Direct Sum of E- Injective N-Groups

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ABSTRACT

Extending the notion of relative injectivity of modules to near-ring groups, we define E-injective near-ring groups and characterize such near-ring groups. The nature of E-injective N-groups under direct sum is studied in this paper. The notion of dominance of an element of an N-group by another N-group plays an important role in establishing the fact that the direct sum of a family of E-injective N-groups is also E-injective. If near-ring group E satisfies chain conditions on its substructures, then inheritance of E-injective character of direct sum of E-injective N-groups is also established.

1. INTRODUCTION

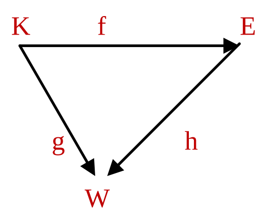
Several researchers like Faith and Utumi [7], Page and Yousif [16], Boyle [3], Goodearl [10], Armendariz [2], Hirano [11], Fuller [9], Fuller and Anderson [1], Thoang [20] and many others have investigated different properties and relations of quasi- injective modules, relative injective modules satisfying chain conditions. Injective modules and near-ring groups have been studied by Mason et al [13], Faith et al [8], Seth and Tiwari [19], Meldrum [14], Oswald [15]. Of these Oswald and Mason have studied injective and projective near-ring modules. Mason [12] studied injective near-ring modules and defined the concepts like n-injective, loosely injective and almost injective near-ring modules. De et al [6] have shown how the n-injectivity and weakly n-injectivity character together with Beidleman's condition exhibit some interesting phenomena in case of m-simple near-ring groups. Saikia and Misra [18] have studied p-injective near-rings and weakly quasi injective near-ring groups. In this paper we attempt to extend some characteristics of quasi- injectivity, relative injectivity in near-rings. The study of quasi injective modules and their endomorphism rings motivates us to extend these concepts to near-ring groups. The objective of this generalization is to investigate whether analogous results can be obtained in near-ring groups.

2. PRELIMINARIES

Throughout the paper we consider all N-groups as unitary N-groups unless otherwise specified. All basic concepts used in this paper are available in Pilz [8]**.** By a dgnr we mean a distributively generated near-ring. This section deals with some basic definitions and results which are used in the later sections.

2.1 Definition

Let E and W be N-groups. W is called E- injective or W is injective relative to E if for each N-monomorphism $f : K \rightarrow E$, every N –homomorphism from K into W can be extended to an N- homomorphism from E into W. i.e. The diagram



commutes which means g = hf.

2.2 Definition

An N-group A is injective if it is E-injective for every N-group E of N. So if an N-group A is injective it is E-injective for any N-group E.

2.3 Definition

If N-group W is E-injective then E is said to be a WI-N-group. If a commutative N-group W is E-injective then E is called a W_CI-N-group.

2.4 Proposition

If N is a dgnr and $\{Ne\}_{e\in E}$ is an independent family of normal N-subgroups of N-group E then E is a homomorphic image of $\oplus_{e\in E} Ne$.

Proof: Let f_e: Ne $\rightarrow$ E be defined by $f_e(ne) = ne$. Then f_e is an N-homomorphism.

If we define $f = \Sigma_{e\in E} f_e : \oplus_{e\in E} Ne \rightarrow E$ by $(\Sigma_{e\in E} f_e)(\Sigma_{e\in E} ne) = (\Sigma_{e\in E} f_e(ne))$, $n \in N$, it is an N-homomorphism. Obviously it is an N-monomorphism. Again for any $e_k \in E$ we get $e_k \in Ne_k \in \oplus_{e\in E} Ne$. So f is onto. Hence E is a homomorphic image of $\oplus_{e\in E} Ne$.

2.5 Proposition

Let N be a dgnr, E be an N-group and F be a commutative N-group. Then the set $Hom_N(E, F) = \{f / f: E\rightarrow F$ is an N-homomorphism$\}$ is an abelian group where addition is defined as: $(f + g)(e) = f(e) + g(e)$, for all $f, g \in Hom_N(E, F)$.

2.6 Proposition

Let B, M be two N-groups and C an ideal of B. For N-homomorphism

$f : B \rightarrow M$ $\exists$ a unique homomorphism $\bar{f} : \frac{B}{C} \rightarrow M$ such that $\bar{f}(\bar{b}) = f(b)$, $\forall\, C \subseteq Kerf$.

