ON THE STABILITY OF AGE SPECIFIC PREY AND PREDATORS SYSTEM WITH TWO FUNCTIONAL RESPONSES

Anuradha Devi, Dept. of Basic Sciences & Humanities, Royal Group Of Institutions
Munmi Saikia, Dept. of Basic Sciences & Humanities, Royal Group Of Institutions

Abstract

In this paper, a mathematical model is proposed to describe and analyse the dynamics of two age specific prey and one predator system. The dynamics of a prey- predator system, where prey population has two stages juvenile and adult with adult prey having anti-predator behaviour modelled by a system of autonomous ordinary differential equations. The main feature of the model is that two different functional response of the predator are incorporated in the model to represent the difference in the way the predator feeds on each of the prey species. Juvenile prey (which is considered as weak) is the easy to capture prey and the predator's response to the juvenile prey is Holling type II. The adult prey (which is considered as strong) having anti predator behaviour is hard to capture and this behaviour is represented by the ratio-dependent response of the predator. Interspecific competition among the prey species is included in the model. In the absence of the predator logistic growth of the prey population are also included in the prey equations. Thus the model presented here is a three variable model with Juvenile prey population, adult prey population and one predator population. Existence of equilibria, their nature is analysed. The stability criteria of the equilibria and conditions for stability are also analysed.

1.Introduction

The biosphere is an important zone for biological activities that are mainly responsible for the changes in ecology and environment. The co- existence of interacting biological species has been of great interest in the past few decades and has been studied extensively using mathematical models by many researchers. Many biological species have been driven to extinction and many others are at the verge of extinction due to several external forces such as overexploitation, over predation, environmental pollution, mismanagement of the habitat, etc. A milestone progress in the study of predator–prey interactions was the discovery of the now well known ''paradox of enrichment'' by Hairston et al [12] and by Rosenzweig [19], which states that according to the Lotka–Volterra type predator–prey theory with Michaelis–Menten Holling type functional response enriching a predator-prey system will cause an increase in the equilibrium density of the predator but not in that of the prey, and will destabilize the community equilibrium. Another similar paradox is the so called ''biological control paradox'', which was discussed by Luck [17] that, it is not possible to have both a low and stable prey equilibrium density. Both these two paradoxes can be illustrated by a standard local stability analysis for the following well studied Lotka–Volterra type pred-ator–prey model with Michaelis–Menten (or Michaelis–Menten–Holling) type functional response (Freedman [9], May[18], Maynard Smith [20]. The classical models are mostly variations of the Lotka–Volterra model. There is a growing explicit biological and physiological evidences (Arditi and Saiah [5], Arditi et al. [4], Gutierrez [10], that in many situations, especially when predators have to search for food and therefore have to share or compete for food, a more suitable general predator-prey theory should be based on the ratio-dependent theory, which states that, the per capita predator growth rate should be a function of the ratio of prey to predator abundance. This is strongly supported by numerous field and laboratory experiments and observations (Arditi and Ginzburg [2], Arditi et al.[3], Hanski [11]. Generally, a ratio-dependent predator–Prey model takes the form,

$$x' = xf(x) - yp\left(\frac{x}{y}\right) \tag{1.1}$$

$$y' = cyq\left(\frac{x}{y}\right) - dy \tag{1.2}$$

While the traditional (or prey dependent) model takes the form

$$x' = xf(x) - yp(x) \tag{1.3}$$

$$y' = cyq(x) - dy \tag{1.4}$$

Here p(x) is the predator functional responses. Often, q(x) is replaced by p(x), in which case c becomes the conversion rate. p(x), q(x) satisfy the usual properties such as being nonnegative and increasing, and equal to zero at zero. Local stability analyses and simulations (Arditi and Ginzburg [2], Berryman [7] show that the ratio-dependent models are capable of producing richer and more reasonable or acceptable dynamics. In many aspects, the ratio dependent models provide the richest dynamics, while the prey dependent ones provide the least in dynamical behaviour (Kuang and Beretta[14].

2. Model Description

Here, a prey- predator model with two age specific prey populations (Juvenile & Adult) and a predator population formulated. Two different functional responses of the predator are considered to represent the difference in the way the predator feeds on each of the prey populations. The assumptions considered while formulating the model are:

(i)   The Juvenile population of the prey is the easy to capture and the predator's response to capture prey is Holling Type II.

(ii)   The adult prey population has adopted an anti-predator behaviour and is hard to capture. This behaviour is represented by the ratio-dependent response of the predator.

(iii)   Both Juvenile and adult prey population follow logistic growth in absence of the predator.

(iv)   There exist interspecific competition among the prey populations.

