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Partition identities-I and their Combinatorial interpretation

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Abstract

The authors give the set theoretic (combinatorial) interpretation of different modulei in the infinite product, which are not available (derive identies related to modulus 11, 17, 19) in this form in the literature using multiple summations.

Key words: Basic Hypergoemetric series, Partition, Jacobi's Triple Product Identity, Combinatorial Interpretation, Rogers-Ramanujan Type Identities.

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1.0 Introduction

Work on the Rogers-Ramanujan identities have been going on for the past 100 years. The original Rogers-Ramanujan identities were:

$$(1)\quad \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty}\left(1-q^{5n-1}\right)^{-1}\left(1-q^{5n-4}\right)^{-1}$$

$$(2)\quad \sum_{n=0}^{\infty} \frac{q^{n^2-n}}{(q;q)_n} = \prod_{n=1}^{\infty}\left(1-q^{5n-2}\right)^{-1}\left(1-q^{5n-3}\right)^{-1}$$

Since then many eminent mathematicians have done work relating to different moduli in the infinite product. Prominent among them are G.E. Andrews, D.M. Bressoud, G. Watson and Bailey, to mention a few. Here we derive identities related to modulus 11,17,19, which are not available in this form, in the literature, using multiple summations. These identities have an interesting set theoretic interpretation related to partition theory.

In the second half of this paper, we give the set theoretic (combinatorial) interpretation.

Definitions and Notations:

1.1. For $|q| < 1$, the q- shifted factorial is defined by $(a;q)_0 = 1$

$$(a;q)_n = \prod_{k=0}^{n-1}\left(1-aq^k\right) \qquad \text{for } n \geq 1$$

$$(a;q)_\infty = \prod_{k=0}^{\infty}\left(1-aq^k\right)$$

The multiple q- shifted factorial is

$$(a_1,a_2,\ldots\ldots a_n;q)_n=(a_1;q)_n(a_2;q)_n\ldots\ldots(a_n;q)_n$$

$$(a_1,a_2,\ldots\ldots a_m;q)_n=(a_1;q)_\infty(a_2;q)_\infty\ldots\ldots(a_m;q)_\infty$$

1.2 The basic hypergeometric series is

$${}_{p+1}\phi_{p+r}\left(\begin{matrix}a_1a_2\ldots\ldots a_{p+1};q;x\\ b_1,b_2\ldots\ldots b_{p+r}\end{matrix}\right)$$

$$=\sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\ldots\ldots(a_{p+1};q)_n x^n(-1)^{nr}q^{\frac{n(n-1)r}{2}}}{(q;q)_n(b_1;q)_n(b_1;q)_n\ldots\ldots(b_{p+r};q)_n}$$

The series ${}_{p+1}\phi_{p+r}$ converges $\forall$ positive integers r and $\forall x$. For r=0, it converges for $|x|<1$

1.3. Jacobi's Triple product Identity.

$$\left(zq^{\frac{1}{2}},z^{-1}q^{\frac{1}{2}},q;q\right)_\infty=\sum_{n=-\infty}^{\infty}(-1)^n z^n q^{\frac{n^2}{2}}$$

and its corollary,

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{(2k+1)n(n+1)}{2}-in}$$

$$=\sum_{n=0}^{\infty}(-1)^n q^{\frac{(2k+1)n(n+1)}{2}-in}\left(1-q^{(2n+1)i}\right)$$

$$=\prod_{n=0}^{\infty}\left(q^{(2k+1)n(n+1)}\right)\left(1-q^{(2k+1)n+i}\right)\left(1-q^{(2k+1)(n+1)-i}\right)$$

1.4. Bailey's Lemma:

If p is a non-negative integer, then

$$(aq;q)_\infty\sum_{n=0}^{\infty}a^n q^{n^2-pn}\beta_n$$

$$=\sum_{j=0}^{p}\frac{\left(q^{-p};q\right)_j(-a)^j q^{j(j+1)/2}}{(q;q)_j}.$$

$$\sum_{j=0}^{p}a^n q^{n^2-pn+2nj}\alpha_n$$

where $\beta_n=\sum_{r=0}^{n}\frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}}$