If f is an epimorphism, then $\bar{f}$ defined as above is also an epimorphism.

2.7 Definition

For an N-group A an element $x\in A$ is said to be dominated by N-group E if $Ann_N(x) \supset Ann_N(e)$ for some $e \in E$.

Let $\{A_\alpha\}_{\alpha\in J}$ be a family of N-groups. Let x be the element of $\Pi_{\alpha\in J} A_\alpha$ whose α- component is x_α.

We define $I_x = \{n \in N | nx \in \oplus_{\alpha\in J} A_\alpha\}$.

Then $x \in \Pi_{\alpha\in J} A_\alpha$ is called a special element if $I_x x_\alpha = 0$ for almost all α. In other words there exists a finite subset F of J such that $nx_\alpha = 0$ for all $n \in I_x$ and for all $\alpha \in F$.

2.8 Proposition

Let U be a commutative N-group and f: L $\rightarrow$ M be an N-homomorphism. We can define a mapping

$f^* = Hom_N(f, U) : Hom_N(M, U) \rightarrow Hom_N(L, U)$ by $Hom_N(f, U) : \gamma \rightarrow \gamma f$ i.e. $f^*\gamma = \gamma f$

then $Hom_N(f, U)$ is an N-homomorphism.

2.9 Proposition

If U is a commutative N-group, then for every exact sequence

$$0 \rightarrow K \xrightarrow{f} E \xrightarrow{g} L \rightarrow 0$$

the sequence $0 \rightarrow Hom_N(L, U) \xrightarrow{g^*} Hom_N(E, U) \xrightarrow{f^*} Hom_N(K, U)$ is exact.

2.10 Proposition

A commutative N-group U is E-injective if and only if $Hom_N(- , U)$ is exact.

3. E-INJECTIVE N-GROUPS

In this section we discuss some properties of E-injective N-groups needed in the sequel.

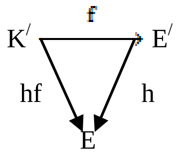
3.1 Proposition

N-subgroups of a WI N-group are again WI N-groups.

Proof: Let E be a WI N-group. Thus W is E-injective.

And let $E^/$ be any N-subgroup of E.

Let $h: E^/ \rightarrow E$ be an N-monomorphism and $K^/$ be an N-subgroup of $E^/$ and $f : K^/ \rightarrow E^/$ be any N-monomorphism. Then hf is also an N-monomorphism, $hf: K^/ \rightarrow E$.

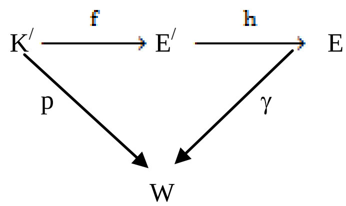


Now W is E-injective, so for any N-subgroup K of E, the N-monomorphism $i: K \rightarrow E$ and any N-homomorphism

$k : K \rightarrow W$, $\exists$ an N- homomorphism $\gamma : E \rightarrow W$ such that $k = \gamma i$.

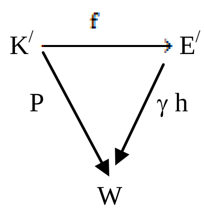
Since W is E-injective, so for N- monomorphism $hf : K^/ \rightarrow E$ and $p : K^/ \rightarrow W$ we get

$\gamma : E \rightarrow W$ such that $\gamma(hf) = p$. That is the following diagram commutes.



Now $f : K^/ \rightarrow E^/$ is an N-monomorphism and for any N- homomorphism $p : K^/ \rightarrow W$,

we get $\gamma h : E^/ \rightarrow W$ such that the diagram commutes.



That is $p = (\gamma h) f$. Therefore W is $E^/$- injective.

3.2 Proposition

If W is E- injective then W is Ne –injective for all $e \in E$.

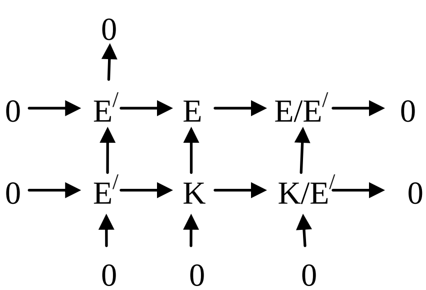
Proof: Since Ne is an N-subgroup of E. As W is E- injective, proposition 2.3 implies W is Ne-injective.