Model equations

The model to represent the dynamics of predator, adult prey and juvenile prey populations:

$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1}\right) - \alpha_1 N_1 N_2 - \left(\frac{a_1 N_1}{1 + b_1 N_1}\right) P$$

$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - \frac{N_2}{K_2}\right) - \alpha_2 N_1 N_2 - \left(\frac{c N_2}{1 + d_1 N_2 + d_2 P}\right) P \qquad (2.1)$$

$$\frac{dP}{dt} = -eP + \lambda_1 \left(\frac{a_1 N_1}{1 + b_1 N_1}\right) P + \lambda_2 \left(\frac{c N_2}{1 + d_1 N_2 + d_2 P}\right) P$$

Where N_1, N_2, P denote juvenile prey, adult prey and predator population respectively at time t, and $r_1, r_2, \alpha_1, \alpha_2, a_1, c, d_1, d_2, \lambda_1, \lambda_2, e, K_1, K_2$ are all positive constants and $\lambda_1 > \lambda_2$. Here r_1, r_2 represent per capita intrinsic growth rates for prey N_1, N_2, respectively, K_1, K_2 are carrying capacities for prey N_1, N_2, respectively and α_1, α_2 are coefficients for interspecific competition between prey N_1 and N_2 respec-

tively; a_1 and c are capturing rates of predator P on N_1, and N_2 respectively; b_1/a_1 and d_1/c are the predator's handling time on prey N_1 and N_2 respectively; e is natural death rate of predator P; d_2 measures the anti-predator behavior of prey N_2; λ_1, λ_2 are coefficients which measure the predator's efficiency to convert prey biomass of N_1, N_2 respectively into fertility.

For ease of computations, non- dimensionalization of the model represented by equations (1) is done so as to reduce the number of parameters as follows:

Let

$$X_1 = b_1 N_1, \qquad X_2 = b_2 N_2, \qquad Y = d_2 P;$$

equations (2.1) reduce to

$$\frac{dX_1}{dt} = r_1 X_1 \left[\left(1 - \frac{X_1}{b_1 K_1}\right) - \frac{\alpha_1 X_2}{r_1 d_1} - \frac{a_1 Y}{(1 + X_1) r_1 d_2}\right]$$

$$\frac{dX_2}{dt} = r_2 X_2 \left[\left(1 - \frac{X_2}{d_1 K_2}\right) - \frac{\alpha_2 X_1}{r_2 b_1} - \frac{cY}{(1 + X_2 + Y) r_2 d_2}\right] \qquad (2.2)$$

$$\frac{dY}{dt} = eY \left[-1 + \frac{\lambda_1 a_1 X_1}{eb_1 (1 + X_1)} + \frac{\lambda_2 c X_2}{ed_1 (1 + X_2 + Y)}\right]$$

Also considering,

$$\beta_1 = \frac{1}{K_1 b_1}, \delta_1 = \frac{\alpha_1}{r_1 d_1}, g_1 = \frac{a_1}{r_1 d_2}, \beta_2 = \frac{1}{K_2 d_1}, \delta_2 = \frac{\alpha_2}{r_2 b_1}, h_1 = \frac{\lambda_1 a_1}{eb_1}, h_2 = \frac{\lambda_1 a_1}{ed_1}$$

give,

$$\frac{dX_1}{dt} = r_1 X_1 \left[(1 - \beta_1 X_1) - \delta_1 X_2 - \frac{g_1 Y}{1 + X_1}\right]$$

$$\frac{dX_2}{dt} = r_2 X_2 \left[(1 - \beta_2 X_2) - \delta_2 X_1 - \frac{g_2 Y}{1 + X_2 + Y}\right] \qquad (2.3)$$

$$\frac{dY}{dt} = eY \left[-1 + \frac{h_1 X_1}{1 + X_1} + \frac{h_2 X_2}{1 + X_2 + Y}\right]$$

3. Analysis of the Model:

Equilibrium points

Equating the governing equation of the model (2.3) to zero, the equilibrium points of the system (2.3) are derived. They are:

i)   The trivial equilibrium $P_0(0,0,0)$.

ii) Equilibrium in the absence of adult prey and predator $(X_2 = 0, Y = 0), P_1(X_1^*, 0, 0)$.

iii) Equilibrium in the absence of juvenile prey and predator $(X_1 = 0, Y = 0), P_2(0, X_2^*, 0)$.

iv) Equilibrium in the absence of predator $(Y = 0), P_3(X_1^*, X_2^*, 0)$

v) Equilibrium in the absence of adult prey $(X_2 = 0), P_4(X_1^*, 0, Y^*)$

vi) Equilibrium in the absence of juvenile prey $(X_1 = 0), P_5(0, X_2^*, Y^*)$

vii) The co-existence equilibrium $P_6(X_1^*, X_2^*, Y^*)$.

Existence of Equilibrium points

The existence of $P_0(0, 0, 0)$ is trivial. We show the existence of other equilibria as follows:

(i) Existence of $P_1(X_1^*, 0, 0)$

Let $X_2 = 0$ and $Y = 0$. Then equations (2.3) give, $X_1^* = \frac{1}{\beta_1}$. Therefore,

$P_1(X_1^*, 0, 0) = P_1\left(\frac{1}{\beta_1}, 0, 0\right)$. $X_1^* = \frac{1}{\beta_1}$ implies, $N_1 = K_1$. Thus in the absence of adult prey N_2 and predator P, the density of juvenile prey N_1 will increase or decrease until it reaches the carrying capacity K_1 of the area.

(ii) Existence of $P_2(0, X_2^*, 0)$

Let $X_1 = 0$ and $Y = 0$. Then equations (2.3) give $X_2^* = \frac{1}{\beta_2}$. Thus,

$P_2(0, X_2^*, 0) = P_2\left(0, \frac{1}{\beta_2}, 0\right)$. $X_2^* = \frac{1}{\beta_2}$ implies $N_2 = K_2$. Thus in the absence of juvenile prey N_1 and predator P, the density of adult prey N_2 will increase or decrease until it reaches the carrying capacity K_2 of the area.