2.0 Derivation of Rogers-Ramanujan Type Identities modulo 11 :

Using (1.4) of [1] and Bailey's Lemma, we get the following transformation

$$\left(a^2q;q\right)_\infty\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{a^{2n+4r}q^{n^2+2nr+2r^2-pn-pr}}{(q;q)_r(q;q)_n\left(qa^2;q^2\right)_r}$$

$$=\sum_{j=0}^{p}\frac{\left(q^{-p};q\right)_j(-1)^j a^{2j}q^{j(j+1)/2}}{(q;q)_j}.$$

$$\sum_{n=0}^{\infty}\frac{\left(a^2;q^2\right)_n\left(1-a^2q^{4n}\right)(-1)^n a^{10n}q^{11n^2-2pn-4nj-\ldots}}{\left(q^2;q^2\right)_n\left(1-a^2\right)}\qquad(2.$$

Using (1.2) and letting $b,x,y\to\infty$ and replacing a by a^2 in (1.4) [1] we get

$$\sum_{r=0}^{[n/2]}\frac{(a^2;q^2)_r\left(1-a^2q^{4r}\right)(-1)^r q^{r(7r-1)}}{(q^2;q^2)_r\left(1-a^2\right)\left(a^2q;q\right)_{n+2r}(q;q)_{n-2r}}$$

$$=\sum_{r=0}^{n}\frac{a^{2r}q^{r^2}}{(q;q)_r(q;q)_{n-r}\left(a^2q;q^2\right)_r} \qquad (2.2)$$

The LHS of Bailey's lemma (1.4) for $a=a^2$

$$\alpha_{2r}=\frac{\left(a^2;q^2\right)_r\left(1-a^2q^{4r}\right)(-1)^r q^{r(7r-1)}a^{6r}}{\left(q^2;q^2\right)_r\left(1-a^2\right)}$$

$\alpha_{2r+1}=0,\alpha_0=1$ gives

$$\left(a^2q;q\right)_\infty\sum_{n=0}^{\infty}a^{2n}q^{n^2-pn}$$

$$\sum_{r=0}^{[n/2]}\frac{\left(a^2;q^2\right)_r\left(1-a^2q^{4r}\right)(-1)^r q^{r(7r-1)}a^{6r}}{\left(q^2;q^2\right)_r\left(1-a^2\right)(q;q)_{n-2r}\left(a^2q;q\right)_{n+2r}}$$

$$=\left(a^2q;q\right)_\infty\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{a^{2n+4r}q^{n^2+2r^2+2nr-pn-pr}}{(q;q)_r(q;q)_n\left(a^2q;q\right)_r} \qquad (2.3)$$

The corresponding RHS of Bailey's lemma for the same, yields.

$$\sum_{j=0}^{p}\frac{\left(q^{-p};q\right)_j(-1)^j a^{2j}q^{j(j+1)/2}}{(q;q)_j}$$

$$\sum_{n=0}^{\infty}a^{4n}q^{4n^2-2pn+4nj}\alpha_{2n}$$

$$=\sum_{j=0}^{p}\frac{\left(q^{-p};q\right)_j(-1)^j a^{2j}q^{j(j+1)/2}}{(q;q)_j}$$

$$\sum_{n=0}^{\infty}\frac{\left(a^2;q^2\right)_n\left(1-a^2q^{4n}\right)(-1)^n a^{10n}q^{11n^2-2pn+4nj-n}}{\left(q^2;q^2\right)_n\left(1-a^2\right)} \qquad (2.4)$$

Equating (2.3) and (2.4) we get our required result (2.1)

Taking x,y,N,c,d $\rightarrow\infty$ in 2.30 of [1] we get

$$(aq;q)_\infty\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{\left(a;q^3\right)_{2r+n}a^{4r+n}q^{12r^2+n^2+6nr-6r-n}}{(q;q)_n\left(q^3;q^3\right)_r(a;q)_{6r+2n}}$$

$$=\sum_{n=0}^{\infty}\frac{\left(a;q^3\right)_n\left(1-a^2q^{12n}\right)(-1)^n a^{5n}q^{3n(11n-3)/2}}{\left(q^3;q^3\right)_n(1-a)} \qquad (2.5)$$

