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Abstract The necessary and sufficient conditions (NASC) for subnormal backward extension of a 1-variable weighted shift was first given by Curto(Curto, 19990, Prof 8). Later an improved version of this result was given by Curto and Yoon (2006, Prof 1.5). In the seme paper, they have also given the NASC for subnormal back ward extension of a 2-variable weighted shift(Curto and Yoon, 2006, Prof 2.9).However, these results only deal with 1-step extension. In this paper we extend these results to 2-step extension, and following a similar technique we propose NASC for n-step backward extension of 1-variable and 2-variable weighted shifts. In the last section we show how these results can also be derived applying Schur product technique.

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BACK-STEP EXTENSION OF WEIGHTED SHIFTS

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ABSTRACT. The necessary and sufficient conditions (NASC) for subnormal backward extension of a 1-variable weighted shift was first given by Curto [1, Prop 8]. Later an improved version of this result was given by Curto and Yoon [4, Prop 1.5]. In the same paper, they have also given the NASC for subnormal backward extension of a 2-variable weighted shift [4, Prop 2.9]. However, these results only deal with 1-step extension. In this paper we extend these results to 2-step extension, and following a similar technique we propose NASC for n-step backward extension of 1-variable and 2-variable weighted shifts. In the last section we show how these results can also be derived applying Schur product technique.

1. INTRODUCTION

Let H be a separable infinite dimensional complex Hilbert space and $B(H)$ denote the algebra of bounded linear operators on H. For $A, B \in B(H)$, let $[A, B] := AB - BA$. We say that an n-tuple $\mathbf{T} = (T_1, \ldots, T_n)$ of operators on H is (jointly) hyponormal if the operator matrix $([T_j^*, T_i])_{i,j=1}^n$ is positive on the direct sum of n copies of H. The n-tuple $\mathbf{T}$ is said to be normal if $\mathbf{T}$ is commuting and each T_i is normal, and $\mathbf{T}$ is subnormal if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace. Clearly, normal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal.

For $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$, a bounded sequence of positive real numbers (called weights), let $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated unilateral weighted shift defined by $W_\alpha e_n = \alpha_n e_{n+1}$ $(\forall\, n \geq 0)$, where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis for $\ell^2(\mathbb{Z}_+)$. The moments of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \begin{cases} 1 & \text{if } k = 0 \\ \alpha_0^2 \ldots \alpha_{k-1}^2 & \text{if } k > 0 \end{cases}$$

It is easy to see that W_α is never normal and that it is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}$ $(\forall\, n \geq 0)$. Berger's theorem states that : W_α is subnormal if and only if there exists a probability measure η supported in $[0, \|W_\alpha\|^2]$ (called the Berger measure of W_α), with $\|W_\alpha\|^2 \in supp\eta$ such that $\gamma_k(\alpha) := \alpha_0^2 \ldots \alpha_{k-1}^2 = \int t^k d\eta(t)$ $(\forall\, k \geq 1)$.

Similarly, consider double indexed positive bounded sequences $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2)$, $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$ and let $\ell^2(\mathbb{Z}_+^2)$ be the Hilbert Space of square summable complex sequences indexed by $\mathbb{Z}_+^2$. We define the 2-variable weighted shift

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$\mathbf{T} = (T_1, T_2)$ by

$$T_1 e_{(k_1,k_2)} = \alpha_{(k_1,k_2)} e_{(k_1+1,k_2)}$$
$$T_2 e_{(k_1,k_2)} = \beta_{(k_1,k_2)} e_{(k_1,k_2+1)}$$

Clearly,

(1.1) $T_1T_2 = T_2T_1 \Longleftrightarrow \beta_{(k_1+1,k_2)}\alpha_{(k_1,k_2)} = \alpha_{(k_1,k_2+1)}\beta_{(k_1,k_2)}$, $\forall\, (k_1,k_2) \in \mathbb{Z}_+^2$.

For $\mathbf{k} \equiv (k_1,k_2) \in \mathbb{Z}_+^2$, the moment of (α,β) of order $\mathbf{k}$ is

$$\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}(\alpha,\beta) := \begin{cases} 1 & \text{if } \mathbf{k} = 0 \\ \alpha^2_{(0,0)} \cdots \alpha^2_{(k_1-1,0)} & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta^2_{(0,0)} \cdots \beta^2_{(0,k_2-1)} & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \beta^2_{(0,0)} \cdots \beta^2_{(0,k_2-1)}\alpha^2_{(0,k_2)} \cdots \alpha^2_{(k_1-1,k_2)} & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1 \end{cases}$$

Due to the commutativity condition (1.1), $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0,0)$ to (k_1,k_2). Moreover, $\mathbf{T} = (T_1,T_2)$ is subnormal if and only if there is a regular Borel probability measure μ defined on the 2-dimensional rectangle $R = [0,a_1] \times [0,a_2]$, $(a_i := \|T_i\|^2)$ such that $\gamma_{\mathbf{k}} = \int\int_R \mathbf{t}^{\mathbf{k}} d\mu(\mathbf{t}) := \int\int_R t_1^{k_1} t_2^{k_2} d\mu(t_1,t_2)$ $(\forall\, \mathbf{k} \in \mathbb{Z}_+^2)$.

Using the Berger's theorem, Curto and Yoon have given necessary and sufficient conditions for 1-step subnormal backward extension of the 1-variable and 2-variable weighted shifts [1], [4]. In this paper we give the necessary and sufficient conditions for 2-step subnormal backward extension of the 1-variable and 2-variable weighted shifts. In §5 it is shown how Schur product techniques can also be applied to establish the results proved in §4.

2. Some prior definitions and results

Definition 2.1. [4] Let μ and ν be two positive measures on $\mathbb{R}_+$. We say that $\mu \leq \nu$ on $X := \mathbb{R}_+$ if $\mu(E) \leq \nu(E)$ for all Borel subset $E \subseteq \mathbb{R}_+$; equivalently, $\mu \leq \nu$ if and only if $\int f d\mu \leq \int f d\nu$ for all $f \in C(X)$ such that $f \geq 0$ on $\mathbb{R}_+$.

Definition 2.2. [4] Let μ be the positive measures on $X \times Y \equiv \mathbb{R}_+ \times \mathbb{R}_+$, and assume $\frac{1}{t} \in L^1(\mu)$. The extremal measure μ_{ext} (which is a probability measure) on $X \times Y$ is given by $d\mu_{ext}(s,t) := (1-\delta_0(t))\frac{1}{t\,\|\frac{1}{t}\|_{L^1(\mu)}} d\mu(s,t)$.

Definition 2.3. [4] Given a measure μ on $X \times Y$, the marginal measure μ^X is given by $\mu^X := \mu\, o\, \pi_X^{-1}$, where $\pi_X : X \times Y \to X$ is the canonical projection on X. Thus $\mu^X(E) = \mu(E \times Y)$, for every $E \subseteq X$. If μ is a probability measure, then so is μ^X.

Lemma 2.4. [4] *Let μ be the Berger measure of 2-variable weighted shift $\mathbf{T}$ and let ξ be the Berger measure of the $shift(\alpha_{(0,0)}, \alpha_{(1,0)}, \ldots)$. Then $\xi = \mu^X$. As a consequence $\int\int f(s) d\mu(s,t) = \int f(s) d\mu^X(s)$ for all $f \in C(X)$.*

Corollary 2.5. [4] *Let μ be the Berger measure of a 2-variable weighted shift $\mathbf{T}$. For $j \geq 1$, let $d\mu_j(s,t) = \frac{1}{\gamma_{(0,j)}} t^j d\mu(s,t)$. Then the Berger measure of the $shift(\alpha_{(0,j)}, \alpha_{(1,j)}, \ldots)$ is $\xi_j = \mu_j^X$.*

Lemma 2.6. [4] *Let μ and ω be two measures on $X \times Y$, and assume that $\mu \leq \omega$. Then $\mu^X \leq \omega^X$.*

3. Back-step extension of 1-variable weighted shifts

In this section we propose the NASC for n-step subnormal backward extension of a 1-variable weighted shift. This will become obvious once we prove the NASC for 2-step backward extension. We begin the section by stating the 1-step subnormal backward extension of a 1-variable weighted shift.

