}^2\alpha_k^2}{x^2} - \alpha_{k-2\varepsilon_1}^2\right)(\beta_{k-2\varepsilon_1+\varepsilon_2}^2 - \beta_{k-2\varepsilon_1}^2) \\ &\quad -(\alpha_{k-2\varepsilon_1+\varepsilon_2}\beta_{k-\varepsilon_1} - \alpha_{k-2\varepsilon_1}\beta_{k-2\varepsilon_1})^2 . \end{aligned}$$

So,

$$f_3'(x) = -2\frac{\alpha_{k-\varepsilon_1}^2\alpha_k^2}{x^3}(\beta_{k-2\varepsilon_1+\varepsilon_2}^2 - \beta_{k-2\varepsilon_1}^2)$$

$$\therefore\ f_3'(\alpha_k) = -2\frac{\alpha_{k-\varepsilon_1}^2}{\alpha_k}(\beta_{k-2\varepsilon_1+\varepsilon_2}^2 - \beta_{k-2\varepsilon_1}^2) < 0.$$

Now from the continuity of f_3 we can make the conclusion C3:

(1) If $\det \Delta_{k-2\varepsilon_1} > 0$ then there exists $\delta_k > 0$ such that $\tilde{\Delta}_{k-2\varepsilon_1} \geq 0$ for all $x \in (\alpha_k - \delta_k, \alpha_k + \delta_k)$.

(2) If $\det \Delta_{k-2\varepsilon_1} = 0$ then there exists $\delta_k > 0$ such that $\tilde{\Delta}_{k-2\varepsilon_2} \geq 0$ for all $x \in (\alpha_k - \delta_k, \alpha_k)$, and $\tilde{\Delta}_{k-2\varepsilon_1} \ngeq 0$ for $x \in (\alpha_k, \alpha_k + \delta_k)$.

For positivity of $\tilde{\Delta}_{k-\varepsilon_2}$, we consider

$$
\begin{aligned}
f_4(x) :&= \det \tilde{\Delta}_{k-\varepsilon_2} \\
&= (\alpha^2_{k+\varepsilon_1-\varepsilon_2} - \alpha^2_{k-\varepsilon_2})(y^2 - t^2) - (x\beta_{k+\varepsilon_1-\varepsilon_2} - t\alpha_{k-\varepsilon_2})^2 \\
&= (\alpha^2_{k+\varepsilon_1-\varepsilon_2} - \alpha^2_{k-\varepsilon_2})\left(\frac{\beta_k^2 x^2}{\alpha_k^2} - \frac{\beta^2_{k-\varepsilon_2}\alpha_k^2}{x^2}\right) - \left(x\beta_{k+\varepsilon_1-\varepsilon_2} - \frac{\beta_{k-\varepsilon_2}\alpha_k\alpha_{k-\varepsilon_2}}{x}\right)^2 \\
&= \frac{1}{\alpha_k^2}\Big[x^2\{\beta_k^2(\alpha^2_{k+\varepsilon_1-\varepsilon_2} - \alpha^2_{k-\varepsilon_2}) - \alpha_k^2\beta^2_{k+\varepsilon_1-\varepsilon_2}\} \\
&\qquad + 2\beta_{k-\varepsilon_2}\alpha_k^3\alpha_{k-\varepsilon_2}\beta_{k+\varepsilon_1-\varepsilon_2} - \alpha^2_{k+\varepsilon_1-\varepsilon_2}\frac{\beta^2_{k-\varepsilon_2}\alpha_k^4}{x^2}\Big]
\end{aligned}
$$

If $\det \Delta_{k-\varepsilon_2} = 0$ then $f_4(\alpha_k) = \det \Delta_{k-\varepsilon_2} = 0$. Therefore,

$$\lambda := \beta_k^2(\alpha^2_{k+\varepsilon_1-\varepsilon_2} - \alpha^2_{k-\varepsilon_2}) - \alpha_k^2\beta^2_{k+\varepsilon_1-\varepsilon_2} = \beta^2_{k-\varepsilon_2}(\alpha^2_{k+\varepsilon_1-\varepsilon_2} - 2\alpha_k^2)$$

Again,

$$
\begin{aligned}
f_4'(x) &= \frac{1}{\alpha_k^2}\left[2x\lambda + \frac{2\alpha_k^4\alpha^2_{k+\varepsilon_1-\varepsilon_2}\beta^2_{k-\varepsilon_2}}{x^3}\right] \\
&= \frac{2}{\alpha_k^2 x}\left[x^2\lambda + \frac{\alpha_k^4\alpha^2_{k+\varepsilon_1-\varepsilon_2}\beta^2_{k-\varepsilon_2}}{x^2}\right] \\
f_4'(\alpha_k) &= \frac{2}{\alpha_k}\left[\lambda + \alpha^2_{k+\varepsilon_1-\varepsilon_2}\beta^2_{k-\varepsilon_2}\right] \\
&= \frac{4\beta^2_{k-\varepsilon_2}}{\alpha_k}(\alpha^2_{k+\varepsilon_1-\varepsilon_2} - \alpha_k^2) \\
&\begin{cases} > 0, & \text{if } \alpha_{k+\varepsilon_1-\varepsilon_2} > \alpha_k \\ < 0, & \text{if } \alpha_{k+\varepsilon_1-\varepsilon_2} < \alpha_k \\ = 0, & \text{if } \alpha_{k+\varepsilon_1-\varepsilon_2} = \alpha_k \end{cases}
\end{aligned}
$$

Now since f_4 is a continuous function, therefore conclusion C4:

(1) If $\det\Delta_{k-\varepsilon_2} > 0$ or $\alpha_{k+\varepsilon_1-\varepsilon_2} = \alpha_k$ then there exists $\delta_k > 0$ such that $\tilde{\Delta}_{k-\varepsilon_2} \geq 0$ for all $x \in (\alpha_k - \delta_k, \alpha_k + \delta_k)$.