Now equation (2.1) for a=1 with p=0 replacing q by $q^{\frac{1}{2}}$ and using Jacobi's Triple product identity we get

$$\frac{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_\infty}{(q;q)_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{q^{\frac{n^2}{2}+nr+n^2}}{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_r\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_n\left(q^{\frac{1}{2}};q\right)_r}$$

$$=\prod\left(1-q^n\right)^{-1}$$

$n\not\equiv 0,\pm 5 \pmod{11}$ (2.6)

Proof:

For a=1 with p=0 replacing q by $q^{\frac{1}{2}}$ in (2.1) we get

$$\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_\infty \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{q^{\frac{1}{2}n^2+nr+r^2}}{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_r\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_n\left(q^{\frac{1}{2}};q\right)_r}$$

$$=\sum_{n=0}^{\infty}(-1)^n\left(1+q^n\right)q^{\frac{1}{2}\left(11n^2-n\right)}$$

$$=\sum_{n=0}^{\infty}(-1)^n\left\{q^{\frac{1}{2}\left(11n^2-n\right)}+q^{\frac{1}{2}\left(11n^2+n\right)}\right\}$$

$$=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{11n^2+n}{2}}$$

$$=\left(q^6,q^5,q^{11};q^{11}\right)_\infty \text{ [using(1.3)]}$$

$$\Rightarrow\frac{\left(q^{\frac{1}{2}},q^{\frac{1}{2}}\right)_\infty}{(q,q)_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{q^{\frac{n^2}{2}+nr+r^2}}{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_r\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_n\left(q^{\frac{1}{2}};q\right)_r}$$

$$=\frac{\left(q^6,q^5,q^{11},q^{11}\right)_\infty}{(q;q)_\infty}$$

$$=\prod_{n\not\equiv 0,\pm 5 \pmod{11}}\left(1-q^n\right)^{-1}$$

Hence the proof.

Equation (2.1) for a =1 with p =1 replacing q by $q^{\frac{1}{2}}$ we have

$$\frac{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_{\infty}}{(q;q)_{\infty}}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{q^{\frac{n^2}{2}+nr+n^2-n-r}\left(1-q^{n+r}\right)}{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_r\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_n\left(q^{\frac{1}{2}};q\right)_r}$$

$$=\prod_{n\not\equiv 0,\pm 4\,(\mathrm{mod}\ 11)}\left(1-q^n\right)^{-1} \qquad (2.8)$$

Proof

Equation (2.1) for a=1 with p=1 and replacing q by $q^{\frac{1}{2}}$ we get

$$\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_{\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{q^{\frac{n^2}{2}+nr+r^2-n-r}}{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_r\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_n\left(q^{\frac{1}{2}};q\right)_r}$$

$$=\sum_{n=0}^{\infty}(-1)^n\left(1+q^n\right)^{\frac{11n^2-3n}{2}}+\sum_{n=0}^{\infty}(-1)^n\left(1+q^n\right)q^{\frac{11n^2+n}{2}}$$

$$=\sum_{n=0}^{\infty}(-1)^n\left(q^{\frac{11n^2-3n}{2}}+q^{\frac{11n^2-n}{2}}\right)+\sum_{n=0}^{\infty}(-1)^n\left\{q^{\frac{11n^2+3n}{2}}+q^{\frac{11n^2+n}{2}}\right\}$$

$$=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{11n^2+3n}{2}}+\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{11n^2+n}{2}}$$

$$=\left(q^7,q^4,q^{11};q^{11}\right)_{\infty}+\left(q^6,q^5,q^{11};q^{11}\right)_{\infty}$$

$$\Rightarrow\frac{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_{\infty}}{(q;q)_{\infty}}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{q^{\frac{n^2}{2}+nr+r^2-n-r}}{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_r\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_n\left(q^{\frac{1}{2}};q\right)_r}$$

$$= \frac{\left(q^7,q^4,q^{11};q^{11}\right)_\infty}{(q;q)_\infty} + \frac{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_\infty}{(q;q)_\infty} \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{q^{\frac{n^2}{2}+nr+r^2}}{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_r \left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_n \left(q^{\frac{1}{2}};q\right)_r}$$

(using 2.7

$$\Rightarrow \frac{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_\infty}{(q;q)_\infty} \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{q^{\frac{n^2}{2}+nr+r^2-n-r}\left(1-q^{n+r}\right)}{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_r \left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_n \left(q^{\frac{1}{2}};q\right)_r}$$

$$= \prod_{n \not\equiv 0, \pm 4 \pmod{11}} \left(1-q^n\right)^{-1}$$

Hence the proof .

