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Quadratic Hyponormality and Positive Quadratic Hyponormality are not Preserved under Weighted Completion and Restriction

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ABSTRACT

Let $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ *and* $\beta = \{\beta_n\}_{n=0}^{\infty}$ *be weighted sequences with positive weights, and* W_α *and* W_β *be the weighted shifts on* $l^2(Z_+)$ *with weight sequences* α *and* β *respectively. If* β *is a subsequence of, we call a weighted extension of, or equivalently, is a weighted restriction of . In this paper we show that the properties of quadratic hyponormality and positive quadratic hyponormality are not preserved under weighted extension and weighted restriction.*

Keywords: Quadratic hyponormality, Positive quadratic hyponormality, Weighted shift.
AMS Subject Classification: 47B37, 47B20

INTRODUCTION

Let H be a separable infinite dimensional complex Hilbert space and B(H) denote the algebra of bounded linear operators on H. For $A, B \in B(H)$, let $[A,B] := AB-BA$. We say that an n-tuple $T=(T_1,\ldots,T_n)$ of operators on H is hyponormal if the operator matrix $([T_j^*, T_i])_{i,j=0}^{n}$ is positive on the direct sum of n-copies of H. For $k \geq 1$ and $T \in B(H)$, T is k-hyponormal if $(1,T,\ldots,T^k)$ is hyponormal. Again, T is weakly k- hyponormal if $p(T)$ is hyponormal for every polynomial p of degree $\leq$ k. It can be shown easily that k- hyponormality of T implies weak k-hyponormality of T.

For $k = 2$, weak 2-hyponormality, often referred to as quadratic hyponormality (for which we write q.h.) was first considered in detail by Curto in [1]. The definition of positive quadratically hyponormal operators (for which we write p.q.h.) was introduced by Curto-Fialkow in [2]. Subsequently, it has been shown that 2-*hyponormality* $\Rightarrow$ *p.q.h.* $\Rightarrow$ *q.h.* though the reverse is not always true.

In this paper we consider a weighted shift W_α with a positive weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$. We prove the following results:

1. If W_α is q.h. (or p.q.h.) then it is not necessary that W_β is also q.h. (or p.q.h.), whenever β is a subsequence of α.
2. If W_α is not q.h. (or p.q.h.) then it is not necessary that the same is true for W_β, whenever β is a subsequence of α.

PRELIMINARIES AND NOTATIONS

Let $\{e_n\}_{n=0}^{\infty}$ be the canonical orthonormal basis for $\ell^2(Z_+)$ and let $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ be a bounded sequence of positive numbers. Let W_α be a unilateral weighted shift defined by $W_\alpha e_n = \alpha_n e_{n+1}$ for $n \geq 0$. Then W_α is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}$ $\forall$ $n \geq 0$.

We recall here some terminologies and notations from [2]. Also we note that an operator T is quadratically hyponormal if $T + sT^2$ is hyponormal for every $s \in X$.,. Let W_α be a hyponormal weighted shift. For $s \in X$.,, we let $D(s) := [(W_\alpha + s W_\alpha^2)^*, (W_\alpha + s W_\alpha^2)]$ and let P_n be the projection onto $\vee_{i=0}^{n} \{e_i\}$. We define

$$D_n := D_n(s) = P_n [(W_\alpha + s W_\alpha^2)^*, (W_\alpha + s W_\alpha^2)] P_n$$

$$= \begin{pmatrix} q_0 & \bar{r}_0 & 0 & . & . & 0 & 0 \\ r_0 & q_1 & \bar{r}_1 & . & . & 0 & 0 \\ 0 & r_1 & q_2 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & q_{n-1} & \bar{r}_{n-1} \\ 0 & 0 & 0 & . & . & r_{n-1} & q_n \end{pmatrix}$$

where $q_k := u_k + |s|^2 v_k$, $r_k := s\sqrt{w_k}$, $u_k := \alpha_k^2 - \alpha_{k-1}^2$, $v_k := \alpha_k^2 \alpha_{k+1}^2 - \alpha_{k-1}^2 \alpha_{k-2}^2$, $w_k := \alpha_k^2 (\alpha_{k+1}^2 - \alpha_{k-1}^2)^2$ for $k \geq 0$ and $\alpha_{-1} = \alpha_{-2} := 0$.