(iii) Existence of $P_3(X_1^*, X_2^*, 0)$

Let $Y = 0$. Then equations (2.3) give

$$X_1^* = \frac{\delta_1 - \beta_2}{\delta_1 \delta_2 - \beta_1 \beta_2}, \quad X_2^* = \frac{\delta_2 - \beta_1}{\delta_1 \delta_2 - \beta_1 \beta_2}.$$

Hence, $P_3(X_1^*, X_2^*, 0) = P_3\left(\frac{\delta_1 - \beta_2}{\delta_1 \delta_2 - \beta_1 \beta_2}, \frac{\delta_2 - \beta_1}{\delta_1 \delta_2 - \beta_1 \beta_2}, 0\right)$.

This exists if,

for, $\beta_2 > \delta_1$ and $\beta_1 > \delta_2$,

$$\beta_1 \beta_2 > \delta_1 \delta_2 \qquad (3.1)$$

and for, $\beta_2 < \delta_1$ and $\beta_1 < \delta_2$,

$$\beta_1 \beta_2 < \delta_1 \delta_2 \qquad (3.2)$$

Condition (3.1) implies $\alpha_1 \alpha_2 < \frac{r_1}{K_1} \frac{r_2}{K_2}$. This condition shows the essential parameters for existence of the two prey species are per capita intrinsic growth rates of the prey, carrying capacities of the prey and interspecific competition among the prey.

(iv) Existence of $P_4(X_1^*, 0, Y^*)$

Let $X_2 = 0$. Then equations (2.3) implies

$$r_1 X_1 \left[1 - \beta_1 X_1 - \frac{g_1 Y}{1 + X_1}\right] = 0 \qquad (3.3)$$

$$eY\left[-1 + \frac{h_1 X_1}{1 + X_1}\right] = 0 \qquad (3.4)$$

Solving (3.3) and (3.4) for $Y \neq 0$ and $X_1 \neq 0$, we have,

$$X_1^* = \frac{1}{h_1 - 1} \text{ provided } h_1 > 1 \qquad (3.5)$$

$$Y^* = \frac{h_1(h_1 - 1 - \beta_1)}{g_1(h_1 - 1)^2} \qquad (3.6)$$

Therefore $P_4(X_1^*, 0, Y^*) = P_4\left(\frac{1}{h_1 - 1}, 0, \frac{h_1(h_1 - 1 - \beta_1)}{g_1(h_1 - 1)^2}\right)$ exists if

$$h_1 > 1 \qquad (3.7)$$

$$h_1 - 1 > \beta_1 \qquad (3.8)$$

From condition (3.7) $\lambda_1 > \frac{eb_1}{a_1}$ i.e., λ_1 the proportion of biomass of prey N_1 converted into food by the predator P must be greater than the product of the predator's natural death rate, e and the time it takes to handle the prey $\frac{b_1}{a_1}$.

V) Existence of $P_5(0, X_2^*, Y^*)$ with $X_2^* > 0, Y^* > 0$

Let $X_1 = 0$ Then (3.3) implies

$$r_2 X_2 \left[1 - \beta_2 X_2 - \frac{g_2 Y}{1 + X_2 + Y}\right] = 0 \qquad (3.9)$$

$$eY\left[-1 + \frac{h_2 X_2}{1 + X_2 + Y}\right] = 0 \qquad (3.10)$$

From (3.10), for $X_2 \neq 0$ and $Y \neq 0$,

30

$$Y = h_2 X_2 - X_2 - 1 \qquad (3.11)$$

From (3.9) and (3.11),
$$\beta_2 h_2 X_2^2 + (g_2 h_2 - h_2 - g_2)X_2 - g_2 = 0 \qquad (3.12)$$
Which is of the form $AX_2^2 + BX_2 + C = 0$
with $A = \beta_2 h_2, B = (g_2 h_2 - h_2 - g_2), C = -g_2$.
This gives,

$$X_2^* = \frac{-B+\sqrt{B^2-4AC}}{2A} \qquad (3.13)$$

Since $A > 0, C < 0$ and $B > 0, X_2^*$ is positive and therefore exists. Thus, from (3.11) ,
$Y^* = X_2^*(h_2 - 1) - 1$ is positive if, $X_2^* > \frac{1}{h_2-1}$ provided

$$h_2 > 1 \qquad (3.14)$$

The condition $h_2 > 1$ implies, $\lambda_2 > e^{\frac{d_1}{c}}$. This means λ_2 the proportion of biomass of prey N_2 converted into fertility by the predator P must be greater than the product of the predator's natural death rate e and the time it takes to handle the prey $\frac{d_1}{c}$.