Equation (2.1) for $a=q^{\frac{1}{2}}$ with p=0, 1,2 successively gives upon replacing q by $q^{\frac{1}{2}}$ and using (1.3) the following:

$$\left(q^{\frac{3}{2}};q\right)_\infty \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{q^{\frac{n^2}{2}+r^2+nr+n+2r}}{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_r \left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_n \left(q^{\frac{3}{2}};q\right)_r}$$

$$= \prod_{n \not\equiv 0, \pm 1 \pmod{11}} \left(1-q^n\right)^{-1}$$

$$\left(q^{\frac{3}{2}};q\right)_\infty \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{q^{\frac{n^2}{2}+\frac{5r}{2}+nr+r^2+\frac{n}{2}}}{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_r \left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_n \left(q^{\frac{3}{2}};q\right)_r}$$

$$= \prod_{n \not\equiv 0, \pm 2 \pmod{11}} \left(1-q^n\right)^{-1} \qquad (2.10$$

$$\left(q^{\frac{3}{2}};q\right)_\infty \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{q^{\frac{n^2}{2}+r+r^2+nr}\left(1-q^{n+r+\frac{1}{2}}\right)}{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_r\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_n\left(q^{\frac{3}{2}};q\right)_r}$$

$$=\prod_{n\equiv 0,\pm 3 \,(\mathrm{mod}\ 11)}\left(1-q^n\right)^{-1} \tag{2.12}$$

Now equation (2.5) for a=1 replacing q by $q^{\frac{1}{3}}$ and using (1.3)we get

$$\frac{\left(q^{\frac{1}{3}};q^{\frac{1}{3}}\right)_\infty}{(q;q)_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(q;q)_{2r+n-1}\,q^{4r^2+\frac{n^2}{3}-nr-\frac{n}{3}}}{\left(q^{\frac{1}{3}};q^{\frac{1}{3}}\right)_n(q;q)_r\left(q^{\frac{1}{3}};q^{\frac{1}{3}}\right)_{6r+2n-1}}$$

$$=\prod_{n\not\equiv 0,+4\,(\mathrm{mod}\,11)}\left(1-q^n\right)^{-1}+\prod_{n\not\equiv 0,+5\,(\mathrm{mod}\,11)}\left(1-q^n\right)^{-1} \tag{2.13}$$

3.1 Derivation of Rogers- Ramanujan Type Identities Related to Modulo 17 :

Here taking $b\rightarrow 0$ in (2.25) of [1] we get the transformation

$$\left(a^6q^3;q^3\right)_\infty\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(a^2q;q\right)_{3n+2r}a^{6n+8r}q^{3n^2+6nr+\frac{9r^2}{2}-\frac{r}{2}-3pn-3pr}}{(q;q)_r\left(q^3;q^3\right)_n\left(a^2q;q\right)_r\left(a^6q^3;q^3\right)_{2n+2r}}$$

$$=\sum_{j=0}^{p}\frac{\left(q^{-3p};q^3\right)_j(-1)^ja^{6j}q^{\frac{3j(j+1)}{2}}}{\left(q^3;q^3\right)_j}\cdot\sum_{n=0}^{\infty}\frac{\left(a^2;q^2\right)_n\left(1-a^2q^{4n}\right)(-1)^na^{16n}q^{17n^2-n-6pn+12nj}}{\left(q^2;q^2\right)_n\left(1-a^2\right)} \tag{3.1}$$

Again we consider the transformation

$$\left(a^6q^3;q^3\right)_\infty \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(a^2q;q\right)_{3n-2r}\, a^{6n-8r}\, q^{3n^2-6nr-\frac{3r^2}{2}+\ldots-3pn-3pr}}{(q;q)_r\left(q^3;q^3\right)_n\left(a^2q;q^2\right)_r\left(a^6q^3;q^3\right)_{2n+2r+1}}$$

$$=\sum_{j=0}^{p}\frac{\left(q^{-3p};q^3\right)_j(-1)^j a^{6j} q^{\frac{3j(j-1)}{2}-6\ldots}}{\left(q^3;q^3\right)_j}\sum_{n=0}^{\infty}\frac{\left(a^2q^2;q^2\right)_n\left(1-a^2q^{2n+\ldots}\right)(-1)^n a^{16n} q^{17n^2+11n}}{\left(q^2;q^2\right)_n\, q^{\ldots}} \tag{3.2}$$