Clearly, W_α is q.h. if and only if $D_n(s) \geq 0$ for every $s \in X$.,and every $n \geq 0$. Let $d_n(.) := \det(D_n(s))$. Then it follows from [2] that $d_0 = q_0$, $d_1 = q_0 q_1 - |r_0|^2$ and $d_{n+2} = q_{n+2} d_{n+1} - |r_{n+1}|^2 d_n$ for $n \geq 0$, and that d_n is actually a polynomial in $t := |s|^2$ of degree $(n+1)$, with Maclaurin expansion $d_n(t) := \sum_{i=0}^{n+1} c(n,i) t^i$. This gives that for $n \geq 0$ and $1 \leq i \leq n+1$,

1. $c(n, 0) = u_0 u_1 \ldots u_n \geq 0$.
2. $c(n, n+1) = v_0 v_1 \ldots v_n \geq 0$.
3. $c(1, 1) = u_1 v_0 - u_0 v_1 - w_0 \geq 0$.

4. $c(n, i) = u_n c(n-1,i) + v_n c(n-1,i-1) - w_{n-1} c(n-2,i-1)$ for $n \geq 2$.

5. $c(n, 1) = u_n c(n-1,1) + (v_n u_{n-1} - w_{n-1} u_0 u_1 \dots u_{n-1})$ for $n \geq 2$.

Also we recall from Definition 4.2 [2] that W_α is p.q.h. if $c(n,i) \geq 0$ for all $n,i \geq 0$ with $0 \leq i \leq n+1$.

Again, if P_+ denotes the non-negative real numbers, then for $x_0 x_1 \dots x_n$ and s in P_+ we denote by $F_n := F_n(x_0, x_1, \dots, x_n, s)$

$$= \sum_{i=0}^{n} q_i x_i^2 - 2\sum_{i=0}^{n-1} r_i x_i x_{i+1}$$

$$= \sum_{i=0}^{n} u_i x_i^2 - 2s \sum_{i=0}^{n-1} \sqrt{w_i}\, x_i x_{i+1} + s^2 \sum_{i=0}^{n} v_i x_i^2$$

And recall, for further use, the following result :

Theorem 2.1. [7] : *Let W_α be a weighted shift with a weight sequence α. Then the following are equivalent :*

(i) *W_α is quadratically hyponormal;*

(ii) *$F_n(x_0, x_1, \dots, x_n, s) \geq 0$ for any $x_0, x_1, \dots, x_n, s \in P_+$* ($n \geq 2$);

(iii) *There exists a positive integer N such that $F_n(x_0, x_1, \dots, x_n, s) \geq 0$ for any $x_0, x_1, \dots, x_n, s \in P_+$ ($n \geq N$).*

EXAMPLES OF QUADRATICALLY HYPONORMAL WEIGHTED SHIFTS

In this section we show that the weighted shift on $\ell^2 (Z_+)$ with weight sequence α: $\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \sqrt{\frac{43}{80}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \dots$ having a Bergman tail, is quadratically hyponormal. This example is to be used in the next section to prove the statements 1 and 2 claimed in section 1.

Eamxple3.1. *Let α be the positive weight sequence given by α: $\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \sqrt{\frac{43}{80}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \dots$ Then the weighted shift operator W_α with weight sequence α is quadratically hyponormal.*

Proof. Let $\{ \alpha_n \}_{n=0}^{\infty}$ denote the sequence α. Then $\alpha_{n+1} = \sqrt{\frac{n}{n+1}}$, $\forall n \geq 2$ and so, in terms of the notations introduced in section 2, we have $w_n = u_{n+1} v_n, \forall n \geq 5$. In view of Theorem 2.1, it is sufficient to show that $F_n \geq 0, \forall n \geq 5$.