Theorem 3.1. *(1-step backward extension)* [4] *Let T be a weighted shift whose restriction $T_M := T|_M$ to $M := \bigvee\{e_1, e_2, \dots\}$ is subnormal, with associated Berger measure μ_M. Then T is subnormal (with associated Berger measure μ) if and only if*
(i) $\frac{1}{t} \in L^1(\mu_M)$
(ii) $\alpha_0^2 \leq (\|\frac{1}{t}\|_{L^1(\mu_M)})^{-1}$
In this case, $d\mu(t) = \frac{\alpha_0^2}{t} d\mu_M(t) + (1 - \alpha_0^2 \|\frac{1}{t}\|_{L^1(\mu_M)}) d\delta_0(t)$ where δ_0 denotes Dirac measure at 0. In particular, T is never subnormal when $\mu_M(\{0\}) > 0$.

Theorem 3.2. *(2-step backward extension) Let T be a weighted shift whose restriction $T|_{M_2}$ to $M_2 := \bigvee\{e_2, e_3, \dots\}$ is subnormal, with associate Berger measure η_2. Then T is subnormal (with associate Berger measure η) if and only if*
(i) $\frac{1}{t^2} \in L^1(\eta_2)$
(ii) $\alpha_0^2\alpha_1^2 \leq (\|\frac{1}{t^2}\|_{L^1(\eta_2)})^{-1}$
(iii) $\alpha_1^2 = (\|\frac{1}{t}\|_{L^1(\eta_2)})^{-1}$
In this case, $d\eta(t) = \left(1 - \alpha_0^2\alpha_1^2 \|\frac{1}{t^2}\|_{L^1(\eta_2)}\right) d\delta_0(t) + \frac{\alpha_0^2\alpha_1^2}{t^2} d\eta_2(t)$, where δ_0 denotes the Dirac measure at 0. In particular, T is never subnormal if $\eta_2(\{0\}) > 0$.

Proof. $\Longrightarrow$) Assume that T is subnormal, so clearly $T|_{M_2}$ is subnormal. The moments of T and $T|_{M_2}$ are related by the equation

$$\gamma_k(T|_{M_2}) \equiv \alpha_2^2\alpha_3^2 \dots \alpha_{k+1}^2 = \frac{\gamma_{k+2}(T)}{\alpha_0^2\alpha_1^2}$$

so that for all $k \geq 0$,

$$\int t^k d\eta_2(t) = \frac{1}{\alpha_0^2\alpha_1^2} \int t^{k+2} d\eta(t)$$

that is, $d\eta_2(t) = \frac{t^2}{\alpha_0^2\alpha_1^2} d\eta(t)$. Let $\eta(0) = \lambda$, $(\lambda \geq 0)$, so it follows at once that

$$\begin{aligned} d\eta(t) &= \lambda\, d\delta_0(t) + \frac{\alpha_0^2\alpha_1^2}{t^2} d\eta_2(t) \\ \Rightarrow \int d\eta(t) &= \lambda \int d\delta_0(t) + \alpha_0^2\alpha_1^2 \int \frac{1}{t^2} d\eta_2(t) \\ \Rightarrow \quad 1 &= \lambda + \alpha_0^2\alpha_1^2 \left\| \frac{1}{t^2} \right\|_{L^1(\eta_2)} \end{aligned} \tag{3.1}$$

that is $\alpha_0^2\alpha_1^2 \|\frac{1}{t^2}\|_{L^1(\eta_2)} = 1 - \lambda \leq 1$, also $\frac{1}{t^2} \in L^1(\eta_2)$. Also, substituting the value of λ in (3.1), we have $d\eta(t) = (1 - \alpha_0^2\alpha_1^2 \|\frac{1}{t^2}\|_{L^1(\eta_2)})\, d\delta_0(t) + \frac{\alpha_0^2\alpha_1^2}{t^2} d\eta_2(t)$.
Again, suppose η_1 is the measure associated with the shift $T|_{M_1}$, where $M_1 := \bigvee\{e_1, e_2, \dots\}$. Then by Theorem 3.1, subnormality of $T|_{M_1}$ and $T|_{M_2}$ will imply that $\frac{1}{t} \in L^1(\eta_2)$, $\alpha_1^2 \leq (\|\frac{1}{t}\|_{L^1(\eta_2)})^{-1}$ and $d\eta_1(t) = \eta_1(0)\, d\delta_0(t) + \frac{\alpha_1^2}{t} d\eta_2(t)$, where $\eta_1(0) = (1 - \alpha_1^2 \|\frac{1}{t}\|_{L^1(\eta_2)})$.
Now, suppose $\quad \alpha_1^2 < (\|\frac{1}{t}\|_{L^1(\eta_2)})^{-1} \Rightarrow \eta_1(0) > 0$

Which is a contradiction to the initial assumption that T is subnormal. Therefore, $\alpha_1^2 = (\|\frac{1}{t}\|_{L^1(\eta_2)})^{-1}$.
$\Longleftarrow$) Let (i),(ii),(iii) hold and

$$d\eta(t) = (1 - \alpha_0^2\alpha_1^2\left\|\frac{1}{t^2}\right\|_{L^1(\eta_2)})d\delta_0(t) + \frac{\alpha_0^2\alpha_1^2}{t^2}\, d\eta_2(t) \tag{3.2}$$

For $k = 0$,

$$\int d\eta(t) = (1 - \alpha_0^2\alpha_1^2\left\|\frac{1}{t^2}\right\|_{L^1(\eta_2)})\int d\delta_0(t) + \alpha_0^2\alpha_1^2\int \frac{1}{t^2}\, d\eta_2(t)$$
$$\Rightarrow \int d\eta(t) = (1 - \alpha_0^2\alpha_1^2\left\|\frac{1}{t^2}\right\|_{L^1(\eta_2)}) + \alpha_0^2\alpha_1^2\left\|\frac{1}{t^2}\right\|_{L^1(\eta_2)}$$
$$\Rightarrow \int d\eta(t) = 1 = \gamma_0(T)$$

For $k = 1$, using (3.2) we have

$$\int t\, d\eta(t) = \int \frac{\alpha_0^2\alpha_1^2}{t}\, d\eta_2(t) = \alpha_0^2\,\alpha_1^2\left\|\frac{1}{t}\right\|_{L^1(\eta_2)} = \alpha_0^2 \qquad (\texttt{Since, } \alpha_1^2\left\|\frac{1}{t}\right\|_{L^1(\eta_2)} = 1)$$
$$= \gamma_1(T)$$

For $k \geq 2$,

$$\int t^k\, d\eta(t) = \alpha_0^2\alpha_1^2\int t^{k-2} d\eta_2(t) = \alpha_0^2\alpha_1^2\gamma_{k-2}(T|_{M_2}) = \gamma_k(T)$$

Thus T is subnormal with Berger measure η.
Also if $\eta_2(0) > 0$ will imply that $T|_{M_1}$ is not subnormal, therefore T is not subnormal.
□

A similar argument will yield the NASC for 3-step backward extension, and in general, the n-step subnormal backward extension of a 1-variable weighted shift will be as follows:

Theorem 3.3. *(n-step backward extension) For $n \geq 2$, let T be a weighted shift whose restriction $T|_{M_n}$ to $M_n := \bigvee\{e_n, e_{n+1}, \dots\}$ is subnormal, with associate Berger measure η_n. Then T is subnormal (with associate Berger measure η) if and only if*
(i) $\frac{1}{t^n} \in L^1(\eta_n)$
(ii) $\alpha_0^2\alpha_1^2 \dots \alpha_{n-1}^2 \leq (\|\frac{1}{t^n}\|_{L^1(\eta_n)})^{-1}$
(iii) $\alpha_i^2\alpha_{i+1}^2 \cdots \alpha_{n-1}^2 = (\|\frac{1}{t^{n-i}}\|_{L^1(\eta_n)})^{-1}$ for $1 \leq i \leq n-1$.
In this case, $d\eta(t) = (1-\alpha_0^2\alpha_1^2 \dots \alpha_{n-1}^2\|\frac{1}{t^n}\|_{L^1(\eta_n)})d\delta_0(t) + \frac{\alpha_0^2\alpha_1^2\dots\alpha_{n-1}^2}{t^n}\, d\eta_n(t)$, where δ_0 denotes the Dirac measure at 0. In particular, T is never subnormal if $\eta_n(\{0\}) > 0$.