(v) Co-Existence of Equilibrium point $P_6(X_1^*, X_2^*, Y^*)$

Equating equations (2.3) to zero gives,
$$(1 - \beta_1 X_1) - \delta_1 X_2 - \frac{g_1 Y}{1+X_1} = 0 \qquad (3.15)$$

$$(1 - \beta_2 X_2) - \delta_2 X_1 - \frac{g_2 Y}{1+X_2+Y} = 0 \qquad (3.16)$$

$$-1 + \frac{h_1 X_1}{1+X_1} + \frac{h_2 X_2}{1+X_2+Y} = 0 \qquad (3.17)$$

From (3.15), $Y = \frac{1+X_1}{g_1}(1 - \beta_1 X_1 - \delta_1 X_2)$ (3.18)

From (3.16), $Y = \frac{(1+X_2)(1-\beta_2 X_2-\delta_2 X_1)}{g_2-(1-\beta_2 X_2-\delta_2 X_1)}$ (3.19)

From (3.16) and (3.17),
$$Y = \frac{h_2 X_2(1+X_1)(1-\beta_2 X_2-\delta_2 X_1)}{g_2(1+X_1-h_1 X_1)} \qquad (3.20)$$

Equations (3.19) and (3.20) give
$$f(X_1, X_2) = \frac{h_2 X_2(1+X_1)}{g_2(1+X_1-h_1 X_1)} - \frac{(1+X_2)}{g_2-(1-\beta_2 X_2-\delta_2 X_1)} = 0 \quad (3.21)$$

Equations (3.18) and (3.20) implies

$$g(X_1, X_2) = \frac{(1-\beta_1 X_1 - \delta_1 X_2)}{g_1} - \frac{h_2 X_2(1-\beta_2 X_2-\delta_2 X_1)}{g_2(1+X_1-h_1 X_1)}$$

$$= 0 \qquad (3.22)$$

To prove the existence of $P_6(X_1^*, X_2^*, Y^*)$, conditions under which $f(X_1, X_2)$ and $g(X_1, X_2)$ meet in the interior of the positive (X_1, X_2) plane, at a point (X_1^*, X_2^*), are found. Y^* can be obtained from (3.19), provided X_1^* and X_2^* are known.

From (3.21), as $X_1 \to 0, X_2 \to X_{2a}$ where,

$$X_{2a} = \frac{-B_1+\sqrt{B_1^2-4A_1 C_1}}{2A_1} \qquad (3.23)$$

Where, $A_1 = h_2\beta_2, B_1 = h_2 g_2 - h_2 - g_2, C_1 = -g_2$
X_{2a} is positive and real since $C_1 < 0$. We notice that X_{2a} is same as X_2^* of $P_5(0, X_2^*, Y^*)$. Again, from (3.22) as $X_1 \to 0, X_2 \to X_{2b}$ given by,

$$X_{2b} = \frac{-B_2+\sqrt{B_2^2-4A_2 C_2}}{2A_2} \qquad (3.24)$$

$A_2 = h_2\beta_2 g_1, B_2 = -(g_2\delta_1 - h_2 g_1), C_2 = g_2$
X_{2b} is positive and real if $B_2^2 > 4A_2 C_2$
$\Rightarrow (g_2\delta_1 - h_2 g_1)^2 > 4h_2\beta_2 g_1 g_2$
X_{2a} and X_{2b} are the points at which the functions $f(X_1, X_2)$ and $g(X_1, X_2)$ would cut the X_2 axis in the (X_1, X_2) plane respectively. Also from (3.21), $\frac{dX_2}{dX_1} = -\frac{\partial f}{\partial X_1}/\frac{\partial f}{\partial X_2}$ where

$$\frac{\partial f}{\partial X_1} = \frac{h_1 h_2 X_2}{g_2(1+X_1-h_1 X_1)^2} + \frac{\delta_2(1+X_2)}{\{g_2-(1-\beta_2 X_2-\delta_2 X_1)\}^2}$$

$$\frac{\partial f}{\partial X_2} = \frac{h_2(1+X_1)}{g_2(1+X_1-h_1 X_1)} - \frac{(g_2-\beta_2)+(\delta_2 X_1-1)}{\{g_2-(1-\beta_2 X_2-\delta_2 X_1)\}^2}$$

We observe that $\frac{dX_2}{dX_1} > 0$ if $\frac{\partial f}{\partial X_1} > 0$ and $\frac{\partial f}{\partial X_1} < 0$ and this requires $h_1 > 1$, $g_2 > \beta_2$ and $X_1 > \max\{\frac{1}{\delta_2}, \frac{1}{h_1-1}\}$. Similarly from (3.22), $\frac{dX_2}{dX_1} = -\frac{\partial g}{\partial X_1}/\frac{\partial g}{\partial X_2}$ where

$$\frac{\partial g}{\partial X_1} = -\left[\frac{\beta_1}{g_1} + \frac{h_2 X_2}{g_2}\frac{(1-\beta_2 X_2)(h_1-1)-\delta_2}{(1+X_1-h_1 X_1)^2}\right]$$

$$\frac{\partial g}{\partial X_2} = -\left[\frac{\delta_1}{g_1} + \frac{h_2(1-\delta_2 X_1-2\beta_2 X_2)}{g_2(1+X_1-h_1 X_1)}\right]$$

We also observe that $\frac{dX_2}{dX_1} < 0$ if $\frac{\partial g}{\partial X_1} < 0$ and $\frac{\partial g}{\partial X_2} < 0$ and this requires $h_1 > 1$, $0 < X_2 < \frac{1}{\beta_2}\left(1 - \frac{\delta_2}{h_1-1}\right)$ and $X_1 > \max\{\frac{1}{\delta_2}, \frac{1}{h_1-1}\}$. As for $f(X_1, X_2)$, we have $\frac{dX_2}{dX_1} > 0$ and for $g(X_1, X_2)$, we have $\frac{dX_2}{dX_1} < 0$, then $f(X_1, X_2)$ and $g(X_1, X_2)$ will meet if $X_{2a} < X_{2b}$. Thus the positive equilibrium point $P_6(X_1^*, X_2^*, Y^*)$ will exist if , for $h_1 > 1$, $g_2 > \beta_2$,
$$X_1 > \max\{\tfrac{1}{\delta_2}, \tfrac{1}{h_1-1}\}, 0 < X_2 < \tfrac{1}{\beta_2}\left(1 - \tfrac{\delta_2}{h_1-1}\right), X_{2a} < X_{2b}.$$