For $x_0,\dots,x_5,s$ reals, we define

$$G_5(x_0,\dots,x_5,s) := F_5 - v_5 t x_5^2 \text{ where } s^2 = t.$$

$$= \sum_{i=0}^{4}(u_i + tv_i)x_i^2 - 2\sum_{i=0}^{4}\sqrt{w_i t}\,x_i x_{i+1} + u_5 x_5^2.$$

Then $F_6 = G_5 + (v_5 t - \frac{w_5 t}{u_6 + tv_6})x_5^2 + (\frac{\sqrt{w_5 t}}{\sqrt{u_6 + tv_6}}x_5 - \sqrt{u_6 + tv_6 x_6})^2.$

Hence, $F_6(x_0,\dots,x_6,s) \geq 0$ for any $x_0,\dots,x_6,s \in P_+$

$$\Leftrightarrow G_5(x_0,\dots,x_5,s) + (v_5 t - \frac{w_5 t}{u_6 + tv_6})x_5^2 \geq 0 \text{ for any } x_0,\dots,x_5,s \in P_+$$

$$\Leftrightarrow G_5(x_0,\dots,x_5,s) + \frac{z_6 t}{1 + z_6 t}v_5 t x_5^2 \geq 0 \text{ for any } x_0,\dots,x_5,s \in P_+ .$$

(using $w_n = u_{n+1}v_n, \forall n \geq 5$ and $z_n = \frac{v_n}{u_n}$).

Similarly,

$F_7(x_0,\dots,x_7,s) \geq 0$ for any $x_0,\dots,x_7,s \in P_+$

$$\Leftrightarrow G_5(x_0,\dots,x_5,s) + \frac{z_7 z_6 t^2}{1 + z_7 t + z_7 z_6 t^2}v_5 t x_5^2 \geq 0 \text{ for any } x_0,\dots,x_5,s \in P_+ .$$

So, by mathematical induction, for $n \geq 6$, we have

$$F_n \geq 0 \Leftrightarrow G_5 + \frac{(z_n z_{n-1}\dots z_6 t^{n-5})v_5 t x_5^2}{1 + z_n t + z_n z_{n-1}t^2 + \dots + z_n z_{n-1}\dots z_6 t^{n-5}} \geq 0$$

$$\Leftrightarrow G_5 + \frac{1}{1 + \frac{1}{z_6 t} + \frac{1}{z_6 z_7 t^2} + \dots + \frac{1}{z_6 z_7 \dots z_n t^{n-5}}}v_5 t x_5^2 \geq 0 \qquad (1)$$

Claim1: $G_5(x_0,\dots,x_5,s) \geq 0$ for $0 \leq s \leq \sqrt{0.299}$

The corresponding symmetric matrix to the quadratic form G_5 is

$$A(t) = \begin{pmatrix} u_0 + tv_0 & -\sqrt{w_0 t} & 0 & 0 & 0 & 0 \\ -\sqrt{w_0 t} & u_1 + tv_1 & -\sqrt{w_1 t} & 0 & 0 & 0 \\ 0 & -\sqrt{w_1 t} & u_2 + tv_2 & -\sqrt{w_2 t} & 0 & 0 \\ 0 & 0 & -\sqrt{w_2 t} & u_3 + tv_3 & -\sqrt{w_3 t} & 0 \\ 0 & 0 & 0 & -\sqrt{w_3 t} & u_4 + tv_4 & -\sqrt{w_4 t} \\ 0 & 0 & 0 & 0 & -\sqrt{w_4 t} & u_5 \end{pmatrix}$$