(2) If $\det\Delta_{k-\varepsilon_2} = 0$ then there exists $\delta_k > 0$ such that

 (i) If $\alpha_{k+\varepsilon_1-\varepsilon_2} > \alpha_k$ then $\tilde{\Delta}_{k-\varepsilon_2} \geq 0$ for all $x \in (\alpha_k, \alpha_k + \delta_k)$ and $\tilde{\Delta}_{k-\varepsilon_2} \ngeq 0$ for $x \in (\alpha_k - \delta_k, \alpha_k)$.

 (ii) If $\alpha_{k+\varepsilon_1-\varepsilon_2} < \alpha_k$ then $\tilde{\Delta}_{k-\varepsilon_2} \geq 0$ for all $x \in (\alpha_k - \delta_k, \alpha_k)$ and $\tilde{\Delta}_{k-\varepsilon_2} \ngeq 0$ for $x \in (\alpha_k, \alpha_k + \delta_k)$.

For positivity of $\tilde{\Delta}_{k-\varepsilon_1}$, we consider

$$\begin{aligned} f_5(x) :&= \det\tilde{\Delta}_{k-\varepsilon_1} \\ &= (x^2 - z^2)(\beta^2_{k-\varepsilon_1+\varepsilon_2} - \beta^2_{k-\varepsilon_1}) - (\alpha_{k-\varepsilon_1+\varepsilon_2}y - \beta_{k-\varepsilon_1}z)^2 \\ &= \left(x^2 - \frac{\alpha^2_{k-\varepsilon_1}\alpha^2_k}{x^2}\right)(\beta^2_{k-\varepsilon_1+\varepsilon_2} - \beta^2_{k-\varepsilon_1}) - \left(\frac{\alpha_{k-\varepsilon_1+\varepsilon_2}\beta_k x}{\alpha_k} - \frac{\alpha_k\alpha_{k-\varepsilon_1}\beta_{k-\varepsilon_1}}{x}\right)^2 \\ &= \frac{x^2}{\alpha_k^2}\left[\alpha_k^2(\beta^2_{k-\varepsilon_1+\varepsilon_2} - \beta^2_{k-\varepsilon_1}) - \alpha^2_{k-\varepsilon_1+\varepsilon_2}\beta_k^2\right] \\ &\qquad + 2\alpha_{k-\varepsilon_1+\varepsilon_2}\beta_k\alpha_{k-\varepsilon_1}\beta_{k-\varepsilon_1} - \frac{1}{x^2}(\alpha^2_{k-\varepsilon_1}\alpha_k^2\beta^2_{k-\varepsilon_1+\varepsilon_2}) \end{aligned}$$

If $\det\Delta_{k-\varepsilon_1} = 0$ then $f_5(\alpha_k) = \det\Delta_{k-\varepsilon_1} = 0$. Therefore

$$\mu := \alpha_k^2(\beta^2_{k-\varepsilon_1+\varepsilon_2} - \beta^2_{k-\varepsilon_1}) - \alpha^2_{k-\varepsilon_1+\varepsilon_2}\beta_k^2 = \alpha^2_{k-\varepsilon_1}(\beta^2_{k-\varepsilon_1+\varepsilon_2} - 2\beta_k^2)$$

Now

$$\begin{aligned} f_5'(x) &= \frac{2x\mu}{\alpha_k^2} + \frac{2}{x^3}\alpha^2_{k-\varepsilon_1}\alpha_k^2\beta^2_{k-\varepsilon_1+\varepsilon_2} \\ \therefore\ f_5'(\alpha_k) &= \frac{2}{\alpha_k}(\mu + \alpha^2_{k-\varepsilon_1}\beta^2_{k-\varepsilon_1+\varepsilon_2}) \\ &= \frac{4\alpha^2_{k-\varepsilon_1}}{\alpha_k}(\beta^2_{k-\varepsilon_1+\varepsilon_2} - \beta_k^2) \\ &\begin{cases} > 0, & \text{if } \beta_{k-\varepsilon_1+\varepsilon_2} > \beta_k \\ < 0, & \text{if } \beta_{k-\varepsilon_1+\varepsilon_2} < \beta_k \\ = 0, & \text{if } \beta_{k-\varepsilon_1+\varepsilon_2} = \beta_k \end{cases} \end{aligned}$$

Again from the continuity of f_5, we can make the conclusion C5:

(1) $\det \Delta_{k-\varepsilon_1} > 0$ or $\beta_{k-\varepsilon_1+\varepsilon_2} = \beta_k$ then there exists $\delta_k > 0$ such that $\tilde{\Delta}_{k-\varepsilon_1} \geq 0$ for all $x \in (\alpha_k - \delta_k, \alpha_k + \delta_k)$.