4. Stability analysis:
The two prey one predator model as described by (2.3) is linearised around the equilibrium points . Now the Jacobian of the system is given by

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial X_1} & \frac{\partial f_1}{\partial X_2} & \frac{\partial f_1}{\partial Y} \\ \frac{\partial f_2}{\partial X_1} & \frac{\partial f_2}{\partial X_2} & \frac{\partial f_2}{\partial Y} \\ \frac{\partial f_3}{\partial X_1} & \frac{\partial f_3}{\partial X_2} & \frac{\partial f_3}{\partial Y} \end{bmatrix}$$

$$= \begin{bmatrix} A^* & -\delta_1 r_1 X_1 & -\frac{g_1 r_1 X_1}{1+X_1} \\ -\delta_2 r_2 X_2 & B^* & -\frac{g_2 r_2 X_2(1+X_2)}{(1+X_2+Y)^2} \\ \frac{eYh_1}{(1+X_1)^2} & \frac{eYh_2(1+Y)}{(1+X_2+Y)^2} & C^* \end{bmatrix} \quad (3.25)$$

Where

$$A^* = r_1 - 2r_1\beta_1 X_1 - \delta_1 r_1 X_2 - \frac{g_1 r_1 Y X_1}{(1+X_1)^2}$$

$$B^* = r_2 - 2r_2\beta_2 X_2 - \delta_2 r_2 X_1 - \frac{g_2 r_2 Y(1+Y)}{(1+X_2+Y)^2}$$

$$C^* = -e + \frac{h_1 e X_1}{1+X_1} + \frac{h_2 e X_2(1+X_2)}{(1+X_2+Y)^2}$$

4.1 Stability Analysis for the trivial equilibrium $P_0(0,0,0)$

The Jacobian matrix at $P_0(0,0,0)$ of the system (3) is given by

$$J(P_0) = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & -e \end{bmatrix}$$

With the eigenvalues are r_1, r_2 and $-e$. We see that $r_1, r_2 > 0$ is always positive.
Therefore $P_0(0,0,0)$ is unstable point.

4.2 Stability analysis around the equilibrium $P_1(X_1^*,0,0)$

Now, the Jacobian matrix of the system (3) at $P_1(X_1^*,0,0) = P_1\left(\frac{1}{\beta_1},0,0\right)$ is

$$J(P_1) = \begin{bmatrix} -r_1 & -\frac{\delta_1 r_1}{\beta_1} & -\frac{g_1 r_1}{1+\beta_1} \\ 0 & r_2\left(1-\frac{\delta_2}{\beta_1}\right) & 0 \\ 0 & 0 & -e+\frac{h_1 e}{1+\beta_1} \end{bmatrix}$$

The eigenvalues of are $J(P_1)$ are $-r_1, r_2\left(1-\frac{\delta_2}{\beta_1}\right)$ and $-e+\frac{h_1 e}{1+\beta_1}$.
This shows that the system is locally asymptotically stable if

$$r_2\left(1-\frac{\delta_2}{\beta_1}\right) < 0$$

i.e. if

$$\delta_2 > \beta_1$$

and

$$-e+\frac{h_1 e}{1+\beta_1} < 0$$

i.e. if $\qquad \beta_1 > h_1 - 1$
i.e. if $\qquad h_1 - 1 < \beta_1 < \delta_2 \qquad (3.26)$

and $\qquad\qquad 0 < h_1 < 1 \qquad (3.27)$

The condition $h_1 < 1$ implies $\lambda_1 < e\frac{b_1}{a_1}$. That is , for local asymptotic stability of $P_1(X_1^*,0,0)$, λ_1 which measures the efficiency of the predator to convert the biomass of prey N_1 into fertility must be less than the predator's natural death rate , e and the time it takes to handle prey N_1, $\frac{b_1}{a_1}$.

4.3 Stability analysis around the equilibrium $P_2(0, X_2^*, 0)$

The Jacobian matrix at the equilibrium point $P_2(0, X_2^*, 0) = P_2\left(0, \frac{1}{\beta_2}, 0\right)$ is

$$J(P_2) = \begin{bmatrix} r_1 - \frac{\delta_1 r_1}{\beta_2} & 0 & 0 \\ -\frac{r_2 \delta_2}{\beta_2} & -r_2 & -\frac{g_2 r_2}{\beta_2 + 1} \\ 0 & 0 & -e+\frac{h_2 e}{1+\beta_2} \end{bmatrix}$$

The eigenvalues of the matrix $J(P_2)$ are $r_1 - \frac{\delta_1 r_1}{\beta_2}$, $-r_2$ and $-e+\frac{h_2 e}{1+\beta_2}$.
The system is locally asymptotically stable if

$$r_1 - \frac{\delta_1 r_1}{\beta_2} < 0 \text{ and } -e+\frac{h_2 e}{1+\beta_2} < 0$$

i.e if $\qquad \beta_2 < \delta_1 \qquad (3.28)$

and $\qquad h_2 < \beta_2 + 1 \qquad (3.29)$
Condition (3.29) implies $\qquad \frac{\lambda_2 c}{e} < \frac{1}{K_2} + d_1$. This shows parameters that are essential for local asymptotic stability of $P_2(0, X_2^*, 0)$.