We discuss the positivity of A(t) by Nested Determinant Test. By direct computation, we have

$$d_0=\frac{1}{2}+\frac{1}{4}t,\ d_1=\frac{3t}{320}+\frac{43t^2}{640},\ d_2=\frac{43t^2}{12800}+\frac{559t^3}{76800},\ d_3=\frac{301t^2}{1024000}+\frac{731t^3}{1024000}+\frac{20683t^4}{12288000}$$

$$d_4=\frac{301t^2}{12288000}+\frac{301t^3}{10240000}+\frac{34529t^4}{368640000}+\frac{599807t^5}{1474560000}$$

$$d_5=\frac{301t^2}{245760000}-\frac{301t^3}{122880000}-\frac{35647t^4}{7372800000}-\frac{20683t^5}{9830400000}$$

If $0<t\le 0.299$ then $d_0,\ldots,d_4>0$ and $d_5>0$, which implies that $A(t)\ge 0$ for $0<t\le 0.299$ and $G_5(x_0,\ldots,x_5,s)\ge 0$ for $0<s\le\sqrt{0.299}$ and claim1 is established.

Hence by (1), $F_n(x_0,\ldots,x_n,s)\ge 0$ *for any* $x_0,\ldots,x_n,s\in P_+$ and $0<s\le\sqrt{0.299}$.

As $z_n=\dfrac{v_n}{u_n}=\dfrac{4(n+1)}{n+2},(n\ge 5)$, so $\{z_n\}$ is an increasing sequence and hence

$$1+\frac{1}{z_6t}+\frac{1}{z_6z_7t^2}+\ldots+\frac{1}{z_6z_7\ldots z_nt^{n-5}}\le 1+\frac{1}{z_6t}+\frac{1}{(z_6t)^2}+\ldots+\frac{1}{(z_6t)^{n-5}}\le\frac{1}{1-\dfrac{1}{z_6t}}$$

Hence if $t>0.299$, then

$$G_5+\frac{1}{1+\dfrac{1}{z_6t}+\dfrac{1}{z_6z_7t^2}+\ldots+\dfrac{1}{z_6z_7\ldots z_nt^{n-5}}}v_5tx_5^2\ge G_5+\left(\frac{24t-7}{144}\right)x_5^2\ \left(\because\ z_6=\frac{24}{7}\text{ and }v_5=\frac{1}{6}\right)$$

Now we consider the corresponding symmetric matrix B(t) to the quadratic form $G_5+\left(\dfrac{24t-7}{144}\right)x_5^2$ as follows:

$$B(t)=\begin{pmatrix} u_0+tv_0 & -\sqrt{w_0t} & 0 & 0 & 0 & 0\\ -\sqrt{w_0t} & u_1+tv_1 & -\sqrt{w_1t} & 0 & 0 & 0\\ 0 & -\sqrt{w_1t} & u_2+tv_2 & -\sqrt{w_2t} & 0 & 0\\ 0 & 0 & -\sqrt{w_2t} & u_3+tv_3 & -\sqrt{w_3t} & 0\\ 0 & 0 & 0 & -\sqrt{w_3t} & u_4+tv_4 & -\sqrt{w_4t}\\ 0 & 0 & 0 & 0 & -\sqrt{w_4t} & u_5+\dfrac{24t-7}{144}\end{pmatrix}$$

As was done in claim1, $d_i>0$ for $t\ge 0.299$ and $i=0,1,2,3,4$. Also, d_5 of *B(t)* is

$$\frac{301t^2}{8847360000}+\frac{301t^3}{1474560000}-\frac{1191487t^4}{265420800000}-\frac{6653089t^5}{1061683200000}+\frac{599807t^6}{8847360000}\ge 0$$

for $t\ge 0.299$. This is because d_5 is an increasing graph as is seen from the following mathematica graph of d_5:

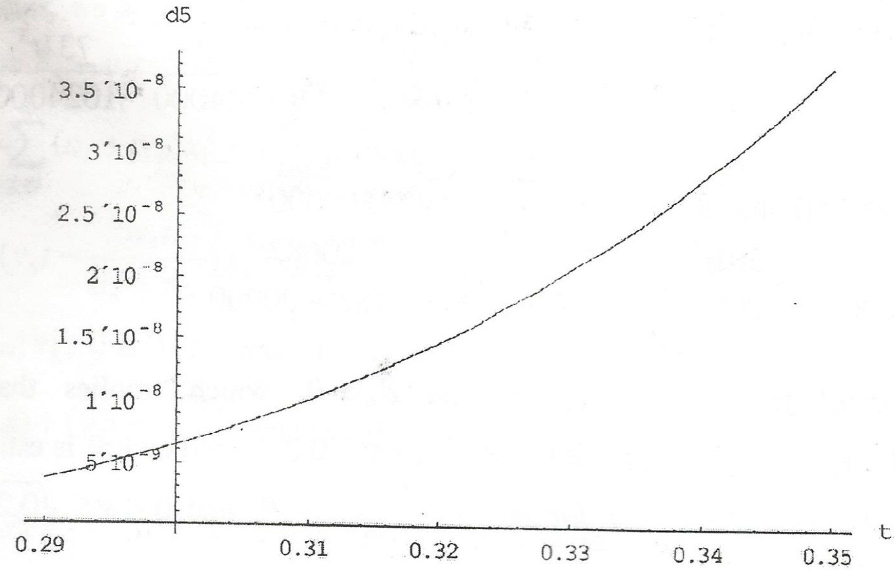

Fig. 1.

Therefore in view of (1), $F_n(x_0,\ldots,x_n,s)\geq 0$ for $n\geq 5$ and $s\geq\sqrt{0.299}$. Thus for all $t\geq 0$, $F_n\geq 0(n\geq 5)$. So by theorem 2.1, W_α is quadratically hyponormal. □

Eamxple 3.2. *Let α be the positive weight sequence given by $\alpha:\sqrt{\frac{1}{2}},\sqrt{\frac{1}{2}},\sqrt{\frac{47}{80}},\sqrt{\frac{3}{4}},\ldots$ Then the weighted shift operator W_α with weight sequence α is quadratically hyponormal.*

This can be shown by a method similar to that used in Example 3.1.

MAIN RESULTS

Definition 4.1. Let α and β be two sequences of positive weights, and W_α *and* W_β be the corresponding weighted shift operators on $\ell^2(\mathbb{Z}_+)$. If β be a subsequence of α we define W_α as a weighted extension of W_β, or equivalently, W_β is a weighted restriction of W_α.
As earlier stated in section1, here we will show that the properties of quadratic hyponormality and positive quadratic hyponormality are not preserved under weighted extension and weighted restriction.

Theorem 4.1. *Let $\alpha=\{\alpha_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers and λ be any positive real number. Also let $\lambda\alpha$ denote the sequence $\{\lambda\alpha_n\}_{n=0}^{\infty}$. Then W_α is quadratically hyponormal if and only if $W_{\lambda\alpha}$ is quadratically hyponormal. Moreover the result is also true if we replace quadratic hyponormality by positive quadratic hyponormality.*

Proof. We have $\alpha=\{\alpha_n\}_{n=0}^{\infty}$ and $\lambda\alpha=\{\lambda\alpha_n\}_{n=0}^{\infty}$ and in terms of the notations defined in section2 we get $\tilde{d}_k=\lambda^{2k+2}d_k$ for $k\geq 0$, where $\tilde{d}_k:=\det(\tilde{D}_n(s))$ and $\tilde{D}_n(s)=P_n[(W_{\lambda\alpha}+sW_{\lambda\alpha}^2)^*,(W_{\lambda\alpha}+sW_{\lambda\alpha}^2)]P_n$. This shows that W_α is q.h. if and only if $W_{\lambda\alpha}$ is q.h.