Now equation (3.1) for a=1 with p=0 replacing q by $q^{\frac{1}{2}}$ and using (1.3) we get

$$\frac{\left(q^{\frac{3}{2}};q^{\frac{3}{2}}\right)_\infty}{(q;q)_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_{3n-2r} q^{\frac{3n^2}{2}-3nr-\frac{3r^2}{4}-\frac{r}{4}}}{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_r\left(q^{\frac{3}{2}};q^{\frac{3}{2}}\right)_n\left(q^{\frac{1}{2}};q\right)_r\left(q^{\frac{3}{2}};q^{\frac{3}{2}}\right)_{2n+2r}}$$

$$=\prod_{n\not\equiv 0,\pm 8 \pmod{17}}\left(1-q^n\right)^{-1} \tag{3.3}$$

and for a=1 with p=1 replacing q by $q^{\frac{1}{2}}$ in (3.1) we get

$$\frac{\left(q^{\frac{3}{2}};q^{\frac{3}{2}}\right)_\infty}{(q;q)_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_{3n-2r} q^{\frac{3n^2}{2}-3nr-\frac{3r^2}{4}-\frac{7r}{4}-\frac{3r}{2}}}{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_r\left(q^{\frac{3}{2}};q^{\frac{3}{2}}\right)_n\left(q^{\frac{1}{2}};q\right)_r\left(q^{\frac{3}{2}};q^{\frac{3}{2}}\right)_{2n+2r}}$$

$$=\prod_{n\not\equiv 0,\pm 5 \pmod{17}}\left(1-q^n\right)^{-1}+\prod_{n\not\equiv 0,\pm 6 \pmod{17}}\left(1-q^n\right)^{-1} \tag{3.4}$$

Equation (3.2) for a=1, p=0 replacing q by $q^{\frac{1}{2}}$ and using (1.3) gives

$$\frac{\left(q^{\frac{3}{2}};q^{\frac{3}{2}}\right)_\infty}{(q;q)_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_{3n+2r} q^{\frac{3n^2}{2}+3nr+\frac{9r^2}{4}+3n+\frac{11r}{4}}}{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_r\left(q^{\frac{3}{2}};q^{\frac{3}{2}}\right)_n\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_r\left(q^{\frac{3}{2}};q^{\frac{3}{2}}\right)_{2n+2r+1}}$$

$$=\prod_{n\equiv 0,\pm 3 \pmod{17}}\left(1-q^n\right)^{-1} \tag{3.5}$$

4. Identities of Rogers Ramanujan Type identities modulo 19

Consider the transformation on (2.25) of [1] & taking $b\rightarrow\infty$ we get

$$\left(a^6q^3;q^3\right)_\infty\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(a^2q;q\right)_{3n+2r}a^{6n+8r}q^{3n^2+6nr+4r^2-3pn-3pr}}{(q;q)_r\left(q^3;q^3\right)_n\left(a^2q;q^2\right)_r\left(a^6q^3;q^3\right)_{2n+2r}}$$

$$=\sum_{j=0}^{p}\frac{\left(q^{-3p};q^3\right)_j(-1)^j a^{6j}q^{3j(j+1)/2}}{\left(q^3;q^3\right)_j}.$$

$$\sum_{n=0}^{\infty}\frac{\left(a^2;q^2\right)_n\left(1-a^2q^{4n}\right)(-1)^n a^{18n}q^{19n^2-n-6pn+12nj}}{\left(q^2;q^2\right)_n\left(1-a^2\right)} \tag{4.1}$$

and

$$\left(a^6q^3;q^3\right)_\infty\sum_{r=0}^{\infty}\frac{\left(a^2q;q\right)_{3n+2r}a^{6n+8r}q^{3n^2+6nr+4r^2+6n+6r-3pn-3pr}}{(q;q)_r\left(a^2q;q^2\right)_r\left(q^3;q^3\right)_r\left(a^6q^3;q^3\right)_{2n+2r+1}}$$