4.4 Stability analysis around the equilibrium $P_3(X_1^*, X_2^*, 0)$

The Jacobian matrix at the equilibrium point $P_3(X_1^*, X_2^*, 0) = P_3\left(\frac{\delta_1 - \beta_2}{\delta_1\delta_2 - \beta_1\beta_2}, \frac{\delta_2 - \beta_1}{\delta_1\delta_2 - \beta_1\beta_2}, 0\right)$ is

$$J(P_3) = \begin{bmatrix} A_1 & B_1 & C_1 \\ D_1 & E_1 & F_1 \\ 0 & 0 & G_1 \end{bmatrix}$$

Where

$A_1 = r_1 - 2r_1\beta_1 X_1^* - \delta_1 r_1 X_2^*, B_1 = -\delta_1 r_1 X_1^*,$

$C_1 = -\frac{g_1 r_1 X_1^*}{1+X_1^*}, D_1 = -r_2\delta_2 X_2^*,$

$E_1 = r_2 - 2r_2\beta_2 X_2^* - \delta_2 r_2 X_1^*$

$F_1 = -\frac{g_2 r_2 X_2^*}{1+X_2^*}, \quad G_1 = -e + \frac{h_1 e X_1^*}{1+X_1^*} + \frac{h_2 e X_2^*}{1+X_2^*}$

The characteristic equation is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

Where, $a_1 = -(A_1 + E_1 + G_1),$
$a_2 = A_1 G_1 + E_1 G_1 + A_1 E_1 - B_1 D_1$
$a_3 = G_1(D_1 B_1 - A_1 E_1)$

By Routh Hurwitz criterion the λ's are negative if $a_1 > 0, a_3 > 0, a_1 a_2 - a_3 > 0$. Each of these conditions are considered next as follows:

a) $a_1 > 0 \Rightarrow -(A_1 + E_1 + G_1)$

$$\Rightarrow (A_1 + E_1 + G_1) < 0 \qquad (3.30)$$

This can be satisfied if $G_1 < 0, A_1 < 0$ & $E_1 < 0$

i) $G_1 < 0$

$$\Rightarrow -e + \frac{h_1 e X_1^*}{1 + X_1^*} + \frac{h_2 e X_2^*}{1 + X_2^*} < 0$$

$$\Rightarrow X_1^* X_2^* (h_1 + h_2 - 1) + X_1^* (h_1 - 1) + X_2^* (h_2 - 1) - 1 < 0$$

This will hold if, for $\beta_2 > \delta_1, \beta_1 > \delta_2$, $\beta_1 \beta_2 > \delta_1 \delta_2$,

$h_1 < 1$ and $h_2 < 1$

$$h_1 + h_2 < 1 \qquad (3.31)$$

Which implies, $\frac{\lambda_1 a_1}{b_1} + \frac{\lambda_2 c}{d_1} < e$

That is the predator's efficiency in converting the biomass of both prey into fertility must be less than the predator's natural death rate and the time it takes to handle both prey.

ii) $A_1 < 0 \Rightarrow r_1 - 2r_1 \beta_1 X_1^* - \delta_1 r_1 X_2^* < 0$

$$\Rightarrow \frac{\beta_1(\beta_2 - \delta_1)}{\delta_1 \delta_2 - \beta_1 \beta_2} < 0$$

Therefore $A_1 < 0$ if $\beta_2 > \delta_1$ and $\beta_1 \beta_2 > \delta_1 \delta_2$.

iii) $E_1 < 0 \Rightarrow r_2 - 2r_2 \beta_2 X_2^* - \delta_2 r_2 X_1^* < 0$

$$\Rightarrow \frac{r_2 \beta_2 (\beta_1 - \delta_2)}{\delta_1 \delta_2 - \beta_1 \beta_2} < 0$$

Therefore $E_1 < 0$ if $\beta_1 > \delta_2$ and $\beta_1 \beta_2 > \delta_1 \delta_2$.

b) $a_3 > 0 \Rightarrow G_1(D_1 B_1 - A_1 E_1) > 0$

This is satisfied if,

$$G_1 < 0, D_1 B_1 - A_1 E_1 < 0 \qquad (3.32)$$

Now, $D_1 B_1 - A_1 E_1 = r_1 r_2 \{X_2^* u + X_1^* v - 1 - 4\beta_1 \beta_2 X_1^* X_2^*\}$,

Where

$$u = 2\beta_2 + \delta_1 - 2\delta_1 \beta_2 X_2^*$$
$$v = 2\beta_1 + \delta_2 - 2\delta_2 \beta_1 X_1^*$$

Therefore $D_1 B_1 - A_1 E_1 < 0$ if $u < 0, v < 0$

$$u = \frac{-2\beta_1 \beta_2^2 + \delta_1(\delta_1 \delta_2 + \beta_1 \beta_2)}{\delta_1 \delta_2 - \beta_1 \beta_2}$$