(2) If $\det \Delta_{k-\varepsilon_1} = 0$ then there exists $\delta_k > 0$ such that

(i) If $\beta_{k-\varepsilon_1+\varepsilon_2} > \beta_k$ then $\tilde{\Delta}_{k-\varepsilon_1} \geq 0$ for all $x \in (\alpha_k, \alpha_k + \delta_k)$ and $\tilde{\Delta}_{k-\varepsilon_1} \ngeq 0$ for $x \in (\alpha_k - \delta_k, \alpha_k)$.

(ii) If $\beta_{k-\varepsilon_1+\varepsilon_2} < \beta_k$ then $\tilde{\Delta}_{k-\varepsilon_1} \geq 0$ for all $x \in (\alpha_k - \delta_k, \alpha_k)$ and $\tilde{\Delta}_{k-\varepsilon_1} \ngeq 0$ for $x \in (\alpha_k, \alpha_k + \delta_k)$.

Finally, to check the positivity of $\tilde{\Delta}_k$ we consider

$$\begin{aligned}
f_6(x) :&= \det \tilde{\Delta}_k \\
&= (\alpha^2_{k+\varepsilon_1} - x^2)(\beta^2_{k+\varepsilon_2} - y^2) - (\alpha_{k-\varepsilon_2}\beta_{k+\varepsilon_1} - xy)^2 \\
&= (\alpha^2_{k+\varepsilon_1} - x^2)\left(\beta^2_{k+\varepsilon_2} - \frac{\beta_k^2 x^2}{\alpha_k^2}\right) - \left(\alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1} - \frac{\beta_k x^2}{\alpha_k}\right)^2 \\
&= x^2\left(- \beta^2_{k+\varepsilon_2} - \frac{\beta_k^2\alpha^2_{k+\varepsilon_1}}{\alpha_k^2} + \frac{2\alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1}\beta_k}{\alpha_k}\right) + (\alpha^2_{k+\varepsilon_1}\beta^2_{k+\varepsilon_2} - \alpha^2_{k+\varepsilon_2}\beta^2_{k+\varepsilon_1}) \\
&= \frac{x^2}{\alpha_k^2}\left(2\alpha_k\alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1}\beta_k - \alpha_k^2\beta^2_{k+\varepsilon_2} - \beta_k^2\alpha^2_{k+\varepsilon_1}\right) + (\alpha^2_{k+\varepsilon_1}\beta^2_{k+\varepsilon_2} - \alpha^2_{k+\varepsilon_2}\beta^2_{k+\varepsilon_1})
\end{aligned}$$

If $\det \Delta_k = 0$ then $f_6(\alpha_k) = \det \Delta_k = 0$. Therefore

$$\gamma := \left(2\alpha_k\alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1}\beta_k - \alpha_k^2\beta^2_{k+\varepsilon_2} - \beta_k^2\alpha^2_{k+\varepsilon_1}\right) = (\alpha^2_{k+\varepsilon_2}\beta^2_{k+\varepsilon_1} - \alpha^2_{k+\varepsilon_1}\beta^2_{k+\varepsilon_2})$$

Again,

$$f_6'(x) = \frac{2x\gamma}{\alpha_k^2} = \frac{2x}{\alpha_k^2}(\alpha^2_{k+\varepsilon_2}\beta^2_{k+\varepsilon_1} - \alpha^2_{k+\varepsilon_1}\beta^2_{k+\varepsilon_2})$$

$$\begin{cases} > 0, & \text{if } \alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1} > \alpha_{k+\varepsilon_1}\beta_{k+\varepsilon_2} \\ < 0, & \text{if } \alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1} < \alpha_{k+\varepsilon_1}\beta_{k+\varepsilon_2} \\ = 0, & \text{if } \alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1} = \alpha_{k+\varepsilon_1}\beta_{k+\varepsilon_2} \end{cases}$$

From the continuity of f_6 we can make the conclusion C6:

(1) $\det \Delta_k > 0$ or $\alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1} = \alpha_{k+\varepsilon_1}\beta_{k+\varepsilon_2}$ then there exists $\delta_k > 0$ such that $\tilde{\Delta}_k \geq 0$ for all $x \in (\alpha_k - \delta_k, \alpha_k + \delta_k)$.

(2) If $\det \Delta_k = 0$ then there exists $\delta_k > 0$ such that

  (i) If $\alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1} > \alpha_{k+\varepsilon_1}\beta_{k+\varepsilon_2}$ then $\tilde{\Delta}_k \geq 0$ for all $x \in (\alpha_k, \alpha_k + \delta_k)$ and $\tilde{\Delta}_k \ngeq 0$ for $x \in (\alpha_k - \delta_k, \alpha_k)$.

  (ii) If $\alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1} < \alpha_{k+\varepsilon_1}\beta_{k+\varepsilon_2}$ then $\tilde{\Delta}_k \geq 0$ for all $x \in (\alpha_k - \delta_k, \alpha_k)$ and $\tilde{\Delta}_k \ngeq 0$ for $x \in (\alpha_k, \alpha_k + \delta_k)$.

From the above analysis we can exhaustively determine whether perturbation of α_k will again result in a hyponormal shift $\tilde{T}$ or not.