3.3 Proposition

Homomorphic images of a $W_C I$ N-groups are again $W_C I$ N-groups.

Proof: Given $0 \rightarrow E^/ \xrightarrow{h} E \xrightarrow{k} E^{//} \rightarrow 0$ is exact and commutative N-group W is E-injective.

We show W is $E^{//}$-injective. Let $E^/ \leq K \leq E$ and that $E^{//} = E/E^/$. Now we consider the canonical diagram



Now applying $Hom_N(- , W)$ we get the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & Hom_N(E/E^{/}, W) & \rightarrow & Hom_N(E, W) & \rightarrow & Hom_N(E^{/}, W) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & Hom_N(K/E^{/}, W) & \rightarrow & Hom_N(K, W) & \rightarrow & Hom_N(E^{/}, W) & \rightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

Since $Hom_N(E/E^{/}, W) \xrightarrow{\phi} Hom_N(K/E^{/}, W)$ is epic, for all $\gamma \in Hom_N(K/E^{/}, W)$ $\exists$ $\alpha \in Hom_N(E/E^{/}, W)$ such that $\phi(\alpha) = \gamma$

$\Rightarrow \alpha f = \gamma$, where $f : K/E^{/} \rightarrow E/E^{/}$ is an N-monomorphism and $\phi = Hom_N(f , W)$. Thus W is $E/E^{/}$-injective $\Rightarrow E^{//}$ is W_cI N-group of E.

4. DIRECT SUM OF N-GROUPS WITH INJECTIVITY AND E-INJECTIVITY

4.1 Proposition

Let N be a dgnr. If E_α is a WI N-group for all $\alpha \in A$, then $E = \oplus_{\alpha \in A} E_\alpha$ is a WI N-group, where E is commutative.

Proof: Let $E = \oplus_{\alpha \in A} E_\alpha$ and E_α is WI N-group

$\Rightarrow$ W is E_α-injective for all $\alpha \in A$.

We consider an N-subgroup K of E and the N-homomorphism $h: K \rightarrow W$.

Let $\Omega = \{f: L \rightarrow W / K \leq L \leq E \text{ and } (f|K) = h\}$.

Let $g: A \rightarrow W$, $h : B \rightarrow W \in \Omega$. $g \leq h$ if $A \subseteq B \subseteq E$.

Then Ω is ordered set by set inclusion. Ω is clearly inductive.

Let $\bar{h} : M \rightarrow W$ be a maximal element in Ω.

To get the proof it is sufficient to show that each E_α is contained in M.

Let $K_\alpha = E_\alpha \cap M$. Then $(\bar{h} \mid K_\alpha) : K_\alpha \rightarrow W$, so since $K_\alpha \leq E_\alpha$ and W is E_α- injective, there is an N-homomorphism $\overline{h_\alpha} : E_\alpha \rightarrow W$ with $(\overline{h_\alpha} \mid K_\alpha) = (\bar{h} \mid K_\alpha)$.

If $e_\alpha \in E_\alpha$ and $m \in M$ such that $e_\alpha + m = 0$, then $e_\alpha = - m \in K_\alpha$ and $\overline{h_\alpha} (e_\alpha) + \bar{h} (m)$

$= \bar{h} (- m) + \bar{h} (m) = 0$.

Thus $f: e_\alpha + m \rightarrow \overline{h_\alpha} (e_\alpha) + \bar{h} (m)$ is a well defined N-homomorphism $f : E_\alpha + M \rightarrow W$.

But $(f|M) = \bar{h}$, so by maximality of $\bar{h}$, $E_\alpha \subseteq M$.

4.2 Proposition

Let N be a dgnr. If W is a commutative N-group and $\{Ne\}_{e \in E}$ is an independent family of normal N-subgroups of N-group E, W is Ne –injective for all $e \in E$, then W is E- injective

Proof: W is Ne –injective for all $e \in E$. So by proposition 3.2, W is $\oplus_{e \in E} Ne$ –injective.