Thus $u < 0$ if for $\beta_1 \beta_2 > \delta_1 \delta_2$

$$\frac{2\beta_1 \beta_2^2}{\delta_1 \delta_2 + \beta_1 \beta_2} < \delta_1 \qquad (3.33)$$

Similarly,

$$v = \frac{-2\beta_2 \beta_1^2 + \delta_2(\delta_1 \delta_2 + \beta_1 \beta_2)}{\delta_1 \delta_2 - \beta_1 \beta_2}$$

Thus $v < 0$ if for $\beta_1 \beta_2 > \delta_1 \delta_2$

$$\frac{2\beta_2 \beta_1^2}{\delta_1 \delta_2 + \beta_1 \beta_2} < \delta_2 \qquad (3.34)$$

Therefore $D_1 B_1 - A_1 E_1 < 0$ if for $\beta_1 > \delta_2, \ \beta_2 > \delta_1$ and $\beta_1 \beta_2 > \delta_1 \delta_2$:

$$\frac{2\beta_2 \beta_1^2}{\delta_1 \delta_2 + \beta_1 \beta_2} < \delta_2$$
$$\frac{2\beta_1 \beta_2^2}{\delta_1 \delta_2 + \beta_1 \beta_2} < \delta_1 \qquad (3.35)$$

(c) $a_1 a_2 - a_3 > 0$ implies,

$(A_1 + E_1)\{(B_1 D_1 - A_1 E_1) - G_1(A_1 + E_1 + G_1)\} > 0$

This is satisfied if $A_1 < 0, \ B_1 < 0, \ G_1 < 0$ and $B_1 D_1 - A_1 E_1 < 0$.

Therefore $P_3(X_1^*, X_2^*, 0)$ is locally asymptotically stable if conditions (3.2), (3.31) and (3.35) hold.

4.4 Stability analysis around the equilibrium $P_4(X_1^*, 0, Y^*)$

The Jacobian matrix at

$$P_4(X_1^*, 0, Y^*) = P_4\left(\frac{1}{h_1 - 1}, 0, \frac{h_1(h_1 - 1 - \beta_1)}{g_1(h_1 - 1)^2}\right)$$

$$J(P_4) = \begin{bmatrix} A_2 & -\delta_1 r_1 X_1^* & -\frac{g_1 r_1 X_1^*}{1 + X_1^*} \\ 0 & B_2 & 0 \\ \frac{h_1 e Y^*}{(1 + X_1^*)^2} & \frac{h_2 e Y^*}{(1 + Y^*)^2} & 0 \end{bmatrix}$$

Where, $A_2 = r_1 - 2r_1 \beta_1 X_1^* - \frac{g_1 r_1 Y^*}{(1 + X_1^*)^2}$

$$B_2 = r_2 - \delta_2 r_2 X_1^* - \frac{r_2 g_2 Y^*}{(1 + Y^*)}$$

The eigenvalues of $J(P_4)$ have negative real parts if $A_2 < 0$ and $B_2 < 0$. P_4 is locally asymptotically if, for $h_1 > 1$:

$$\beta_1 < h_1 - 1 < 2\beta_1 \qquad (3.36)$$
$$0 < h_1 < \delta_2 + 1 \qquad (3.37)$$

4.5 Stability analysis around the equilibrium $P_5(0, X_2^*, Y^*)$

The Jacobian matrix at $P_5(0, X_2^*, Y^*)$ is

$$J(P_5) = \begin{bmatrix} A_3 & 0 & 0 \\ -r_2 \delta_2 X_2^* & B_3 & -\frac{g_2 r_2 X_2^*(1 + X_2^*)}{(h_2 X_2^*)^2} \\ h_1 e Y^* & h_2 e Y^*(1 + Y^*) & C_3 \end{bmatrix}$$

Where, $A_3 = r_1 - \delta_1 r_1 X_2^*$,

$$B_3 = r_2 - 2r_2 \beta_2 X_2^* - \frac{g_2 r_2 Y^*(1 + Y^*)}{(h_2 X_2^*)^2} \text{ and }$$

$$C_3 = -e + \frac{e h_2 X_2^*(1 + X_2^*)}{(h_2 X_2^*)^2}$$

The matrix $J(P_5)$ will have eigen values with negative real parts if $A_3 < 0, B_3 < 0$ and $C_3 < 0$. These conditions will hold if for $h_2 > 1$ and $X_2^* > \min\left\{\frac{1}{\beta_2}, \frac{1}{h_2-1}\right\}$ (3.38)

4.5 Stability analysis around the equilibrium $P_6(X_1^*, X_2^*, Y^*)$

The Jacobian matrix at $P_6(X_1^*, X_2^*, Y^*)$ is

$$J(P_6) = \begin{bmatrix} A_4 & -\delta_1 r_1 x_1^* & -\frac{g_1 r_1 x_1^*}{1+x_1^*} \\ -r_2 \delta_2 X_2^* & B_4 & -\frac{g_2 r_2 X_2^*(1+X_2^*)}{(1+X_2^*+y^*)^2} \\ \frac{h_1 e Y^*}{(1+x_1^*)^2} & \frac{h_2 e Y^*(1+Y^*)}{(1+X_2^*+y^*)^2} & C_4 \end{bmatrix}$$

Where

$$A_4 = r_1 - 2r_1\beta_1 X_1^* - \delta_1 r_1 X_2^* - \frac{g_1 r_1 Y^* X_1^*}{(1+X_1^*)^2}$$

$$B_4 = r_2 - 2r_2\beta_2 X_2^* - \delta_2 r_2 X_1^* - \frac{g_2 r_2 Y^*(1+Y^*)}{(1+X_2^*+Y^*)^2}$$

$$C_4 = -e + \frac{h_1 e X_1^*}{1+X_1^*} + \frac{h_2 e X_2^*(1+X_2^*)}{(1+X_2^*+Y^*)^2}$$

The stability of the point $P_6(X_1^*, X_2^*, Y^*)$ depends on the determinant and trace of the above Jacobian $J(P_6)$. The point is stable if det $J(P_6) > 0$ and Trace $J(P_6) < 0$.