For illustration let us consider the following examples:

EXAMPLE 2.1 *Let $T = (T_1, T_2)$ be hyponormal with $\Delta_{(0,3)} > 0$, $\Delta_{(0,5)} > 0$, $\Delta_{(0,4)} = 0$ and $\alpha_{(1,4)} < \alpha_{(0,5)}$. We want to perturb $\alpha_{(0,5)}$.*

Applying $C1(1)$, $C6(1)$ and $C4(2)(ii)$ we conclude that $\tilde{T}$ will still be hyponormal for a slight left perturbation of $\alpha_{(0,5)}$, but will not be hyponormal for any right perturbation of $\alpha_{(0,5)}$.

EXAMPLE 2.2 *We want to perturb $\alpha_{(7,11)}$. Hence we need to consider $\Delta_{(7,9)}$, $\Delta_{(6,10)}$, $\Delta_{(5,11)}$, $\Delta_{(7,10)}$, $\Delta_{(6,11)}$, $\Delta_{(7,11)}$. Suppose $\Delta_{(6,10)}$, $\Delta_{(5,11)}$, $\Delta_{(7,10)}$ $\Delta_{(7,11)} > 0$ and $\Delta_{(7,9)} = \Delta_{(6,11)} = 0$. So by $C1(2)$ and $C5$, we make the following conclusions:*

(1) *If $\beta_{(6,12)} \leq \beta_{(7,11)}$ then $\tilde{T}$ will be hyponomal for a slight left perturbation of $\alpha_{(7,11)}$, but will not be hyponormal for any right perturbation of $\alpha_{(7,11)}$.*

(2) *If $\beta_{(6,12)} > \beta_{(7,11)}$ then for any slight perturbation of $\alpha_{(7,11)}$, $\tilde{T}$ will fail to be hyponormal.*

3. Weak hyponormality. For $\tau = (\tau_1, \tau_2) \in \mathbb{Z}_+^2$ and $|\tau| = \tau_1 + \tau_2$, let $a_\tau := \alpha_{\tau+\varepsilon_1}^2 - \alpha_\tau^2$, $b_\tau := \alpha_{\tau+\varepsilon_2}\beta_{\tau+\varepsilon_1} - \alpha_\tau\beta_\tau$, $d_\tau := \beta_{\tau+\varepsilon_2}^2 - \beta_\tau^2$, so that $\Delta_\tau = \begin{pmatrix} a_\tau & b_\tau \\ b_\tau & d_\tau \end{pmatrix}$.

LEMMA 3.1 *$T = (T_1, T_2)$ is weakly hyponormal if and only if for all $x = \sum_{k \in \mathbb{Z}_+^2} c_k e_k \in \ell^2(\mathbb{Z}_+^2)$ and for all $\lambda \in \mathbb{C}$, we have*

$$\sum_{j=0}^{\infty} |c_{(0,j)}|^2 \alpha_{(0,j)}^2 + |\lambda|^2 \sum_{i=0}^{\infty} |c_{(i,0)}|^2 \beta_{(i,0)}^2 + \sum_{k \in \mathbb{Z}_+^2} \left\langle \Delta_k \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda c_{k+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda c_{k+\varepsilon_2} \end{pmatrix} \right\rangle \geq 0.$$

Proof: T is weakly hyponormal

$$\Leftrightarrow\quad T_1+\bar{\lambda}T_2 \text{ is hyponormal};\ \ \forall\,\lambda\in\mathbb{C}$$

$$\Leftrightarrow\quad \left\langle \begin{pmatrix} \langle [T_1^*,T_1]x,x\rangle & \langle [T_2^*,T_1]x,x\rangle \\ \langle [T_1^*,T_2]x,x\rangle & \langle [T_2^*,T_2]x,x\rangle \end{pmatrix}\begin{pmatrix}1\\ \lambda\end{pmatrix},\begin{pmatrix}1\\ \lambda\end{pmatrix}\right\rangle \geq 0;\ \forall\,\lambda\in\mathbb{C} \text{ and } \forall x\in \ell^2(\mathbb{Z}_+^2)$$

We have

$$\begin{aligned}
[T_1^*,T_1]e_k &= (\alpha_k^2-\alpha_{k-\varepsilon_1}^2)e_k\\
[T_2^*,T_1]e_k &= (\alpha_k\beta_{k+\varepsilon_1-\varepsilon_2}-\alpha_{k-\varepsilon_2}\beta_{k-\varepsilon_2})e_{k+\varepsilon_1-\varepsilon_2}\\
[T_1^*,T_2]\,e_k &= \big(\alpha_{k-\varepsilon_1+\varepsilon_2}\,\beta_k-\alpha_{k-\varepsilon_1}\beta_{k-\varepsilon_1}\big)e_{k-\varepsilon_1+\varepsilon_2}\\
[T_2^*,T_2]\,e_k &= \big(\beta_k^2-\beta_{k-\varepsilon_2}^2\big)\,e_k
\end{aligned}$$

assuming $\alpha_{(t_1,t_2)}=0$ for all $t_1<0,\ t_2\in\mathbb{Z}_+$ and $\beta_{(t_1,t_2)}=0$ for all $t_1\in\mathbb{Z}_+,\ t_2<0$.