Corollary 3.4. *Let T be a subnormal weighted shift and for $j \geq 2$, let $M_j := \bigvee\{e_j, e_{j+1}, \dots\}$. Let η_j denote the Berger measure of $T|_{M_j}$. Then $\alpha_1, \alpha_2, \dots, \alpha_{j-1}$ is completely determined by η_j that is, $\alpha_{j-1}^2 = (\|\frac{1}{t}\|_{L^1(\eta_j)})^{-1}$.*
Also, if T is subnormal then condition (iii) of Theorem 3.3 imply that

$$\left\|\frac{1}{t^{n-i}}\right\|_{L^1(\eta_n)} = \left\|\frac{1}{t}\right\|_{L^1(\eta_{i+1})}\left\|\frac{1}{t}\right\|_{L^1(\eta_{i+2})}\cdots\left\|\frac{1}{t}\right\|_{L^1(\eta_n)} \quad \textit{for } 1 \leq i \leq n.$$

4. Back-step extension of 2-variable weighted shifts

Lemma 4.1. *Let μ be a positive measure on $\mathbb{R}_+ \times \mathbb{R}_+$ such that $\mu(E \times \{0\}) = 0$ for all Borel sets $E \subseteq \mathbb{R}_+$. For $n \geq 1$, let $\frac{1}{t^n} \in L^1(\mu)$. Then the extremal measure $\mu_{(ext)^n}$ on $\mathbb{R}_+ \times \mathbb{R}_+$ is given by*

$$d\mu_{(ext)^n}(s,t) := \frac{1-\delta_0(t)}{t^n \|\frac{1}{t^n}\|_{L^1(\mu)}}\, d\mu(s,t).$$

Proof. For $n = 1$, $\frac{1}{t} \in L^1(\mu)$ and we have

$$d\mu_{(ext)}(s,t) := \frac{1-\delta_0(t)}{t\|\frac{1}{t}\|_{L^1(\mu)}}\, d\mu(s,t) \text{ (by Definition 2.2)}$$

Suppose result is true for n i.e.,

$$d\mu_{(ext)^n}(s,t) := \frac{1-\delta_0(t)}{t^n \|\frac{1}{t^n}\|_{L^1(\mu)}}\, d\mu(s,t).$$

Let $\frac{1}{t^{n+1}} \in L^1(\mu)$. Then,

$$\begin{aligned}
\iint \frac{1}{t}\, d\mu_{(ext)^n}(s,t) &= \iint \frac{1-\delta_0(t)}{t^{n+1}\|\frac{1}{t^n}\|_{L^1(\mu)}}\, d\mu(s,t) \\
&= \iint \frac{1}{t^{n+1}\|\frac{1}{t^n}\|_{L^1(\mu)}}\, d\mu(s,t) \\
&\quad (\text{since } \mu(E \times \{0\}) = 0,\ \forall E \subseteq \mathbb{R}_+) \\
&= \frac{\|\frac{1}{t^{n+1}}\|_{L^1(\mu)}}{\|\frac{1}{t^n}\|_{L^1(\mu)}} < \infty
\end{aligned}$$

$$\Rightarrow \frac{1}{t} \in L^1(\mu)_{(ext)^n} \quad \text{and} \quad \left\|\frac{1}{t}\right\|_{L^1(\mu)_{(ext)^n}} \left\|\frac{1}{t^n}\right\|_{L^1(\mu)} = \left\|\frac{1}{t^{n+1}}\right\|_{L^1(\mu)}.$$

Now as $\frac{1}{t} \in L^1(\mu)_{(ext)^n}$, so by Definition 2.2,

$$\begin{aligned}
d\mu_{(ext)^{n+1}}(s,t) &:= \frac{1-\delta_0(t)}{t\|\frac{1}{t}\|_{L^1(\mu)_{(ext)^n}}}\, d(\mu)_{(ext)^n}(s,t) \\
&= \frac{1-\delta_0(t)}{t^{n+1}\|\frac{1}{t}\|_{L^1(\mu)_{(ext)^n}}\|\frac{1}{t^n}\|_{L^1(\mu)}}\, d\mu(s,t) \\
&= \frac{1-\delta_0(t)}{t^{n+1}\|\frac{1}{t^{n+1}}\|_{L^1(\mu)}}\, d\mu(s,t)
\end{aligned}$$

Thus the result hold (by induction) for all $n = 1, 2, \ldots$ □

Theorem 4.2. *(1-step backward extension) [4] Let $\boldsymbol{T} = (T_1, T_2)$ be a 2-variable weighted shift and M be the subspace of $\ell^2(\mathbb{Z}_+^2)$ associated to indices $\boldsymbol{k} = (k_1, k_2)$ with $k_2 \geq 1$. Let $\boldsymbol{T}_M := \boldsymbol{T}|_M$ be subnormal with associated Berger measure μ_M and let $W_0 := shift(\alpha_{00}, \alpha_{10}, \ldots)$ is subnormal with associated Berger measure ν. Then $\boldsymbol{T}$ is subnormal if and only if*
(i) $\frac{1}{t} \in L^1(\mu_M)$
(ii) $\beta_{00}^2 \leq (\|\frac{1}{t}\|_{L^1(\mu_M)})^{-1}$
(iii) $\beta_{00}^2 \|\frac{1}{t}\|_{L^1(\mu_M)} (\mu_M)_{ext}^X \leq \nu$

Moreover, if $\beta_{00}^2 \|\frac{1}{t}\|_{L^1(\mu_M)} = 1$, then $(\mu_M)_{ext}^X = \nu$. In the case when $\mathbf{T}$ is subnormal, the Berger measure μ of $\mathbf{T}$ is given by
$d\mu(s,t) = \beta_{00}^2 \|\frac{1}{t}\|_{L^1(\mu_M)} d(\mu_M)_{ext}(s,t) + (d\nu(s) - \beta_{00}^2 \|\frac{1}{t}\|_{L^1(\mu_M)} d(\mu_m)_{ext}^X(s)) d\delta_0(t)$

Theorem 4.3. *(2-step backward extension) Let $\mathbf{T}$ be a 2-variable weighted shift with the weight sequences α and β. Assume that $\mathbf{T}|_{M_2}$ the restriction of $\mathbf{T}$ to $M_2 := \bigvee\{e_{(k_1,k_2)} : k_2 \geq 2\}$ is subnormal with associated Berger measure μ_2. Let $W_0 := shift(\alpha_{(0,0)}, \alpha_{(1,0)}, \ldots)$ and $W_1 := shift(\alpha_{(0,1)}, \alpha_{(1,1)}, \ldots)$ be subnormal with associated measures ξ_0 and ξ_1 respectively. Then $\mathbf{T}$ is subnormal with associated Berger measure μ if and only if*
(i) $\frac{1}{t^2} \in L^1(\mu_2)$
(ii) $\beta_{(0,0)}^2 \beta_{(0,1)}^2 \|\frac{1}{t^2}\|_{L^1(\mu_2)} \leq 1$
(iii) $\beta_{(0,0)}^2 \beta_{(0,1)}^2 \|\frac{1}{t^2}\|_{L^1(\mu_2)} (\mu_2)_{(ext)^2}^X \leq \xi_0$
(iv) $\beta_{(0,1)}^2 \|\frac{1}{t}\|_{L^1(\mu_2)} = 1$
(v) $(\mu_2)_{ext}^X = \xi_1$
Moreover, if $\beta_{(0,0)}^2 \beta_{(0,1)}^2 \|\frac{1}{t^2}\|_{L^1(\mu_2)} = 1$, then $(\mu_2)_{(ext)^2}^X = \xi_0$. In the case when $\mathbf{T}$ is subnormal, the Berger measure μ of $\mathbf{T}$ is given by,

$$\mu = \beta_{(0,0)}^2 \beta_{(0,1)}^2 \left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} (\mu_2)_{(ext)^2} + \left(\xi_0 - \beta_{(0,0)}^2 \beta_{(0,1)}^2 \left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} d(\mu_2)_{(ext)^2}^X\right) \times \delta_0$$

Proof. $\Longrightarrow$) Let $\mathbf{T}$ be subnormal. Then $\mathbf{T}|_{M_1}$ and $\mathbf{T}|_{M_2}$ are also subnormal with the corresponding Berger measures μ_1 and μ_2 respectively. The moments are related as follows:

$$\gamma_{(k_1,k_2+1)}(\mathbf{T}) = \beta_{(0,0)}^2 \gamma_{(k_1,k_2)}(\mathbf{T}|_{M_1})$$
$$\gamma_{(k_1,k_2+2)}(\mathbf{T}) = \beta_{(0,0)}^2 \beta_{(0,1)}^2 \gamma_{(k_1,k_2)}(\mathbf{T}|_{M_2})$$

Therefore, the subnormality of $\mathbf{T}$, $\mathbf{T}|_{M_1}$ and $\mathbf{T}|_{M_2}$ imply that

$$t\, d\mu(s,t) = \beta_{(0,0)}^2 d\mu_1(s,t) \tag{4.1}$$
$$t^2\, d\mu(s,t) = \beta_{(0,0)}^2 \beta_{(0,1)}^2 d\mu_2(s,t) \tag{4.2}$$

Therefore, $\mu_1(E \times \{0\}) = 0$, $\mu_2(E \times \{0\}) = 0$, $\forall\, E \subseteq \mathbb{R}_+$.
Now,

$$\begin{aligned} \iint \frac{1}{t^2} d\mu_2(s,t) = \iint_{t>0} \frac{1}{t^2} d\mu_2(s,t) &= \frac{1}{\beta_{(0,0)}^2 \beta_{(0,1)}^2} \iint_{t>0} d\mu(s,t) \\ &= \frac{1}{\beta_{(0,0)}^2 \beta_{(0,1)}^2} \mu(t>0) \\ &\leq \frac{1}{\beta_{(0,0)}^2 \beta_{(0,1)}^2} \end{aligned} \tag{4.3}$$

So, $\frac{1}{t^2} \in L^1(\mu_2)$ and $\beta_{(0,0)}^2 \beta_{(0,1)}^2 \|\frac{1}{t^2}\|_{L^1(\mu_2)} \leq 1$, which establishes (i) and (ii).