$$=\sum_{j=0}^{p}\frac{\left(q^{-3p};q^3\right)_j(-1)^j a^{6j}q^{3j(j+1)/2+6j}}{\left(q^3;q^3\right)_j}$$

$$\sum_{n=0}^{\infty}\frac{\left(a^2q^2;q^2\right)_n\left(1-a^2q^{12n+\ldots}\right)(-1)^{\ldots}a^{18n}q^{19n^2+43n-3pn-42n}}{\left(q^2;q^2\right)_{\ldots}} \quad (4.2)$$

Now for a=1with p=0 replacing q by $q^{\ldots}$ and using (1.3) and (4.1) we get

$$\frac{\left(q^{\frac{3}{2}};q^{\frac{3}{2}}\right)_{\infty}}{(q;q)_{\infty}}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(q^{\ldots};q^{\ldots}\right)_{3n+2r}q^{\ldots+3nr+2r^2-3n-3r}}{\left(q^{\ldots};q^{\ldots}\right)_{\ldots}\left(q^{\ldots};q^{\ldots}\right)_n\left(q^{\ldots};q\right)_r\left(q^{\ldots};q^{\ldots}\right)_{2n-2r}}$$

$$=\prod_{n\not\equiv 0,\pm 9\ (\mathrm{mod}\ 19)}\left(1-q^n\right)^{-1} \quad (4.3)$$

For a=1,p=0 and replacing q by $q^{\ldots}$ in (4.2) we get

$$\frac{\left(q^{\frac{3}{2}};q^{\frac{3}{2}}\right)_{\infty}}{(q;q)_{\infty}}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(q^{\ldots};q^{\ldots}\right)_{3n+2r}q^{\ldots+3nr+2r^2+3n+3r}}{\left(q^{\ldots};q^{\ldots}\right)_{\ldots}\left(q^{\ldots};q^{\ldots}\right)_{\ldots}\left(q^{\ldots};q^{\ldots}\right)_n\left(q^{\ldots};q^{\ldots}\right)_{2n-2r-1}}$$

$$=\prod_{n\not\equiv 0,\pm 3\ (\mathrm{mod}\ 19)}\left(1-q^n\right)^{-1} \quad (4.4)$$

Combinatorial Interpretation of Some Identities Related to modulo 11, 17 and 19

To give combinatorial Interpretation we require the following results:

Theorem (1) (Basil Gordon)[2]:

Let $1\le a\le k$ be integers and $B_{k,a}(n)$ denote the number of partitions of n of the form $n=b_1+b_2+\ldots+b_s$ where $b_i\ge b_{i+1}$, $b_i-b_{i+k-1}\ge 2$ and 1 appears as a summand atmost (k-a) times. Let $A_{k,a}(n)$ denote the number of partitions of n into parts not congruent to $0,\pm a$ (mod 2k+1). Then $A_{k,a}(n)=B_{k,a}(n)$ $\forall n\in N$.

Theorem (2) (Andrews)[2]:

Let $B_{\lambda,k,a}(n)$ denote the number of partitions of n of the form $n=b_1+b_2+....+b_t$ where $b_i \geq b_{i+1}$ only parts divisible by $\lambda+1$ may be repeated $b_i - b_{i+k-1} \geq \lambda+1$ (with strict inequality if $(\lambda+1)|b_i$), and the total number of appearances of summands in the set $\{1,2,.......,\lambda+1\}$ is atmost a-1. If λ is even, let $A_{\lambda,k,a}(n)$ denote the number of partition of n into parts such that no part $\neq 0$ (mod $\lambda+1$) may be repeated, and no part is congruent to 0, $\pm \left(a-\frac{1}{2}\lambda\right)$ $(\lambda+1)$ (mod $(2k-\lambda+1)$ $(\lambda-1)$).

If λ is odd, let $A_{\lambda,k,a}(n)$ denote the number of partitions of n into parts such that no part $\not\equiv 0$,(mod $\frac{1}{2}(\lambda+1)$ may be repeated, no part is congruent to $(\lambda+1)$ (mod $(2\lambda+1)$), and no part is congruent to 0, $\pm$ $(2a-\lambda)\frac{1}{2}(\lambda+1)$ (mod $(2k-\lambda+1)(\lambda+1)$).