Thus for $x=\sum_{k\in\mathbb{Z}_+^2} c_k\,e_k\in\ell^2(\mathbb{Z}_+^2)$, we have

$$\begin{aligned}
\langle [T_1^*,T_1]x,x\rangle &= \sum_{j=0}^{\infty}|c_{(0,j)}|^2\alpha_{(0,j)}^2+\sum_{k\in\mathbb{Z}_+^2} a_k|c_{k+\varepsilon_1}|^2\\
\langle [T_2^*,T_1]x,x\rangle &= \sum_{k\in\mathbb{Z}_+^2} b_k\bar{c}_{k+\varepsilon_1}c_{k+\varepsilon_2}\\
\langle [T_1^*,T_2]x,x\rangle &= \sum_{k\in\mathbb{Z}_+^2} b_k c_{k+\varepsilon_1}\bar{c}_{k+\varepsilon_2}\\
\langle [T_2^*,T_2]x,x\rangle &= \sum_{i=0}^{\infty}|c_{(i,\,0)}|^2\beta_{(i,\,0)}^2+\sum_{k\in\mathbb{Z}_+^2} d_k|c_{k+\varepsilon_2}|^2
\end{aligned}$$

Thus, T is weakly hyponormal

$$\Leftrightarrow\quad \left\langle \begin{pmatrix} \sum\limits_{j=0}^{\infty}|c_{(0,j)}|^2\alpha_{(0,j)}^2 & 0\\ 0 & \sum\limits_{i=0}^{\infty}|c_{(i,\,0)}|^2\beta_{(i,\,0)}^2\end{pmatrix}\begin{pmatrix}1\\ \lambda\end{pmatrix},\begin{pmatrix}1\\ \lambda\end{pmatrix}\right\rangle$$

$$+\left\langle \begin{pmatrix} \sum_{k\in\mathbb{Z}_+^2} a_k |c_{k+\varepsilon_1}|^2 & \sum_{k\in\mathbb{Z}_+^2} b_k \bar{c}_{k+\varepsilon_1} c_{k+\varepsilon_2} \\ \sum_{k\in\mathbb{Z}_+^2} b_k c_{k+\varepsilon_1} \bar{c}_{k+\varepsilon_2} & \sum_{k\in\mathbb{Z}_+^2} d_k |c_{k+\varepsilon_2}|^2 \end{pmatrix} \begin{pmatrix} 1 \\ \lambda \end{pmatrix}, \begin{pmatrix} 1 \\ \lambda \end{pmatrix} \right\rangle \geq 0;$$

$$\text{for all } \lambda \in \mathbb{C} \text{ and } x = \sum_{k\in\mathbb{Z}_+^2} c_k\, e_k \in \ell^2(\mathbb{Z}_+^2)$$

$$\Leftrightarrow \quad \sum_{j=0}^{\infty} |c_{(0,j)}|^2 \alpha_{(0,j)}^2 + |\lambda|^2 \sum_{i=0}^{\infty} |c_{(i,0)}|^2 \beta_{(i,0)}^2$$

$$+ \sum_{k\in\mathbb{Z}_+^2} \left\langle \Delta_k \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix} \right\rangle \geq 0;$$

$$\text{for all } \lambda \in \mathbb{C} \text{ and } x = \sum_{k\in\mathbb{Z}_+^2} c_k\, e_k \in \ell^2(\mathbb{Z}_+^2).$$

REMARKS 3.1 *If* $T = (T_1, T_2)$ *is hyponormal then* $\Delta_k \geq 0$, $\forall\, k \in \mathbb{Z}_+^2$ *and hence (by Lemma 3.1) T is also weakly hyponormal.*

For $s \geq 0$ define L_s as follows:

$$\begin{pmatrix} a_{(s,0)} + |\lambda|^2\beta_{(s+1,0)}^2 & b_{(s,0)} & 0 & 0 & \cdots & 0 & 0 \\ b_{(s,0)} & d_{(s,0)} + \frac{a_{(s-1,1)}}{|\lambda|^2} & \frac{b_{(s-1,1)}}{|\lambda|^2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{b_{(s-1,1)}}{|\lambda|^2} & \frac{d_{(s-1,1)}}{|\lambda|^2} + \frac{a_{(s-2,2)}}{|\lambda|^4} & \frac{b_{(s-2,2)}}{|\lambda|^4} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{d_{(1,s-1)}}{|\lambda|^{2(s-1)}} + \frac{a_{(0,s)}}{|\lambda|^{2s}} & \frac{b_{(0,s)}}{|\lambda|^{2s}} \\ 0 & 0 & 0 & 0 & \cdots & \frac{b_{(0,s)}}{|\lambda|^{2s}} & \frac{d_{(0,s)}}{|\lambda|^{2s}} + \frac{\alpha_{(0,s+1)}^2}{|\lambda|^{2(s+1)}} \end{pmatrix}$$

LEMMA 3.2 *A 2-variable weighted shift* $T = (T_1, T_2)$ *with weight sequences* $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}_+^2}$ *and* $\beta = \{\beta_k\}_{k\in\mathbb{Z}_+^2}$ *is weakly hyponormal if and only if for all* $x = \sum_{k\in\mathbb{Z}_+^2} c_k\, e_k \in \ell^2(\mathbb{Z}_+^2)$, *and* $\lambda \in \mathbb{C}$

we have $|c_{(0,0)}|^2(\alpha^2_{(0,0)} + |\lambda|^2\beta^2_{(0,0)}) + \sum\limits_{n=0}^{\infty} \langle L_n X_n, X_n\rangle \geq 0$, *where*

$$X_n = \begin{pmatrix} c_{(n+1,0)} \\ \lambda c_{(n,1)} \\ \lambda^2 c_{(n-1,2)} \\ \vdots \\ \lambda^{n+1} c_{(0,n+1)} \end{pmatrix}$$