For arbitrary Borel sets $E \subseteq \mathbb{R}_+$ and $F \subseteq \mathbb{R}_+$, we have

$$
\begin{aligned}
&\beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)}(\mu_2)_{(ext)^2}(E\times F)\\
&\qquad=\beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)}\iint_{E\times F} d(\mu_2)_{(ext)^2}(s,t)\\
&\qquad=\beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)}\iint_{E\times F}(1-\delta_0(t))\frac{1}{t^2\,\|\frac{1}{t^2}\|_{L^1(\mu_2)}}\,d\mu_2(s,t)\\
&\qquad=\iint_{E\times(F\setminus\{0\})}\beta^2_{(0,0)}\beta^2_{(0,1)}\frac{1}{t^2}\,d\mu_2(s,t)\\
&\qquad=\iint_{E\times(F\setminus\{0\})} d\mu(s,t)\\
&\qquad=\mu(E\times(F\setminus\{0\}))\leq\mu\,(E\times F) \qquad (4.4)
\end{aligned}
$$

and by Lemmas 2.6 and 2.4, $\beta^2_{(0,0)}\beta^2_{(0,1)}\|\frac{1}{t^2}\|_{L^1(\mu_2)}(\mu_2)^X_{(ext)^2}\leq\mu^X=\xi_0$.

If $\beta^2_{(0,0)}\beta^2_{(0,1)}\|\frac{1}{t^2}\|_{L^1(\mu_2)}=1$ then by (4.3) $\mu(t>0)=1$, and so $\mu(E\times(F\setminus\{0\}))=\mu(E\times F)$. Therefore, from (4.4) we get $(\mu_2)_{(ext)^2}=\mu\Rightarrow(\mu_2)^X_{(ext)^2}=\xi_0$.

Again,

$$
\begin{aligned}
\left\|\frac{1}{t}\right\|_{L^1(\mu_2)}&=\iint\frac{1}{t}\,d\mu_2(s,t)=\frac{1}{\beta^2_{(0,0)}\beta^2_{(0,1)}}\iint t\,d\mu(s,t)\\
&=\frac{\gamma_{(0,1)}(\mathbf{T})}{\beta^2_{(0,0)}\beta^2_{(0,1)}}=\frac{\beta^2_{(0,0)}}{\beta^2_{(0,0)}\beta^2_{(0,1)}}=\frac{1}{\beta^2_{(0,1)}}
\end{aligned}
$$

which gives $\beta^2_{(0,1)}\|\frac{1}{t}\|_{L^1(\mu_2)}=1$, proving (iv).

Since $\mathbf{T}|_{M_1}$ is a 1-step subnormal extension of $\mathbf{T}|_{M_2}$, and also $\beta^2_{(0,1)}\|\frac{1}{t}\|_{L^1(\mu_2)}=1$, so by Theorem 4.2, we have $\xi_1=(\mu_2)^X_{ext}$.

Finally from (4.2) we have $t^2\,d\mu(s,t)=\beta^2_{(0,0)}\beta^2_{(0,1)}d\mu_2(s,t)$. So if $\mu(s,0)=\lambda(s)$ then

$$
\begin{aligned}
&d\mu(s,t)=d\lambda(s)\,d\delta_0(t)+\frac{\beta^2_{(0,0)}\beta^2_{(0,1)}}{t^2}d\mu_2(s,t)\\
&\Rightarrow d\mu(s,t)=d\lambda(s)\,d\delta_0(t)+\beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)}d(\mu_2)_{(ext)^2}(s,t)\\
&\Rightarrow\iint d\mu(s,t)=\int d\lambda(s)\int d\delta_0(t)+\beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)}\iint d(\mu_2)_{(ext)^2}(s,t)\\
&\Rightarrow\int d\mu^X(s)=\int d\lambda(s)+\beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)}\int d(\mu_2)^X_{(ext)^2}(s)\\
&\Rightarrow\int d\xi_0(s)=\int d\lambda(s)+\beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)}\int d(\mu_2)^X_{(ext)^2}(s)\\
&\Rightarrow d\xi_0(s)=d\lambda(s)+\beta^2_{(0,0)}\beta^2_{(0,1)}\|\frac{1}{t^2}\|_{L^1(\mu_2)}d(\mu_2)^X_{(ext)^2}(s)
\end{aligned}
$$

Therefore,

$$d\mu(s,t) = \left(d\xi_0(s) - \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} d(\mu_2)^X_{(ext)^2}(s)\right) d\delta_0(t) + \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} d(\mu_2)_{(ext)^2}(s,t).$$

$\Longleftarrow$) Conditions (iv) and (v) imply that T_{M_1} is subnormal with measure μ_1 such that $\mu_1(E\times\{0\}) = 0$ for all Borel sets $E \subseteq \mathbb{R}_+$.
Given conditions (i) to (v), let

$$\mu(s,t) := \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} (\mu_2)_{(ext)^2}(s,t) + \left(\xi_0(s) - \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} (\mu_2)^X_{(ext)^2}(s)\right)\delta_0(t).$$

If $\beta^2_{(0,0)}\beta^2_{(0,1)}\|\frac{1}{t^2}\|_{L^1(\mu_2)} = 1$ then total mass of the second summand is zero, and so $\mu := (\mu_2)_{(ext)^2}$.

For $j = 0$,

$$\begin{aligned}\iint s^i\, d(\mu)(s,t) &= \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} \iint s^i\, d(\mu_2)_{(ext)^2}(s,t)\\ &\quad + \int s^i d\xi_0(s) - \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} \int d(\mu_2)^X_{(ext)^2}(s)\\ &= \int s^i d\xi_0(s) \text{ (using Lemma 2.4)}\\ &= \gamma_{(i,0)}(\mathbf{T})\end{aligned}$$

For $j = 1$,

$$\begin{aligned}\iint s^i t\, d(\mu)(s,t) &= \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} \iint s^i t\, d(\mu_2)_{(ext)^2}(s,t)\\ &= \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} \iint s^i t \frac{(1-\delta_0(t))}{t\,\|\frac{1}{t}\|_{L^1(\mu_2)_{ext}}}\, d(\mu_2)_{ext}(s,t)\\ &= \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t}\right\|_{L^1(\mu_2)} \iint s^i(1-\delta_0(t))\, d(\mu_2)_{ext}(s,t)\\ &= \beta^2_{(0,0)} \int s^i\, d(\mu_2)^X_{ext}(s) \text{ (using } (iv))\\ &= \beta^2_{(0,0)} \int s^i\, d\xi_1(s) = \beta^2_{(0,0)}\alpha^2_{(0,1)}\cdots\alpha^2_{(i-1,1)}\\ &= \gamma_{(i,1)}(\mathbf{T})\end{aligned}$$

For $j > 1$,

$$\begin{aligned}\iint s^i t^j d(\mu)(s,t) &= \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} \iint s^i t^j d(\mu_2)_{(ext)^2}(s,t) \\ &= \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} \iint s^i t^j (1-\delta_0(t))\frac{1}{t^2\,\|\frac{1}{t^2}\|_{L^1(\mu_2)}}\,d\mu_2(s,t) \\ &= \beta^2_{(0,0)}\beta^2_{(0,1)} \iint s^i t^{j-2} d\mu_2(s,t) \\ &= \beta^2_{(0,0)}\beta^2_{(0,1)}\gamma_{(i,j-2)}(\mathbf{T}|_{M_2}) = \gamma_{(i,j)}(\mathbf{T})\end{aligned}$$

Hence, it follows that $\mathbf{T}$ is subnormal with Berger measure μ.