Also by an easy calculation we get the relations
$\tilde{c}(k,0)=\lambda^{2k+2}c(k,0)$, $\tilde{c}(k,1)=\lambda^{2k+4}c(k,1),\ldots,\ \tilde{c}(k,k)=\lambda^{4k+2}c(k,k)$,
$\tilde{c}(k,k+1)=\lambda^{4k+4}c(k,k+1)$
and so for $k\geq 0$ and $0<i\leq k+1$, $c(k,i)\geq 0$ if and only if $\tilde{c}(k,i)\geq 0$. Hence W_α is p.q.h. if and only if $W_{\lambda\alpha}$ is p.q.h. □

Question1: Is q.h. (or, p.q.h.) preserved under weighted restriction? That is , if β is any subsequence of a sequence α of positive numbers, then does q.h. (or, p.q.h.) of W_α necessarily imply q.h. (or, p.q.h.) of W_β ?

This question is answered in the negative as is seen from the following examples:

Example 4.1. *If α denotes the weight sequence α: $\sqrt{\frac{1}{2}},\sqrt{\frac{1}{2}},\sqrt{\frac{43}{80}},\sqrt{\frac{2}{3}},\sqrt{\frac{3}{4}},\ldots$,then W_α is q.h. as shown in Example 3.1. But if β: $\sqrt{\frac{1}{2}},\sqrt{\frac{1}{2}},\sqrt{\frac{2}{3}},\sqrt{\frac{3}{4}},\ldots$then β* is a subsequence *of α, and W_β is not q.h.(Example 3.10. [3])*

Example 4.2. *If α denote the weight sequence α: $\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}},\sqrt{\frac{3}{4}},\sqrt{\frac{4}{5}},\ldots$,then W_α is p.q.h. (proposition 7.[1]). But if β: $\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}},\sqrt{\frac{4}{5}},\sqrt{\frac{6}{7}},\sqrt{\frac{8}{9}},\ldots$,then β* is a subsequence of *α, and W_β is not p.q.h. since c(4,3)<0.*

Question 2: Is q.h. (or, p.q.h.) preserved under weighted extension? That is , if β is any subsequence of a sequence α of positive reals, then does q.h. (or, p.q.h.) of W_β imply q.h. (or, p.q.h.) of W_α ?

This question is answered in the negative as is seen from the following examples:

Example 4.3. *By Example 3.8 [3] if $\alpha=\{\alpha_n\}$ is sequence of positive weights such that $0<\alpha_0\leq\alpha_1=\alpha_2=\alpha_3=\alpha_4<\alpha_5\leq\ldots$, then W_α is not q.h. Thus if we choose α as the sequence*

$\alpha: \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \ldots$, *then* W_α *is not q.h. Let* β be the subsequence of α *given by* $\beta: \sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \ldots$ *Then* W_β *is q.h. (proposition7. [1]).*

Example 4.4. *Let* α *be the sequence* $\alpha: \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \ldots$ *Then* W_α *is not q.h. (Example 3.10.[2]) and hence not p.q.h.. If* β is a subsequence of α *given by* β : $\sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \ldots$,*then* W_β *is p.q.h. (Proposition 7. [1]).*

CONCLUSION AND OPEN PROBLEM

For a positive real number z, let $\alpha(z)$ denote the sequence $\alpha(z): \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \sqrt{z}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \ldots$

In examples 3.1. and 3.2. we have shown that $W_{\alpha(z)}$ is q.h. if $z = \frac{43}{80}$ and $\frac{47}{80}$. In view of the questions answered in section 4, one should be able to determine all the values of z such that $W_{\alpha(z)}$ is q.h. We have reasons to believe that it is q.h. for all z such that $\frac{43}{80} \leq z \leq \frac{47}{80}$. However, this has to be validated by rigorous proof.

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