Proof: Direct calculation shows that:

$$\begin{aligned}\langle L_0 X_0, X_0\rangle &= |\lambda|^2|c_{(1,0)}|^2\beta^2_{(1,0)} + |c_{(1,0)}|^2\alpha^2_{(0,1)} \\ &\quad + \left\langle \Delta_{(0,0)} \begin{pmatrix} c_{(0,0)+\varepsilon_1} \\ \lambda\, c_{(0,0)+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{(0,0)+\varepsilon_1} \\ \lambda\, c_{(0,0)+\varepsilon_2} \end{pmatrix} \right\rangle\end{aligned}$$

$$\begin{aligned}\langle L_1 X_1, X_1\rangle &= |\lambda|^2|c_{(2,0)}|^2\beta^2_{(2,0)} + |c_{(0,2)}|^2\alpha^2_{(0,2)} \\ &\quad + \sum_{|k|=1} \left\langle \Delta_k \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix} \right\rangle\end{aligned}$$

Similarly,

$$\begin{aligned}\langle L_n X_n, X_n\rangle &= |\lambda|^2|c_{(n+1,0)}|^2\beta^2_{(n+1,0)} + |c_{(0,n+1)}|^2\alpha^2_{(0,n+1)} \\ &\quad + \sum_{|k|=n} \left\langle \Delta_k \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix} \right\rangle\end{aligned}$$

Therefore,

$$\begin{aligned}&|c_{(0,0)}|^2(\alpha^2_{(0,0)} + |\lambda|^2\beta^2_{(0,0)}) + \sum_{n=0}^{\infty} \langle L_n X_n, X_n\rangle \\ &= \sum_{j=0}^{\infty} |c_{(0,j)}|^2\alpha^2_{(0,j)} + |\lambda|^2 \sum_{i=0}^{\infty} |c_{(i,0)}|^2\beta^2_{(i,0)} \\ &\quad + \sum_{k\in\mathbb{Z}^2_+} \left\langle \Delta_k \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix} \right\rangle\end{aligned}$$

The result now follows from Lemma 3.1

Let α_k be perturbed to the weight to x. For commutativity, β_k is changed to $y = \dfrac{\beta_k x}{\alpha_k}$, $\alpha_{k-\varepsilon_1}$ is changed to $z = \dfrac{\alpha_{k-\varepsilon_1}\alpha_k}{x}$ and $\beta_{k-\varepsilon_2}$ is changed to $t = \dfrac{\beta_{k-\varepsilon_2}\alpha_k}{x}$.

Let $\tilde{T} = (\tilde{T}_1, \tilde{T}_2)$ be the perturbed shift with weight sequences $\{\tilde{\alpha}_\tau\}_{\tau\in\mathbb{Z}_+^2}$ and $\{\tilde{\beta}_\tau\}_{\tau\in\mathbb{Z}_+^2}$ as defined in section 1. Also just as Δ_τ and L_s are defined with respect to T, in a similar way $\tilde{\Delta}_\tau$ and $\tilde{L}_s$ are defined for $\tilde{T}$. As T is hyponormal so $L_s \geq 0\ \forall s \in \mathbb{Z}_+^2$. Also $\tilde{L}_s = L_s$ for $s < |k| - 2$ and $s > |k|$. So if we can show that $L_s \geq 0$ for $|k| - 2 \leq s \leq |k|$, then by Lemma 3.2 we can conclude that $\tilde{T}$ is weakly hyponormal.

For example if $k = (3, 2)$ then $\tilde{L}_3, \tilde{L}_4, \tilde{L}_5$ can be represented by the following weight diagram:

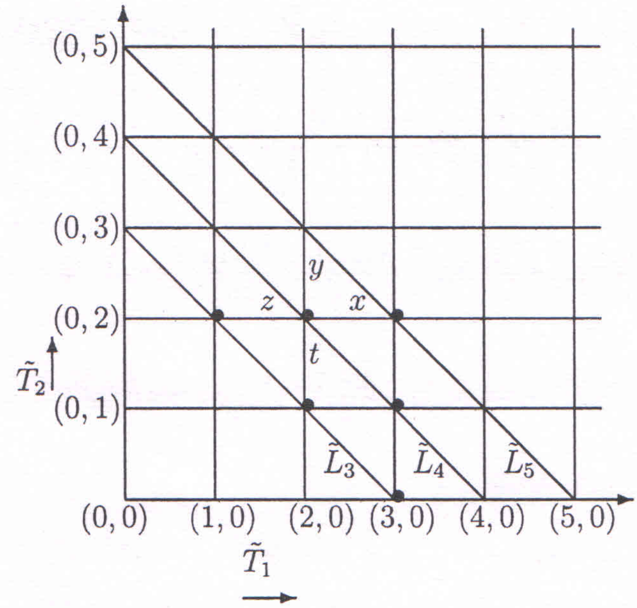

Figure 3

THEOREM 3.1 *Let $T = (T_1, T_2)$ be hyponormal with weight sequences $\{\alpha_\tau\}_{\tau\in Z_+^2}$ and $\{\beta_\tau\}_{\tau\in Z_+^2}$. Then for any $k \in \mathbb{Z}_+^2$, a slight perturbation of the weight α_k makes the perturbed shift $\tilde{T}$ weakly hyponormal (assuming that β_k, $\alpha_{k-\varepsilon_1}$ and $\beta_{k-\varepsilon_2}$ are also necessarily perturbed to preserve commutativity).*