Since E is a homomorphic image of $\oplus_{e\in E}$ Ne, by proposition 2.4 and since homomorphic image of a W_CI N-group is W_CI N-group by proposition 3.3. So W is E-injective.

4.3 Proposition

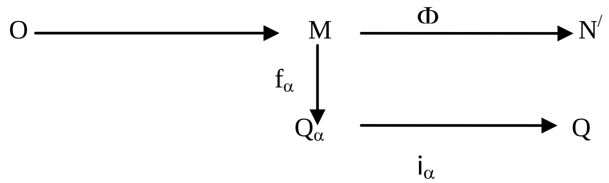
If a finite direct sum of injective normal N-subgroups (ideals) Q_α of E is injective, then each Q_α is injective.

Proof: Let $Q = \oplus Q_\alpha$ be injective N- group and consider the N-monomorphism

$f_\alpha : M \to Q_\alpha$, where M is some N- subgroup of E.

Since, Q is direct sum of Q_α's, for any $\alpha = 1, 2, 3, \ldots \ldots \ldots, n$, so there is the inclusion map $i_\alpha : Q_\alpha \to Q$ and the projection map $\Pi_\alpha : Q \to Q_\alpha$ such that $\Pi_\alpha i_\alpha = 1_{Q_\alpha}$.

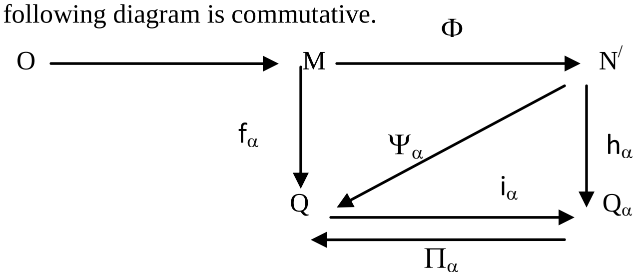
Consider a diagram



with top row exact.

Since Q is injective there exists an N- homomorphism $h_\alpha : N' \to Q$, such that $h_\alpha \Phi = i_\alpha f_\alpha$.

Now we define $\Psi : N' \to Q_\alpha$ by $\Psi_\alpha = \Pi_\alpha h_\alpha$. Since $\Pi_\alpha i_\alpha = 1_{Q_\alpha}$, it follows that $\Psi_\alpha \Phi = \Pi_\alpha h_\alpha \Phi = \Pi_\alpha i_\alpha f_\alpha = f_\alpha$. So, the following diagram is commutative.



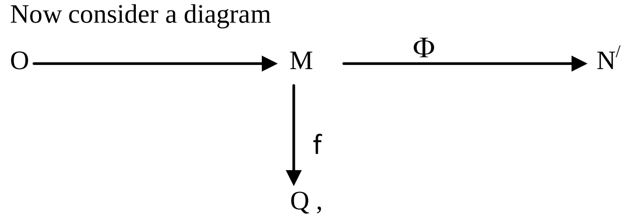
Thus, each Q_α. is injective.

4.4 Proposition

Let N be a dgnr. A finite direct sum of injective normal N-subgroups (ideals) Q_α of E is injective if each Q_α is injective.

Proof: Let $Q = \oplus Q_\alpha$ with each $Q_{\alpha\, I}$ is an injective N- group.

Now consider a diagram



where M, N' are N- subgroups of E with the top row exact.

For any $\alpha = 1, 2, 3, \ldots \ldots \ldots, n$, there is the canonical inclusion $i_\alpha : Q_\alpha \to Q$ and the projection

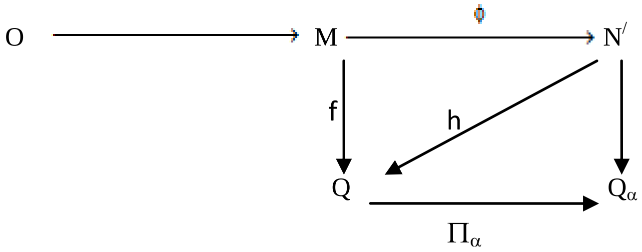
$\Pi_\alpha : Q \to Q_\alpha$, so there are N-homomorphisms $\Pi_\alpha f : M \to Q_\alpha$.