Then provided $k \geq 2\lambda-1$ and $k \geq a \geq \lambda$, we have $A_{\lambda,k,a}(n)=B_{\lambda,k,a}(n)$ for each natural number n.

Note : Basil Gordon's Theorem is a corollary of Andrew's Theorem considering λ =0.

Now we give the combinatorial interpretation of the Rogers Ramanujan Type identity related to modulo 11 (2.3) :

Using Andrews Theorem the combinatorial interpretation of (2.3) is

$$1+\sum_{n=1}^{\infty} A_{5,4}(n)q^n$$

$$=\prod_{n \neq 0, \pm 5 (\mathrm{mod}\, 11)} \left(1-q^n\right)^{-1}$$

$$=\frac{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_\infty}{(q;q)_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{q^{\frac{n2}{2}+nr+r^2}}{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_r\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_n\left(q^{\frac{1}{2}};q\right)_r}$$

$$=1+\sum_{n=1}^{\infty} B_{5,4}(n)q^n \quad \text{with } |q|<1$$

Here considering $\lambda=0$, $A_{5,4}(n)$ is the number of partitions of n into parts such that no part $\equiv 0$ may be repeated, and no part is congruent to 0, $\pm$ 4(mod 11) and $B_{5,4}(n)$ denote the number of partitions of n of the from $n=b_1+b_2+....+b_t$ where $b_i \geq b_{i+1}, b_i - b_{i+4} > 1$ and 1 appears as a summand atmost 3 times.

Then $A_{5,4}(n)=B_{5,4}(n)$ $\forall n \in N$

Interpretation of the identity (2.9) (modulus 11)

Using Basil Gordon Theorem the combinatorial interpretation of (2.9) is

$$1+\sum_{n=1}^{\infty}A_{5,4}(n)q^{n}+1+\sum_{n=1}^{\infty}A_{5.5}(n)q^{n}$$

$$=2+\sum_{n=1}^{\infty}A_{5,4}(n)q^{n}+\sum_{n=1}^{\infty}A_{5.5}(n)q^{n}$$

$$=\prod_{n\not\equiv 0,\pm 4\,(\mathrm{mod}\,11)}\left(1-q^{n}\right)^{-1}+\prod_{n\not\equiv 0,\pm 5\,(\mathrm{mod}\,11)}\left(1-q^{n}\right)^{-1}$$

$$=\frac{\left(q^{\frac{1}{3}};q^{\frac{1}{3}}\right)_{\infty}}{(q;q)_{\infty}}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(q;q)_{2r-n-1}\,q^{4r^{2}-\frac{n^{2}}{3}-nr-\frac{n}{3}}}{\left(q^{\frac{1}{3}};q^{\frac{1}{3}}\right)_{n}(q;q)_{r}\left(q^{\frac{1}{3}};q^{\frac{1}{3}}\right)_{6r-2n-1}}$$

$$=2+\sum_{n=1}^{\infty}B_{5,4}(n)q^{n}+\sum_{n=1}^{\infty}B_{5.5}(n)q^{n}$$

Let $k=5$, $a=4$ then $A_{5,4}(n)$ denotes the number of partitions of n into parts not congruent to 0, ± 4(mod 11) and $B_{5,4}(n)$ denotes the number of partitions of n of the form $n=b_1+b_2+....+b_t$ where $b_i\geq b_{i+1}, b_1-b_{i+4}\geq 2$ and 1 appears as a summand atmost 3 times $\therefore A_{5,4}(n)=B_{5,4}(n)\ \ \forall n\in N$ Again let $k=5$, $a=5$ then. $A_{5,5}(n)$ denotes the number of partitions of n into parts not congruent to 0, ± 5(mod 11) $B_{5,5}(n)$ denotes the number of partitions of n of the form $n=b_1+b_2+....+b_t$ where $b_i\geq b_{i+1}, b_i-b_{i+4}\geq 2$ and 1 appears as a summand atmost 4 times.