Proof: Let $\quad g_0(x) := \tilde{a}_{(s,0)} + |\lambda|^2 \tilde{\beta}^2_{(s+1,0)} > 0.$

$$g_1(x) : = \det \begin{pmatrix} \tilde{a}_{(s,0)} + |\lambda|^2 \tilde{\beta}^2_{(s+1,0)} & \tilde{b}_{(s,0)} \\ \tilde{b}_{(s,0)} & \tilde{d}_{(s,0)} + \frac{\tilde{a}_{(s-1,1)}}{|\lambda|^2} \end{pmatrix}$$

$$= \frac{\tilde{a}_{(s-1,1)}}{|\lambda|^2} g_0(x) + \det \tilde{\Delta}_{(s,0)} + |\lambda|^2 \tilde{\beta}^2_{(s+1,0)} \tilde{d}_{(s,0)}$$

$$g_2(x) : = \det \begin{pmatrix} \tilde{a}_{(s,0)} + |\lambda|^2 \tilde{\beta}^2_{(s+1,0)} & \tilde{b}_{(s,0)} & 0 \\ \tilde{b}_{(s,0)} & \tilde{d}_{(s,0)} + \frac{\tilde{a}_{(s-1,1)}}{|\lambda|^2} & \frac{\tilde{b}_{(s-1,1)}}{|\lambda|^2} \\ 0 & \frac{\tilde{b}_{(s-1,1)}}{|\lambda|^2} & \frac{\tilde{d}_{(s-1,1)}}{|\lambda|^2} + \frac{\tilde{a}_{(s-2,2)}}{|\lambda|^4} \end{pmatrix}$$

$$= \frac{\tilde{\alpha}_{(s-2,2)}}{|\lambda|^4} g_1(x) + \frac{\tilde{d}_{(s-1,1)}}{|\lambda|^2} g_1(x) - \frac{\tilde{b}^2_{(s-1,1)}}{|\lambda|^4} g_0(x)$$

$$= \frac{\tilde{a}_{(s-2,2)}}{|\lambda|^4} g_1(x) + \det \tilde{\Delta}_{(s-1,1)} \frac{g_0(x)}{|\lambda|^4} + \frac{\tilde{d}_{(s-1,1)}}{|\lambda|^2} \left(\det \tilde{\Delta}_{(s,0)} + |\lambda|^2 \tilde{\beta}^2_{(s+1,0)} \tilde{d}_{(s,0)} \right)$$

$$= \frac{\tilde{a}_{(s-2,2)}}{|\lambda|^4} g_1(x) + \det \tilde{\Delta}_{(s-1,1)} \frac{g_0(x)}{|\lambda|^4} + \frac{\tilde{d}_{(s-1,1)}}{|\lambda|^2} \det \tilde{\Delta}_{(s,0)} + \tilde{d}_{(s-1,1)} \tilde{d}_{(s,0)} \tilde{\beta}^2_{(s+1,0)}$$

$$g_3(x) : = \det \begin{pmatrix} \tilde{a}_{(s,0)} + |\lambda|^2 \tilde{\beta}^2_{(s+1,0)} & \tilde{b}_{(s,0)} & 0 & 0 \\ \tilde{b}_{(s,0)} & \tilde{d}_{(s,0)} + \frac{\tilde{a}_{(s-1,1)}}{|\lambda|^2} & \frac{\tilde{b}_{(s-1,1)}}{|\lambda|^2} & 0 \\ 0 & \frac{\tilde{b}_{(s-1,1)}}{|\lambda|^2} & \frac{\tilde{d}_{(s-1,1)}}{|\lambda|^2} + \frac{\tilde{a}_{(s-2,2)}}{|\lambda|^4} & \frac{\tilde{b}_{(s-2,2)}}{|\lambda|^4} \\ 0 & 0 & \frac{\tilde{b}_{(s-2,2)}}{|\lambda|^4} & \frac{\tilde{d}_{(s-2,2)}}{|\lambda|^4} + \frac{\tilde{a}_{(s-3,3)}}{|\lambda|^6} \end{pmatrix}$$

$$= \frac{\tilde{a}_{(s-3,3)}}{|\lambda|^6} g_2(x) + \det \tilde{\Delta}_{(s-2,2)} \frac{g_1(x)}{|\lambda|^8} + \det \tilde{\Delta}_{(s-1,1)} \tilde{d}_{(s-2,2)} \frac{g_0(x)}{|\lambda|^8}$$

$$+ \det \tilde{\Delta}_{(s,0)} \frac{\tilde{d}_{(s-1,1)} \tilde{d}_{(s-2,2)}}{|\lambda|^6} + \tilde{\beta}^2_{(s+1,0)} \frac{\tilde{d}_{(s,0)} \tilde{d}_{(s-1,1)} \tilde{d}_{(s-2,2)}}{|\lambda|^4}.$$