Since Q_α is injective, there exists an N-homomophism $h_\alpha : N' \to Q_\alpha$ such that $h_\alpha \Phi = \Pi_\alpha f$.

Now we define a map h: $N' \to Q$ by the formula $h(x) = (h_1(x) + \ldots \ldots \ldots + h_n(x))$ $\forall x \in N'$.

Then h is an N-homomophism as N is a dgnr.

We shall show the diagram commutes. i.e. $f = h\Phi$.



Since Q is a direct sum, for any $x \in N^{/}$,

$h\phi(x) = (h_1\phi(x) + h_2\phi(x) + \ldots \ldots \ldots + h_n\, \phi(x)) = (\Pi_1 f\,(x) + \Pi_2 f\,(x) + \ldots \ldots \ldots + \Pi_n f\,(x)) = f(x)$

So, $h\,\phi = f$. Thus Q is injective.

4.5 Corollary

Let N be a dgnr. A finite direct sum of injective normal N-subgroups (ideals) Q_α of E is injective if and only if each Q_α is injective.

4.6 Theorem

A finite direct sum of injective N-groups Q_α is injective if and only if each Q_α is injective.

Proof: Proof is same as theorem 4.3., except in this case we define a map $h : N^{/} \rightarrow Q$ by $h(x) = (h_1(x), \ldots, h_n(x))$ $\forall\, x \in N^{/}$.

4.7 Theorem

Let N be a near-ring and $\{Q_i\}_{i\in I}$ a family of E-injective N-groups. Then the product $Q = \Pi_{i\in I} Q_i$ is E- injective.

Proof: Let $A \subseteq E$ be an N-subgroup of E and $f: A \rightarrow Q$ an N-homomorphism.

It is enough to show that f can be extended to E.

For $i\in I$ we denote $\pi_i: Q \rightarrow Q_i$ the projection map. Since Q_i is E-injective for any $i\in I$, so the N-homomorphism $\pi_i.f: A \rightarrow Q_i$ can be extended to $f_i^{/}: E \rightarrow Q_i$.

Then we have $f : E \rightarrow Q$ by $f^{/}(e) = (f_i^{/}(e))_{\,i\in I}$.

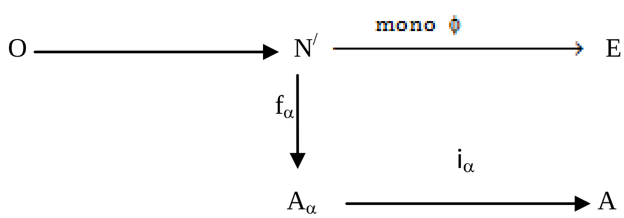
If $a \in A$, then $f\,(a) = f(a)$, so $f^{/}$ is an extension of f. Thus Q is E-injective.

4.8 Theorem

If $\oplus_{\alpha\in J} A_\alpha$ is E -injective then each A_α is E –injective and every element of $\Pi_{\alpha\in J} A_\alpha$ dominated by E is special.

Proof: Let $A = \oplus_{\alpha\in J} A_\alpha$ be E injective. Let $N^{/}$ be an N-group. Consider the N-homomorphism $f_\alpha: N^{/} \rightarrow A_\alpha$. Since A is direct sum of N-groups A_α, $\alpha\in J$, so for any $\alpha \in J$, there is the inclusion map $i_\alpha : A_\alpha \rightarrow A$ and the projection $\pi_\alpha : A \rightarrow A_\alpha$ such that $\pi_\alpha\, i_\alpha = 1_{A_\alpha}$.

Consider a diagram, with top row exact.

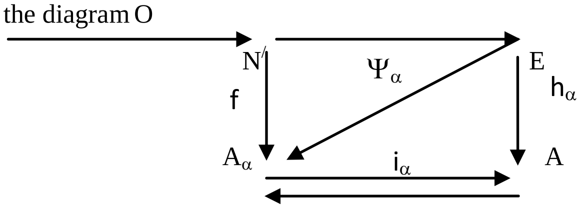


Since A is E - injective, $\exists$ a homomorphism $h_\alpha : E \rightarrow A$ such that $h_\alpha\Phi = i_\alpha\, f_\alpha$.

Now define $\Psi_\alpha: E \rightarrow A_\alpha$ by $\Psi_\alpha = \pi_\alpha h_\alpha$.