Conclusion

In this paper we have analysed a prey –predator model to study the dynamics of two prey population where one prey is Juvenile prey(weak) , easy to capture and another one is adult population (stronger) hard to capture with one predator population. The response functions are Holling Type II and ratio-dependent. Existence and local stability criterion of all equilibrium points are discussed. Here we observed that the existence & stability of the system are affected by parameter values.

Reference:

[1] Arditi, R. and Berryman, A. A. [1991] The biological control paradox, Trends in Ecology and Evolution, **6**, 32.

[2] Arditi, R. and Ginzburg, L. R. [1989] Coupling in predator-prey dynamics: ratio dependence, J.Theoretical Biology, **139**, 311–326.

[3] Arditi, R., Ginzburg, L. R. and Akcakaya, H. R. [1991] Variation in plankton densities among lakes: a case for ratio-dependent models, American Naturalist, **138**, 1287–1296.

[4] Arditi, R., Perrin, N. and Saiah, H. [1991] Functional responses and heterogeneities:

an experimental test with cladocerans, OIKOS, **60**, 69–75.

[5] Arditi, R. and Saiah, H. [1992] Empirical evidence of the role of heterogeneity in ratio-dependent consumption, Ecology, **73**, 1544–1551.

[6] Beretta, E. and Kuang, Y. [1997] Global analyses in some delayed ratio dependent predator-prey systems, Nonlinear Analysis, TMA., in press.

[7] Berryman, A. A. [1992] The origins and evolution of predator-prey theory, Ecology, **73**, 1530–1535.

[8] Edwin A. [2010] modeling and analysis of a two prey-one predator system with harvesting, holling type II & ratio-dependent responses, 1-81.

[9] Freedman, H. I. [1980] Deterministic Mathematical Models in Population Ecology, Marcel Dekker, New York.

[10] Gutierrez, A. P. [1992] The physiological basis of ratio-dependent predator-prey theory: a metabolic pool model of Nicholson's blowflies as an example, Ecology, **73**, 1552–1563.

[11] Hanski, I. [1991] The functional response of predator: worries about scale, TREE, **6**, 141–142.

[12] Hairston, N. G., Smith, F. E. and Slobodkin, L. B. [1960] Community structure, population control and competition, American Naturalist, **94**, 421–425.

[13] Holling,C.S.(1965)," The functional response of predator to prey density and its role in mimicry and population regulation", Mem. Ent. Soc. Canada 45 (1965), pp. 1-60.

[14] Kuang, Y. and Beretta, E. [1996] Global qualitative analysis of a ratio-dependent predator-prey system, J. Math. Biol., in press.

[15] Kuang, Y. and Freedman, H. I. [1988] Uniqueness of limit cycles in Gause-type predator-prey systems, Math. Biosci., **88**, 67–84.

[16]Lotka,A.J.(1925) Elements of Physical Biology, Williams &Wilkins Co., Baltimore.

[17] Luck, R. F. [1990] Evaluation of natural enemies for biological control: a behavior approach, Trends in Ecology and Evolution, **5**, 196–199.

[18] May, R. M. [1974] Stability and Complexity in Model Ecosystems, Princeton Univ. Press.

[19] Rosenzweig, M. L. [1969] Paradox of enrichment: destabilization of exploitation systems in ecological time, Science, **171**, 385–387.

[20] Smith, J. Maynard [1974] Models in Ecology, Cambridge Univ. Press.

[21] Volterra,V.(1926),"Fluctuations in the abundance of a species considered mathematically", Nature, 118, pp. 558-600.

Biographies

ANURADHA DEVI received the B.Sc degree in Mathematics honours from the Gauhati University, Assam, in 1984, the M.Sc. degree in Mathematics from the I.I.T, Kharagpur, West Bengal in 1996, and the Ph.D. degree in Mathematics, Gauhati University, Assam, in 1999respectively. Currently, She is an Professor of Mathematics at Royal Group of Institutions, Guwahati, Assam. Her teaching and research areas include instrumentation,

Mathematical modelling and control of Biological systems. She has co-authored four textbook on Engineering Mathematics. Dr. Anuradha Devi may be reached at devianuradha09@gmail.com

MUNMI SAIKIA is a research scholar at the department of Mathematics of Gauhati University. She obtained M.Sc degree with First class from Gauhati Unviersity in 2007, M.Phil from Gauhati University in 2009. She has been working as an Asst. professor in Royal Group of Institutions since 2011 till date. Her research field is Mathematical Modeling and Control of Ecosystem. She may be reached at munmi19saikia@gmail.com