Similarly,

$$g_4(x) := \frac{\tilde{a}_{(s-4,4)}}{|\lambda|^8} g_3(x) + \det\tilde{\Delta}_{(s-3,3)} \frac{g_2(x)}{|\lambda|^{12}} + \det\tilde{\Delta}_{(s-2,2)}\tilde{d}_{(s-3,3)} \frac{g_1(x)}{|\lambda|^{14}}$$

$$+ \det\tilde{\Delta}_{(s-1,1)}\tilde{d}_{(s-2,2)}\tilde{d}_{(s-3,3)} \frac{g_0(x)}{|\lambda|^{14}} + \det\tilde{\Delta}_{(s,0)} \frac{\tilde{d}_{(s-1,1)}\tilde{d}_{(s-2,2)}\tilde{d}_{(s-3,3)}}{|\lambda|^{12}}$$

$$+\tilde{\beta}^2_{(s+1,0)} \frac{\tilde{d}_{(s,0)}\tilde{d}_{(s-1,1)}\tilde{d}_{(s-2,2)}\tilde{d}_{(s-3,3)}}{|\lambda|^{10}}$$

For $j = 0, 1, \ldots, s+1$, let M_j denote the $(j+1) \times (j+1)$ leading submatrix of $\tilde{L}_s$. If $g_j(x) := \det M_j$ then

$$g_0(x) = \tilde{a}_{(s,0)} + |\lambda|^2 \tilde{\beta}^2_{(s+1,0)},$$

$$g_1(x) = \frac{\tilde{a}_{(s-1,1)}}{|\lambda|^2} g_0(x) + \det\tilde{\Delta}_{(s,0)} + |\lambda|^2 \tilde{\beta}^2_{(s+1,0)}\tilde{d}_{(s,0)}$$

For $j = 2, 3, \ldots, s$, $g_j(x)$ is

$$g_j(x) := \frac{\tilde{a}_{(s-j,j)}}{|\lambda|^{2j}} g_{j-1}(x) + \det\tilde{\Delta}_{(s-j+1,s-1)} \frac{g_{j-2}(x)}{|\lambda|^{4(j-1)}}$$

$$+ \sum_{l=2}^{j-1} (\det\tilde{\Delta}_{(s-j+l,\, j-l)}) \frac{\prod_{r=1}^{l-1} \tilde{d}_{(s-j+r,\, j-r)} g_{(j-l-1)}(x)}{|\lambda|^{4(j-l)+(l-1)(2j+l)}}$$

$$+ \det\tilde{\Delta}_{(s,0)} \frac{\prod_{r=1}^{j-1} \tilde{d}_{(s-r,\, r)}}{|\lambda|^{j(j-1)}} + |\lambda|^2 \tilde{\beta}^2_{(s+1,0)} \frac{\prod_{r=0}^{j-1} \tilde{d}_{(s-r,\, r)}}{|\lambda|^{j(j-1)}}$$

and $$g_{s+1}(x) := \frac{\tilde{\alpha}^2_{(0,s+1)}}{|\lambda|^{2(s+1)}} g_s(x) + \det\tilde{\Delta}_{(0,s)} \frac{g_{s-1}(x)}{|\lambda|^{4s}} + \frac{\tilde{d}_{(0,s)}}{|\lambda|^{2s}} \left(\det\tilde{\Delta}_{(1,s-1)} \frac{g_{s-2}(x)}{|\lambda|^{4(s-1)}} \right.$$

$$+ \sum_{l=2}^{s-1} (\det\tilde{\Delta}_{(l,\, s-l)}) \frac{\prod_{r=1}^{l-1} \tilde{d}_{(r,\, s-r)} g_{(s-l-1)}(x)}{|\lambda|^{4(s-l)+(l-1)(2s+l)}} + \det\tilde{\Delta}_{(s,0)} \frac{\prod_{r=1}^{s-1} \tilde{d}_{(s-r,\, r)}}{|\lambda|^{s(s-1)}}$$

$$\left. +|\lambda|^2 \tilde{\beta}^2_{(s+1,0)} \frac{\prod_{r=0}^{s-1} \tilde{d}_{(s-r,\, r)}}{|\lambda|^{s(s-1)}} \right).$$

At $x = \alpha_k$, we have $\det \tilde{\Delta}_\tau = \det \Delta_\tau \geq 0$ for all $\tau \in \mathbb{Z}_+^2$. Also as $g_0(\alpha_k) > 0$, hence for all $j = 1, \ldots, s$, we have

$$g_j(\alpha_k) \geq \frac{a_{(s-j,j)}}{|\lambda|^{2j}} g_{j-1}(\alpha_k) > 0.$$

Similarly

$$g_{s+1}(\alpha_k) = \frac{\alpha^2_{(0,s+1)}}{|\lambda|^{2(s+1)}} g_s(\alpha_k) > 0.$$

Thus by continuity of g_j there exists $\delta_k > 0$ such that $g_j(x) > 0$ for all $x \in (\alpha_k - \delta_k, \alpha_k + \delta_k)$, which implies that $\tilde{L}_s \geq 0$. So by lemma 3.2, $\tilde{T}$ is weakly hyponormal for any slight perturbation of α_k.

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