$\therefore A_{5,5}(n)=B_{5,5}(n)\ \ \forall n\in N$

Interpretation of the identity (equation 3.3) (modulus 17)

The analytical identity is

$$\frac{\left(q^{\frac{3}{2}};q^{\frac{3}{2}}\right)_{\infty}}{(q;q)_{\infty}}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_{3n+2r}q^{\frac{3n^{2}}{2}+3nr+\frac{9r^{2}}{4}-\frac{r}{4}}}{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_{r}\left(q^{\frac{3}{2}};q^{\frac{3}{2}}\right)_{n}\left(q^{\frac{1}{2}};q\right)_{r}\left(q^{\frac{3}{2}};q^{\frac{3}{2}}\right)_{2n+2r}}$$

$$=\prod_{n\not\equiv 0,\pm 8 \pmod{17}} \left(1-q^n\right)^{-1}$$

We use Basil Gordon's theorem. Let k=8=a then $B_{8,8}(n)$= the number of partitions of n of the form $n=b_1+b_2+....+b_t$, where $b_i \geq b_{i+1}, b_i - b_{i+7} \geq 2$ and 1 appears as a summand atmost 7 times.

Also $A_{8,8}(n)$= the number of partitions of n into parts not congruent to 0, $\pm$ 8(mod 17)

Then $A_{8,8}(n)=B_{8,8}(n)\ \forall n \in N$

and we have for $|q|<1$,

$$1+\sum_{n=1}^{\infty} A_{8,8}(n)q^n$$

$$=\prod_{n\not\equiv 0,\pm 8 \pmod{17}} \left(1-q^n\right)^{-1}$$

$$=\frac{\left(q^{\frac{3}{2}};q^{\frac{3}{2}}\right)_\infty}{(q;q)_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_{3n+2r} q^{\frac{3n^2}{2}+3nr+\frac{9r^2}{4}-\frac{r}{4}}}{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_r\left(q^{\frac{3}{2}};q^{\frac{3}{2}}\right)_n\left(q^{\frac{1}{2}};q\right)_r\left(q^{\frac{3}{2}};q^{\frac{3}{2}}\right)_{2n+2r}}$$

$$=1+\sum_{n=1}^{\infty} B_{8,8}(n)q^n$$

Interpretation of the identity (equation 4.2) (modulus 19)

The analytical identity is

$$\frac{\left(q^{\frac{3}{2}};q^{\frac{3}{2}}\right)_\infty}{(q;q)_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_{3n+2r} q^{\frac{3n^2}{2}+3nr+2r^2-3n-3r}}{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_r\left(q^{\frac{3}{2}};q^{\frac{3}{2}}\right)_n\left(q^{\frac{1}{2}};q\right)_r\left(q^{\frac{3}{2}};q^{\frac{3}{2}}\right)_{2n+2r}}$$

$$=\prod_{n\not\equiv 0,\pm 9 \pmod{19}} \left(1-q^{n}\right)^{-1}$$

We use Basil Gordon's theorem. Let $k = 9 = a$ then $B_{9,9}(n)$ = the number of partitions of n of the form $n=b_1+b_2+....+b_t$, where $b_i \geq b_{i+1}$. $b_i - b_{i+8} \geq 2$ and 1 appears as a summand atmost 8 times.

Also $A_{9,9}(n)$ = the number of partitions of n into parts not congruent to $0, \pm 9$ (mod 19) Then $A_{9,9}(n) = B_{9,9}(n)\ \forall n \in N$

and we have for $|q|<1$

$$1+\sum_{n=1}^{\infty} A_{9,9}(n)q^{n}$$

$$=\prod_{n\not\equiv 0,\pm 9 \pmod{19}} \left(1-q^{n}\right)^{-1}$$

$$\frac{\left(q^{\frac{3}{2}};q^{\frac{3}{2}}\right)_{\infty}}{(q;q)_{\infty}} \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_{3n+2r} q^{\frac{3n^2}{2}+3nr+2r^2-3n-3r}}{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_{r}\left(q^{\frac{3}{2}};q^{\frac{3}{2}}\right)_{n}\left(q^{\frac{1}{2}};q\right)_{r}\left(q^{\frac{3}{2}};q^{\frac{3}{2}}\right)_{2n+2r}}$$

$$=1+\sum_{n=1}^{\infty} B_{9,9}(n)q^{n}$$

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