□

Theorem 4.4. *(n-step subnormal backward extension of a 2-variable weighted shift) Let $\boldsymbol{T} = (T_1, T_2)$ be a 2-variable weighted shift with double indexed weight sequences $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}^2_+}$ and $\beta = \{\beta_k\}_{k\in\mathbb{Z}^2_+}$. For $n \geq 1$, let M_n be the subspace associated to the indices $\mathbf{k} = (k_1, k_2)$ with $k_2 \geq n$. Assume that $\boldsymbol{T}|_{M_n}$ is subnormal with associated Berger measure μ_n. For $0 \leq i \leq n-1$ let $W_i := shift\{\alpha_{(0,i)}, \alpha_{(1,i)}, \dots\}$ be subnormal with associated Berger measures ξ_i respectively. Then $\boldsymbol{T}$ is subnormal if and only if*
(i) $\frac{1}{t^n} \in L^1(\mu_n)$
(ii) $\prod_{j=0}^{n-1}\beta^2_{(0,j)}\|\frac{1}{t^n}\|_{L^1(\mu_n)} \leq 1$
(iii) $\prod_{j=0}^{n-1}\beta^2_{(0,j)}\|\frac{1}{t^n}\|_{L^1(\mu_n)}(\mu_n)^X_{(ext)^n} \leq \xi_0$
(iv) $\prod_{j=i}^{n-1}\beta^2_{(0,j)}\|\frac{1}{t^{n-i}}\|_{L^1(\mu_n)} = 1$ *for* $1 \leq i \leq n-1$
(v) $(\mu_n)^X_{(ext)^i} = \xi_{n-i}$ *for* $1 \leq i \leq n-1$
Moreover, if $\prod_{j=0}^{n-1}\beta^2_{(0,j)}\|\frac{1}{t^n}\|_{L^1(\mu_n)} = 1$, *then* $(\mu_n)^X_{(ext)^n} = \xi_0$. *In the case when $\boldsymbol{T}$ is subnormal, the Berger measure μ of $\boldsymbol{T}$ is given by,*

$$\mu = \prod_{j=0}^{n-1}\beta^2_{(0,j)}\left\|\frac{1}{t^n}\right\|_{L^1(\mu_n)}(\mu_n)_{(ext)^n} + \left(\xi_0 - \prod_{j=0}^{n-1}\beta^2_{(0,j)}\left\|\frac{1}{t^n}\right\|_{L^1(\mu_n)}(\mu_n)^X_{(ext)^n}\right)\times\delta_0$$

The proof being similar to that of Theorem 4.3 is omitted.

5. Derivation of above results using Schur product techniques

In this section we show that the above results can also be derived using Schur product techniques. To show the results for 2-variable case, we used the techniques of Curto-Park results as shown in paper [2] for 1-variable case.

Definition 5.1. [2] Let $\alpha := \{\alpha_n\}_{n=0}^{\infty}$ and $\beta := \{\beta_n\}_{n=0}^{\infty}$. The Schur product of α and β is defined by $\alpha\beta := \{\alpha_n\beta_n\}_{n=0}^{\infty}$.

Definition 5.2. [2] Given integers i and ℓ $(\ell \geq 1)$ and for $0 \leq i \leq \ell - 1$, consider the decomposition $H \equiv \ell^2(\mathbb{Z}_+) = \bigoplus_{j=0}^{\infty}\{e_j\}$, define $H_i := \bigoplus_{j=0}^{\infty}\{e_{\ell j+i}\}$ and for the weight sequence α, define $\alpha(\ell : i) := \left\{\prod_{m=0}^{\ell-1}\alpha_{\ell j+i+m}\right\}_{j=0}^{\infty}$.

Proposition 5.3. [2] *W_α^ℓ is unitarily equivalent to $\bigoplus_{i=0}^{\ell-1} W_{\alpha(\ell:i)}$.*

Theorem 5.4. [2] *Let W_α be a weighted shift whose restriction $W_\alpha|_M$ to $M := \bigvee\{e_1, e_2, \dots\}$ is subnormal. Then the following are equivalent*

(1) W_α^ℓ is k-hyponormal.

(2) $W_{\alpha(\ell:0)}$ is k-hyponormal.

Definition 5.5. [6] $T = (T_1, \dots, T_N)$, where each T_j acts on a Hilbert space H, is said to be unitarily equivalent to $S = (S_1, \dots, S_N)$, where each S_j acts on a Hilbert space K, if there exists a unitary operator $\mathcal{U} : H \to K$ such that $\mathcal{U}^* S_j \mathcal{U} = T_j$ for $1 \le j \le N$.
For $L = (l, m)$ and $I = (i, j)$ in $\mathbb{Z}_+^2$, let $H_I := \bigvee_{(k_1,\, k_2) \in \mathbb{Z}_+^2} \{e_{(i+lk_1,\, j+mk_2)}\}$
In the sequel, we choose $l, m \ge 1$ and $0 \le i \le l-1$, $0 \le j \le m-1$.

Explanation:

If $L = (1,1)$ then $i = j = 0$ and so $H_I = H_{(0,\,0)} = \bigvee_{(k_1,\, k_2) \in \mathbb{Z}_+^2} \{e_{(k_1,\, k_2)}\} = \ell^2(\mathbb{Z}_+^2)$.
If $L = (2,1)$ then $0 \le i \le 1$ and $j = 0$. As $H_{(0,\,0)} = \bigvee_{(k_1,\, k_2) \in \mathbb{Z}_+^2} \{e_{(2k_1,\, k_2)}\}$ and $H_{(1,\,0)} = \bigvee_{(k_1,\, k_2) \in \mathbb{Z}_+^2} \{e_{(1+2k_1,\, k_2)}\}$. So, $\ell^2(\mathbb{Z}_+^2) = H_{(0,\,0)} \bigoplus H_{(1,\,0)}$.
Thus, $\ell^2(\mathbb{Z}_+^2) = \bigoplus_{j=0}^{m-1} \bigoplus_{i=0}^{l-1} H_{(i,\,j)}$.

Definition 5.6. For $\delta = (\delta_{(k_1,k_2)}) \in \ell^\infty(\mathbb{Z}_+^2)$ define $P_{(L:I)} : \ell^\infty(\mathbb{Z}_+^2) \to \ell^\infty(\mathbb{Z}_+^2)$ as $P_{(L:I)}(\delta) = \left\{\prod_{p=0}^{l-1} \delta_{(i+k_1l+p,\, j+k_2m)}\right\}_{(k_1,\, k_2) \in \mathbb{Z}_+^2}$ and $Q_{(L:I)} : \ell^\infty(\mathbb{Z}_+^2) \to \ell^\infty(\mathbb{Z}_+^2)$ as $Q_{(L:I)}(\delta) = \left\{\prod_{p=0}^{m-1} \delta_{(i+k_1l,\, j+k_2m+p)}\right\}_{(k_1,\, k_2) \in \mathbb{Z}_+^2}$.

Definition 5.7. Define S_1 and S_2 on $\ell^\infty(\mathbb{Z}_+^2)$ as $(S_1\gamma)(k_1, k_2) = \gamma(k_1+1, k_2)$ and $(S_2\gamma)(k_1, k_2) = \gamma(k_1, k_2+1)$ for $\gamma = (\gamma_{(k_1,k_2)}) \in \ell^\infty(\mathbb{Z}_+^2)$. Note $S_1S_2 = S_2S_1$.