Since $\pi_\alpha i_\alpha = 1_{A_\alpha}$, it follows that $\Psi_\alpha \Phi = \pi_\alpha h_\alpha \Phi = \pi_\alpha i_\alpha f_\alpha = f_\alpha$

So the diagram O



is commutative.

Thus A_α is E -injective. Π_α

Let $x \in \Pi_\alpha A_\alpha$ be dominated by E . This implies that there is an $e \in E$ such that $Ann_N(x) \supset Ann_N(e)$.

Then it gives an N-homomorphism f: $Ne \rightarrow \Pi A_\alpha$ defined by $\lambda e \rightarrow \lambda x$ ($\lambda \in N$).

Let $(\lambda_1 e), (\lambda_2 e) \in Ne$ and $f(\lambda_1 e) \neq f(\lambda_2 e) \Rightarrow (\lambda_1 x) \neq (\lambda_2 x) \Rightarrow (\lambda_1 - \lambda_2)\, x \neq 0$

$\Rightarrow (\lambda_1 - \lambda_2) \notin Ann_N(x)$

$\Rightarrow (\lambda_1 - \lambda_2) \notin Ann_N(e)$ [since $Ann_N(x) \supset Ann_N(e)$] $\Rightarrow (\lambda_1 - \lambda_2)e \neq 0 \Rightarrow (\lambda_1\, e) \neq (\lambda_2\, e)$.

Therefore, the mapping is well defined .

$f(\lambda_1 e + \lambda_2 e) = f((\lambda_1 + \lambda_2)e) = (\lambda_1 + \lambda_2)x = (\lambda_1 x + \lambda_2 x) = f(\lambda_1 e) + f(\lambda_2 e)$

Next for $n \in N$, $f(n(\lambda_1 e)) = f((n\lambda_1)e) = (n\lambda_1)x = n(\lambda_1 x) = n\, f(\lambda_1 e)$

Thus f is an N- homomorphism.

The image of the N-subgroup $I_x e$ by f is clearly $I_x x$ ($\subset \oplus A_\alpha$).

Thus the restriction of f to $I_x e$ is regarded as an N-homomorphism $I_x e \rightarrow \oplus A_\alpha$.

Since $\oplus A_\alpha$ is E-injective and so Ne-injective by proposition 3.2. So, we get N-homomorphism $Ne \rightarrow \oplus A_\alpha$ which means that there exists a $u \in \oplus A_\alpha$ such that $\lambda x = \lambda u$ (for all $\lambda \in I_x$).

It follows that $I_x x_\alpha = I_x u_\alpha$ for all $\alpha \in J$.

But since $u_\alpha = 0$ for almost all α, it follows that $I_x x_\alpha = 0$ for almost all α too $\Rightarrow$ x is special.

4.9 Theorem

Let N be a dgnr. If $\{Ne\}_{e \in E}$ is an independent family of normal N-subgroups of E, $\oplus_{\alpha \in J} A_\alpha$ is a commutative N-group, then each A_α is E –injective. Moreover, every element of $\Pi_{\alpha \in J} A_\alpha$ dominated by E is special implies $\oplus_{\alpha \in J} A_\alpha$ is E -injective.

Proof: Let each A_α is E -injective and every element of $\Pi_{\alpha \in J} A_\alpha$ dominated by E is special.

Let $e \in E$ and consider the N- subgroup Ne of E. Let J be an N-subgroup of N. Then Je is an N-subgroup of Ne. Let there be given an N- homomorphism h: $Je \rightarrow \oplus A_\alpha$. Then since $\oplus A_\alpha \subset \Pi A_\alpha$ and ΠA_α is E-injective (as each A_α is E-injective, by theorem 4.7) whence Ne- injective (by proposition 3.2), h can be extended to an N-homomorphism $Ne \rightarrow \Pi A_\alpha$.

Let $x \in \Pi A_\alpha$ and we define the N-homomorphism as $\lambda e \rightarrow \lambda x$ ($\lambda \in N$)

Therefore it follows that $Jx = h(Je) \subset \oplus A_\alpha$, whence $J \subset I_x$.

On the otherhand since clearly $Ann_N(e) \subset Ann_N(x)$, x is dominated by E and thus x is special by assumption $\Rightarrow I_x x_\alpha = 0$ whence $Jx_\alpha = 0$ for almost all α.