Proposition 5.8. *$P_{(L:(0,0))}\, S_1^i S_2^j = P_{(L:I)}$ and $Q_{(L:(0,0))}\, S_1^i S_2^j = Q_{(L:I)}$.*

Proof. $S_1^i S_2^j(\delta)(k_1, k_2) = \delta(k_1+i, k_2+j) = \tilde{\delta}(k_1, k_2)$ (say)

$$\begin{aligned} \textit{Then} \qquad P_{(L:(0,0))}\, S_1^i S_2^j(\delta)(k_1, k_2) &= P_{(L:(0,0))}\tilde{\delta}(k_1, k_2) \\ &= \prod_{p=0}^{l-1} \tilde{\delta}(k_1 l + p,\, k_2 m) \\ &= \prod_{p=0}^{l-1} \delta(i + k_1 l + p,\, j + k_2 m) \\ &= P_{(L:I)}(\delta)(k_1, k_2) \end{aligned}$$

Similarly, $Q_{(L:(0,0))}\, S_1^i S_2^j = Q_{(L:I)}$. □

Given, $\alpha = \{\alpha_{(k_1,\, k_2)}\} \in \ell^\infty(\mathbb{Z}_+^2)$ and $\beta = \{\beta_{(k_1,\, k_2)}\} \in \ell^\infty(\mathbb{Z}_+^2)$, let $T = (T_1, T_2)$ be 2-variable weighted shift with weight sequences α and β, defined as

$$T_1\, e_{(k_1,k_2)} = \alpha_{(k_1,k_2)} e_{(k_1+1,\, k_2)}$$
$$T_2\, e_{(k_1,k_2)} = \beta_{(k_1,k_2)} e_{(k_1,\, k_2+1)}$$

Let $T_{(L:I)} = \big((T_{(L:I)})_1, (T_{(L:I)})_2\big)$ be 2-variable weighted shift with weight sequences $P_{(L:I)}(\alpha)$ and $Q_{(L:I)}(\beta)$, defined as

$$(T_{(L:I)})_1\, e_{(k_1,k_2)} = \left\{\prod_{p=0}^{l-1} \alpha_{(i+k_1l+p,\, j+k_2m)}\right\} e_{(k_1+1,\, k_2)}$$

$$(T_{(L:I)})_2\, e_{(k_1,k_2)} = \left\{\prod_{p=0}^{m-1} \beta_{(i+k_1l,\, j+k_2m+p)}\right\} e_{(k_1,\, k_2+1)}$$

Now, $T^L := (T_1^l,\, T_2^m)$ and $T^L|_{H_I} := (T_1^l|_{H_I},\, T_2^m|_{H_I})$.

Proposition 5.9. *T^L is unitarily equivalent to $\bigoplus_{j=0}^{m-1} \bigoplus_{i=0}^{l-1} T_{(L:I)}$.*

Proof. Define $\mathcal{U} : \ell^2(\mathbb{Z}_+^2) \to H_I$ as $\mathcal{U}\, e_{(k_1,\, k_2)} = e_{(i+k_1l,\, j+k_2m)}$. Then for $e_{(k_1,\, k_2)} \in H_I$, $\mathcal{U}^*\, e_{(k_1,\, k_2)} = e_{(\frac{k_1-i}{l},\, \frac{k_2-j}{m})}$ and so $\mathcal{U}\mathcal{U}^* = I = \mathcal{U}^*\mathcal{U}$. Now, $T^{(l,m)} = (T_1^l,\, T_2^m)$ and $T^{(l,\, m)}|_{H_I} = (T_1^l|_{H_I},\, T_2^m|_{H_I})$.
As $\mathcal{U}^* T_1^l|_{H_I} \mathcal{U}\, e_{(k_1,\, k_2)} = \{\prod_{p=0}^{l-1} \alpha_{(i+k_1l+p,\, j+k_2m)}\} e_{(k_1+1,\, k_2)} = (T_{(L:I)})_1\, e_{(k_1,k_2)}$ and similarly, $\mathcal{U}^* T_2^l|_{H_I} \mathcal{U}\, e_{(k_1,\, k_2)} = (T_{(L:I)})_2\, e_{(k_1,k_2)}$, so by Definition 5.5, $(T_1^l|_{H_I},\, T_2^m|_{H_I}) \cong \big((T_{(L:I)})_1,\, (T_{(L:I)})_2\big)$. That is, $T^L|_{H_I} \cong T_{(L:I)}$.
Now as

$$\ell^2(\mathbb{Z}_+^2) = \bigoplus_{j=0}^{m-1} \bigoplus_{i=0}^{l-1} H_I,$$

so

$$T^L = \bigoplus_{j=0}^{m-1} \bigoplus_{i=0}^{l-1} T^L|_{H_I} \cong \bigoplus_{j=0}^{m-1} \bigoplus_{i=0}^{l-1} T_{(L:I)}$$

□

Corollary 5.10. *(a) T^L is k-hyponormal if and only if $T_{(L:I)}$ is k-hyponormal for all $0 \le i \le l-1$, $0 \le j \le m-1$.*

(b) T^L is subnormal if and only if $T_{(L:I)}$ is subnormal for all $0 \le i \le l-1$, $0 \le j \le m-1$.
(c) *T is subnormal $\Rightarrow T^L = (T_1^l,\, T_2^m)$ is subnormal*
$\Rightarrow T^L|_{H_I}$ is subnormal for $0 \le i \le l-1$, $0 \le j \le m-1$
$\Rightarrow T_{(L:I)}$ is subnormal for $0 \le i \le l-1$, $0 \le j \le m-1$.

We are now seek to identify the Berger measure $\mu_{(L:I)}$ corresponding to $T_{(L:I)}$.

Theorem 5.11. *$d\mu_{(L:I)}(s,t) = \frac{s^{i/l}\, t^{j/m}}{\gamma_{(i,j)}(T)}\, d\mu(s^{1/l},\, t^{1/m}) = \frac{s^{i/l}\, t^{j/m}}{\gamma_{(i,j)}(T)}\, d\mu_{(L:(0,0))}(s,\, t)$ for $0 \le i \le l-1$, $0 \le j \le m-1$. If $\mu(s,t) = \nu(s,t) + \rho(s)\delta_0(t)$, where $\nu(E \times \{0\}) = 0\ \forall\ E \subseteq \mathbb{R}_+$, then*

(a) $$d\mu_{(L:(i,0))}(s,t) = \frac{s^{i/l}}{\gamma_{(i,0)}(T)}\, d\mu(s^{1/l},\, t^{1/m})$$

(b) For $1 \le j \le m-1$, $$d\mu_{(L:I)}(s,t) = \frac{s^{i/l}\, t^{j/m}}{\gamma_{(i,j)}(T)}\, d\nu(s^{1/l},\, t^{1/m})$$

Proof. Let $\gamma_{(k_1,k_2)}(T)$ and $\gamma_{(k_1,k_2)}(T_{(L:I)})$ denote the moment sequences related to T and $T_{(L:I)}$ respectively.

Then $$\gamma_{(k_1,k_2)}(T_{(L:I)}) = \frac{\gamma_{(i+k_1 l,\, j+k_2 m)}(T)}{\gamma_{(i,j)}(T)}$$

$$\Rightarrow \iint s^{k_1} t^{k_2} d\mu_{(L:I)}(s,t) = \frac{1}{\gamma_{(i,j)}(T)} \iint s^{i+k_1 l}\, t^{j+k_2 m} d\mu(s,t)$$

$$= \frac{1}{\gamma_{(i,j)}(T)} \iint s^{i/l} s^{k_1} t^{j/m} t^{k_2} d\mu(s^{1/l},\, t^{1/m})$$

$$\Rightarrow \quad d\mu_{(L:I)}(s,t) = \frac{s^{i/l}\, t^{j/m}}{\gamma_{(i,j)}(T)}\, d\mu(s^{1/l},\, t^{1/m}) \tag{5.1}$$

Also, $d\mu_{(L:(0,0))}(s,\, t) = d\mu(s^{1/l},\, t^{1/m})$. So clearly $d\mu_{(L:I)}(s,t) = \frac{s^{i/l}\, t^{j/m}}{\gamma_{(i,j)}(T)}\, d\mu(s^{1/l},\, t^{1/m}) = \frac{s^{i/l}\, t^{j/m}}{\gamma_{(i,j)}(T)}\, d\mu_{(L:(0,0))}(s,\, t)$ for $0 \le i \le l-1,\ 0 \le j \le m-1$.

If $\mu(s,t) = \nu(s,t) + \rho(s)\delta_0(t)$, then from (5.1), we get

$$d\mu_{(L:(i,0))}(s,t) = \frac{s^{i/l}}{\gamma_{(i,0)}(T)}\, d\mu(s^{1/l},\, t^{1/m})$$

For $1 \le j \le m-1$,

$$\iint s^{k_1} t^{k_2} d\mu_{(L:I)}(s,t) = \frac{1}{\gamma_{(i,j)}(T)} \iint s^{i+k_1 l}\, t^{j+k_2 m} d\mu(s,t)$$

$$= \frac{1}{\gamma_{(i,j)}(T)} \iint s^{i+k_1 l}\, t^{j+k_2 m} d\nu(s,\, t)\ (\texttt{since}\ j + k_2 m > 0,\ \forall k_2)$$

$$\Rightarrow \quad d\mu_{(L:I)}(s,t) = \frac{s^{i/l}\, t^{j/m}}{\gamma_{(i,j)}(T)}\, d\nu(s^{1/l},\, t^{1/m})$$

□

Theorem 5.12. *Let $T = (T_1, T_2)$ be 2-variable weighted shift with weight sequences α and β, and $M = \bigvee_{k_2 \ge 1} e_{(k_1,k_2)}$. If $T_M := T|_M$ is subnormal, then for $L = (l, m)$, with $l \ge 1, m \ge 1$, the following are equivalent:*
(a) T^L is k-hyponormal.
(b) $T_{(L:(i,0))}$ is k-hyponormal for $0 \le i \le l-1$

Proof. $(a) \Rightarrow (b)$ is obvious from Corollary 5.10.
$(b) \Rightarrow (a)$: Here

$$T^L \cong \bigoplus_{j=0}^{m-1} \bigoplus_{i=0}^{l-1} T_{(L:I)}.$$

Given that $T_{(L:(i,0))}$ is k-hyponormal for $0 \le i \le l-1$. To show $T_{(L:(i,j))}$ is k-hyponormal for $0 \le i \le l-1$ and $1 \le j \le m-1$.
Define $\tilde{\alpha}_{(k_1,k_2)} = \alpha_{(k_1,k_2+1)}$ and $\tilde{\beta}_{(k_1,k_2)} = \beta_{(k_1,k_2+1)}$.
$(T_M)_{(L:I)}$ is a 2-variable weighted shift with weight sequences

$$P_{(L:I)}(\tilde{\alpha}) = \{\prod_{p=0}^{l-1} \tilde{\alpha}_{(i+k_1 l+p,\, j+k_2 m)}\} = \{\prod_{p=0}^{l-1} \alpha_{(i+k_1 l+p,\, j+k_2 m+1)}\} = P_{(L:(i,j+1))}(\alpha)$$

and

$$Q_{(L:I)}(\tilde{\beta}) = Q_{(L:(i,j+1))}(\beta).$$

Thus $(T_M)_{(L:(i,j))} = T_{(L:(i,j+1))}$ for $0 \le i \le l-1$ and $0 \le j \le m-1$
That is, $(T_M)_{(L:(i,j-1))} = T_{(L:(i,j))}$ for $0 \le i \le l-1$ and $1 \le j \le m$.

Now

$$\begin{aligned} T_M \text{ is subnormal } &\Rightarrow T_M^L \text{ is subnormal} \\ &\Rightarrow (T_M)_{(L:I)} \text{ is subnormal and hence } k\text{-hyponormal} \\ &\quad \text{for } 0 \le i \le l-1,\ 0 \le j \le m-1 \text{ (by Corollary 5.10)} \\ &\Rightarrow (T_M)_{(L:(i,j-1))} \text{ is } k\text{-hyponormal for } 0 \le i \le l-1,\ 1 \le j \le m \\ &\Rightarrow (T)_{(L:I)} \textit{ is } k \text{ -hyponormal for } 0 \le i \le l-1,\ 1 \le j \le m-1 \end{aligned}$$

□

Theorem 5.13. *Let $T = (T_1, T_2)$ be 2-variable weighted shift with weight sequences α and β. Let $M_n = \bigvee_{k_2 \ge n} e_{(k_1, k_2)}$ and $T_{M_n} := T|_{M_n}$ be subnormal. For $L = (l, m)$ with $l \ge 1, m \ge 1, I = (i, j)$ with $0 \le i \le l-1,\ 0 \le j \le m-1$, and $k \ge 1$, then the following are equivalent*
(a) T^L is k-hyponormal.
(b) $T_{(L:(i,0))}, T_{(L:(i,1))}, \dots, T_{(L:(i,n-1))}$ are k-hyponormal for all $0 \le i \le l-1$.

Theorem 5.14. *Let $T = (T_1, T_2)$ be 2-variable weighted shift with weight sequences α and β. Let $M_1 = \bigvee_{k_2 \ge 1} e_{(k_1, k_2)}$ and $T_{M_1} := T|_{M_1}$ be subnormal with the Berger measure $\mu_1(s,t) = \nu_1(s,t) + \rho(s)\,\delta_0(t)$ and $W_0 := shift(\alpha_{(0,0)}, \alpha_{(1,0)}, \dots)$ be subnormal with associated measure ξ_0. Then $T^{(1,2)}$ is subnormal if and only if*

$$\beta^2_{(0,0)} \le \left(\left\| \frac{1}{t} \right\|_{L^1(\nu_1)} \right)^{-1}$$

$$\textit{and} \qquad \beta^2_{(0,0)} \left\| \frac{1}{t} \right\|_{L^1(\nu_1)} (\nu_1)^X_{ext} \le \xi_0$$

If $\rho(s) = 0$, then $T^{(1,2)}$ is subnormal if and only if T is subnormal.

Proof. By the Theorem 5.12, if T_{M_1} is subnormal, then $T^{(1,2)}$ is subnormal if and only if $T_{((1,2):(0,0))}$ is subnormal. So, it suffices to check for $T_{((1,2):(0,0))}$. Again $T_{((1,2):(0,0))}$ is the 1-step back extension of $(T_{M_1})_{((1,2):(0,1))}$.
Since T_{M_1} is subnormal with measure μ_1, so by Corollary 5.10 (c) $(T_{M_1})_{((1,2):(0,1))}$ is also subnormal with measure $(\mu_1)_{((1,2):(0,1))}$. Therefore by Theorem 4.2, $T_{((1,2):(0,0))}$ is subnormal if and only if

$$\beta^2_{(0,0)}\beta^2_{(0,1)} \le \left(\left\| \frac{1}{t} \right\|_{L^1\left((\mu_1)_{((1,2):(0,1))}\right)} \right)^{-1} \tag{5.2}$$

$$\beta^2_{(0,0)}\beta^2_{(0,1)} \left\| \frac{1}{t} \right\|_{L^1\left((\mu_1)_{((1,2):(0,1))}\right)} \left((\mu_1)_{((1,2):(0,1))} \right)^X_{ext} \le \xi_0 \tag{5.3}$$

Now,

$$d(\mu_1)_{((1,2):(0,1))}(s,t) = \frac{t^{1/2}}{\gamma_{(0,1)}(T_{M_1})}\, d\nu_1(s, t^{1/2}) \tag{5.4}$$

and

$$
\begin{aligned}
d(\mu_1)_{\left((1,2):(0,1)\right)_{ext}}(s,t) &= \frac{(1-\delta_0(t))}{t\|\frac{1}{t}\|_{L^1\left((\mu_1)_{((1,2):(0,1))}\right)}}\, d(\mu_1)_{\left((1,2):(0,1)\right)}(s,t)\\
&= \frac{(1-\delta_0(t))}{t^{1/2}\int \frac{1}{t^{1/2}}\, d\nu_1(s,t^{1/2})}\, d\nu_1(s,t^{1/2}) \text{ (using (5.4))}\\
&= \frac{(1-\delta_0(t))}{t\int \frac{1}{t}\, d\nu_1(s,t)}\, d\nu_1(s,t)\\
&= d(\nu_1)_{ext}(s,t)
\end{aligned} \tag{5.5}
$$

Now,

$$
\begin{aligned}
(5.2) \Rightarrow \beta^2_{(0,0)}\beta^2_{(0,1)} &\int \frac{1}{t}\, d(\mu_1)_{\left((1,2):(0,1)\right)}(s,t) \le 1\\
\Rightarrow \beta^2_{(0,0)} &\int \frac{1}{t^{1/2}}\, d\nu_1(s,t^{1/2}) \le 1\\
\Rightarrow \beta^2_{(0,0)} &\int \frac{1}{t}\, d\nu_1(s,t) \le 1\\
\Rightarrow \beta^2_{(0,0)} &\le \left(\left\|\frac{1}{t}\right\|_{L^1(\nu_1)}\right)^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
(5.3) \Rightarrow \beta^2_{(0,0)}\beta^2_{(0,1)} &\int \frac{1}{t}\, d((\mu_1)_{((1,2):(0,1))}(s,t))\ \left((\mu_1)_{((1,2):(0,1))}(s,t)\right)^X_{ext} \le \xi_0(s)\\
\Rightarrow \beta^2_{(0,0)} &\int \frac{1}{t^{1/2}}\, d\nu_1(s,t^{1/2})\ (\nu_1(s,t))^X_{ext} \le \xi_0(s)\\
\Rightarrow \beta^2_{(0,0)} &\int \frac{1}{t}\, d\nu_1(s,t)\ (\nu_1(s,t))^X_{ext} \le \xi_0(s)\\
\Rightarrow \beta^2_{(0,0)} &\left\|\frac{1}{t}\right\|_{L^1(\nu_1)} (\nu_1)^X_{ext} \le \xi_0
\end{aligned}
$$

If $\rho(s)=0$, then $\mu_1(s,t)=\nu_1(s,t)$. Therefore by Theorem 4.2, $T^{(1,2)}$ is subnormal if and only if T is subnormal.