Let u be the element of $\oplus A_\alpha$, whose α-component is x_α or 0 according as $Jx_\alpha \neq 0$ or $Jx_\alpha = 0$.

Then it is clear that $\lambda u = \lambda x$ for all $\lambda \in J$.

Further, it is also clear that $Ann_N(e) \subset Ann_N(x) \subset J$ and therefore the mapping gives an N-homomorphism f: $Ne \rightarrow \oplus A_\alpha$ which is an extension of h, because $f(\lambda e) = \lambda u = \lambda x$ for all $\lambda \in J$.

This implies that $\oplus A_\alpha$ is Ne-injective and so E–injective by proposition 3.1.

4.10 Corollary

Let N be a dgnr. If $\{Ne\}_{e\in E}$ is an independent family of normal N-subgroups of N-group E, $\oplus_{\alpha\in J} A_\alpha$ is commutative N-group then $\oplus_{\alpha\in J} A_\alpha$ is E -injective if and only if each A_α is E-injective and every element of $\Pi_{\alpha\in J} A_\alpha$ dominated by E is special implies $\oplus_{\alpha\in J} A_\alpha$ is E-injective

4.11 Theorem

Suppose $\{A_\alpha\}_{\alpha\in J}$ is a family of E-injective N-groups such that for every countable subset K of J, $\oplus_{\alpha\in K}A_\alpha$ is E - injective. Then $\oplus_{\alpha\in J}A_\alpha$ is itself E - injective.

Proof: Assume that $\oplus_{\alpha\in J}A_\alpha$ is not E - injective.

Then by theorem 4.8, there exists an $x \in \Pi_{\alpha\in J}A_\alpha$ which is dominated by E but is not special $\Longrightarrow I_x x_\alpha \neq 0$ for infinitely many $\alpha\in J$.

Let k be an infinite countable subset of the infinite set $\{\alpha \in J \mid I_x x_\alpha \neq 0\}$.

Let y be element of $\Pi_{\alpha\in k} A_{\alpha,}$ whose α- component y_α is equal to x_α for all $\alpha \in K$.

Then clearly $I_x \subset I_y$, so that it follows that y is dominated by E and $I_y y_\alpha = I_y x_\alpha \neq 0$ for all $\alpha \in K$.

This implies again by theorem 4.8, that $\oplus_{\alpha\in K}A_\alpha$ is not E -injective (because each A_α is E - injective by our assumption). This is a contradiction and so the proof is complete.

4.12 Theorem

Let N be dgnr. If $\{Ne\}_{e\in E}$ is an independent family of normal N-subgroups of N-group E, $\oplus_{\alpha\in J} A_\alpha$ is commutative N-group then direct sum of any family $\{A_\alpha\}$ of E-injective N- groups is E - injective if E is Noetherian.

Proof: Let $\{A_\alpha\}$ be a family of E-injective N- group. Let x be an element of ΠA_α, dominated by e. Then there is an $e \in E$ such that $Ann_N(e) \subset Ann_N(x)$. Consider $I_x e$.

Since clearly $Ann_N(x) \subset I_x$, whence $Ann_N(e) \subset I_x$, it follows that $I_x/ Ann_N(e) \cong I_x e$

On the other hand $I_x e$ is a N-subgroup of $Ne_,$ so N-subgroup of Noetherian N-group E.

Hence, $I_x/ Ann_N(e)$ is finitely generated $\Rightarrow$ there exists a finite number of elements $\lambda_1, \lambda_{2,} \ldots \ldots \ldots, \lambda_n$ of I_x such that $I_x = N\lambda_1 + N\lambda_2 + \ldots \ldots \ldots + N\lambda_n + Ann_N(e)$

It follows therefore $I_x x_\alpha = N\lambda_1 x_\alpha + N\lambda_2 x_\alpha + \ldots \ldots \ldots + N\lambda_n x_\alpha$ for all components x_α.

Since however for each i, $\lambda_i x_\alpha = 0$, for almost all α, it follows that $I_x x_\alpha = 0$ for almost all $\alpha \Rightarrow$ x is special. Thus $\oplus A_\alpha$ is E-injective by theorem 4.9.

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