□

Theorem 5.15. *Let T be a 2-variable weighted shift with the weight sequences α and β. Assume that $T_{M_2} := T|_{M_2}$ the restriction of T to $M_2 := \bigvee\{e_{(k_1,k_2)} : k_2 \ge 2\}$ is subnormal with associated measure μ_2. Let $W_0 := shift(\alpha_{(0,0)}, \alpha_{(1,0),\ldots})$ and $W_1 := shift(\alpha_{(0,1)}, \alpha_{(1,1),\ldots})$ be subnormal with associated measures ξ_0 and ξ_1 respectively. Then T is subnormal with associated measure μ if and only if*

(i) $\frac{1}{t} \in L^1(\mu_2)$ *and* $\frac{1}{t^2} \in L^1(\mu_2)$

(ii) $\beta^2_{(0,0)}\beta^2_{(0,1)}\|\frac{1}{t^2}\|_{L^1(\mu_2)} \le 1$

(iii) $\beta^2_{(0,0)}\beta^2_{(0,1)}\|\frac{1}{t^2}\|_{L^1(\mu_2)}(\mu_2)^X_{(ext)^2} \le \xi_0$

(iv) $\beta^2_{(0,1)}\|\frac{1}{t}\|_{L^1(\mu_2)} = 1$

(v) $(\mu_2)^X_{ext} = \xi_1$

Proof. Assume that T be subnormal. Since T_{M_1} is a subnormal weighted shift possessing a subnormal extension T, so $\beta^2_{(0,1)} = \left(\|\frac{1}{t}\|_{L^1(\mu_2)}\right)^{-1}$ and $(\mu_2)^X_{ext} =$

ξ_1, Moreover, if μ_1 is a Berger measure of T_{M_1}, then $\mu_1 = (\mu_2)_{ext}$. Since T is subnormal so by Corollary 5.10 (c), $T_{((1,2):(0,0))}$ is also subnormal. Again $T_{((1,2):(0,0))}$ is the 1-step extension of $(T_{M_1})_{((1,2):(0,1))}$. Therefore by Theorem 4.2, $T_{((1,2):(0,0))}$ is subnormal if and only if

$$\frac{1}{t} \in L^1((\mu_1)_{((1,2):(0,1))}) \tag{5.6}$$

$$\beta^2_{(0,0)}\beta^2_{(0,1)} \left\|\frac{1}{t}\right\|_{L^1\left((\mu_1)_{((1,2):(0,1))}\right)} \leq 1 \tag{5.7}$$

$$\beta^2_{(0,0)}\beta^2_{(0,1)} \left\|\frac{1}{t}\right\|_{L^1\left((\mu_1)_{((1,2):(0,1))}\right)} \left((\mu_1)_{((1,2):(0,1))}\right)^X_{ext} \leq \xi_0 \tag{5.8}$$

Now,

$$d(\mu_1)_{((1,2):(0,1))}(s,t) = \frac{t^{1/2}}{\gamma_{(0,1)}(T_{M_1})}\, d\mu_1(s,t^{1/2}) = d\mu_2(s,t^{1/2}). \tag{5.9}$$

So, (5.6) implies that $\frac{1}{t^2} \in L^1(\mu_2(s,t))$ and so also $\frac{1}{t} \in L^1(\mu_2(s,t))$. Also, $\mu_1(E \times \{0\}) = 0$, $\mu_2(E\times\{0\}) = 0 \;\forall E \subseteq \mathbb{R}_+$.

and
$$\begin{aligned} d(\mu_1)_{\left((1,2):(0,1)\right)_{ext}}(s,t) &= \frac{(1-\delta_0(t))}{t\|\frac{1}{t}\|_{L^1\left((\mu_1)_{((1,2):(0,1))}(s,t)\right)}}\, d(\mu_1)_{\left((1,2):(0,1)\right)}(s,t) \\ &= \frac{(1-\delta_0(t))}{t\|\frac{1}{t}\|_{L^1\left(\mu_2(s,t^{1/2})\right)}}\, d\mu_2(s,t^{1/2}) \text{ (using (5.9))} \\ &= \frac{(1-\delta_0(t))}{t^2\|\frac{1}{t^2}\|_{L^1\left(\mu_2(s,t)\right)}}\, d\mu_2(s,t) \\ &= d(\mu_2)_{(ext)^2}(s,t) \end{aligned} \tag{5.10}$$

Again from (5.7), we get
$$\beta^2_{(0,0)}\beta^2_{(0,1)} \left\|\frac{1}{t}\right\|_{L^1\left(\mu_2(s,t^{1/2})\right)} \leq 1$$
$$\Rightarrow \beta^2_{(0,0)}\beta^2_{(0,1)} \left\|\frac{1}{t^2}\right\|_{L^1\left(\mu_2(s,t)\right)} \leq 1$$

and from (5.8), we get

$$\beta^2_{(0,0)}\beta^2_{(0,1)} \left\|\frac{1}{t}\right\|_{L^1\left(\mu_2(s,t^{1/2})\right)} (\mu_2(s,t))^X_{(ext)^2} \leq \xi_0(s) \text{ (using(5.9) and (5.10))}$$
$$\Rightarrow \beta^2_{(0,0)}\beta^2_{(0,1)} \left\|\frac{1}{t^2}\right\|_{L^1\left(\mu_2(s,t)\right)} (\mu_2(s,t))^X_{(ext)^2} \leq \xi_0(s)$$

($\Longleftarrow$) Suppose all the conditions are hold. To show T is subnormal. From conditions (i), (iv) and since T_{M_2} is subnormal so by Theorem 4.2, T_{M_1} is subnormal with the Berger measure μ_1 such that $\mu_1(E\times\{0\}) = 0$ for all $E \subseteq \mathbb{R}_+$ and $\mu_1 = (\mu_2)_{ext}$. So by Theorem 5.14 to check the subnormality of T, it suffices to check the subnormality of $T^{(1,2)}$ and by Theorem 5.12 this reduces to verifying the subnormality of $T_{((1,2):(0,0))}$. Again $T_{((1,2):(0,0))}$ is the 1-step extension of $(T_{M_1})_{((1,2):(0,1))}$ (which is subnormal).

Now, since $(T_{M_1})_{((1,2):(0,1))}$, T_{M_1} and T_{M_2} are subnormal with measures $(\mu_1)_{((1,2):(0,1))}$, μ_1 and μ_2 respectively. So, we can establish as above that $d(\mu_1)_{\left((1,2):(0,1)\right)}(s,t) = d\mu_2(s,t^{1/2})$ and $d(\mu_1)_{((1,2):(0,1))_{ext}}(s,t) = d(\mu_2)_{(ext)^2}(s,t)$.

So, condition (i) implies that $\frac{1}{t} \in L^1((\mu_1)_{((1,2):(0,1))})$. From condition (ii) we will get

$$\beta^2_{(0,0)}\beta^2_{(0,1)} \left\|\frac{1}{t^2}\right\|_{L^1\left((\mu_1)_{((1,2):(0,1))}(s,t^2)\right)} \leq 1$$

$$\Rightarrow \beta^2_{(0,0)}\beta^2_{(0,1)} \left\|\frac{1}{t}\right\|_{L^1\left((\mu_1)_{((1,2):(0,1))}(s,t)\right)} \leq 1$$

and condition (iii) will give,

$$\beta^2_{(0,0)}\beta^2_{(0,1)} \left\|\frac{1}{t^2}\right\|_{L^1\left((\mu_1)_{((1,2):(0,1))}(s,t^2)\right)} \left((\mu_1)_{((1,2):(0,1))}(s,t)\right)^X_{ext} \leq \xi_0(s)$$

$$\Rightarrow \beta^2_{(0,0)}\beta^2_{(0,1)} \left\|\frac{1}{t}\right\|_{L^1\left((\mu_1)_{((1,2):(0,1))}(s,t)\right)} \left((\mu_1)_{((1,2):(0,1))}(s,t)\right)^X_{ext} \leq \xi_0(s)$$

Thus by Theorem 4.2, $T_{((1,2):(0,0))}$ is subnormal and hence T is subnormal. □

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