Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 10, Number 3 (2014), pp. 395-400
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On Weakly Noetherian N-Groups and Ascending Chain Conditions on Essential Ideals

Navalakhi Hazarika[1] and Helen K Saikia[2]

[1]*Department of Mathematics, Royal School of Engineering and Technology, Guwahati-781035*
[2]*Department of Mathematics, Gauhati University, Guwahati-781014*
[1]*Email: navalakhmi@gmail.com*
[2]*Email: hsaikia@yahoo.com*

ABSTRACT

Several researchers like Armendariz, Efraim, Dunget al, Page, S.S, Yousif, Fuller, Anderson have been studied chain conditions on essential ideals in modules. We extend the concept of ascending chain condition to Near-ring groups defining weakly Noetherian N-group. Introducing almost weakly Noetherian N-group we established that for dgnr N, N-group E is almost weakly Noetherian, E has A.C.C. on essential ideals and E/M is weakly Noetherian for every essential ideal M of E are equivalent. These relations motivate us to study the relation between weakly Noetherian N-group E and A.C.C. on essential ideals of E. Finally we have shown Near-ring N is weakly Noetherian if $\oplus_{i\in I}E_i$ of injective N-groups is injective.

Keywords: near-ring group, weakly Noetherian N-group, almost weakly Noetherian N-group, essential ideal.

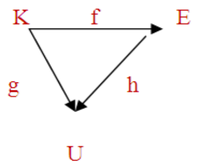
Mathematics subject classification (AMS) 2010: 16Y30

PREREQUISITES:

All basic concepts used in this paper are available in Pilz[3]. Throughout the paper we consider all N-groups as unitary N-groups. By a dgnr we mean a distributively generated near-ring. This section deals with some basic definitions and results which are used in the next section.

Definition1.1: N-group E is said to be weakly Noetherian if every strict ascending chain of ideals or normal N-subgroups $A_1 \subset A_2 \subset$ …...... of E terminates after finitely many steps or equivalently for each chain $A_1 \subseteq A_2 \subseteq$ …...... of E, $\exists$ n $\in$N such that $A_n = A_{n+1} =$ …......

Definition 1.2: Let E and U be N-groups. U is called E-injective or U is injective relative to E if for each N-monomorphism $f : K \to E$, every N –homomorphism from K into U can be extended to an N-homomorphism from E into U. i.e. The diagram



commutes. i.e. $g = hf$.

An N-group A is injective if it is E-injectve for every N-group E of N. So if an N-group A is injective it is E-injectve for any N-group E.

In [5] V. Seth and K. Tiwari proved that if N left dgnr, with identity and M right N-group then M is injective if and only if for every right ideal U of N and every N-homomorphism $f : U \longrightarrow M$, there exists an element m in M such that f (a) = ma for all a in U. But in [11] A. Oswald claimed that converse of the above is not always true.

Theorem 1.3: [Seth, Tiwari]: N near-ring with identity and M N-group. If M is injective then for for every right ideal U of N and every N-homomorphism $f : U \longrightarrow M$, there exists an element m in M such that f (a) = ma for all a in U.

Proposition 1.4: Let $0 \to A \to E \to B \to 0$ be a short exact sequence of N-groups where A is N-subgroup (ideal) of E. Then E is Noetherian (weakly Noetherian) if and only if both A and B are Noetherian (weakly Noetherian).

Proof: First let E be Noetherian. Then since A is isomorphic to an N-subgroup of E, so by definition A is Noetherian. Again let $g : E \to B$ be the N-epimorphism. Then $E/Kerg \cong B$. Kerg is ideal of E and E is Noetherian, so $E/Kerg \cong B$ is Noetherian.

Conversely let A and B are both Noetherian, to show E is Noetherian. If we assume A is an ideal of E and B = E/A. If A is an N-subgroup of E, $E/Kerg \cong B$ is Noetherian. Imf = Kerg, Kerg is ideal of E. Now, A is Noetherian and $A/Kerf \cong Imf$. A is Noetherian $\Longrightarrow$ A/Kerf is Noetherian $\Longrightarrow$ Imf is Noetherian $\Longrightarrow$ Kerg is Noetherian, so E/Kerg, Kerg is Noetherian $\Longrightarrow$ E is Noetherian.

Corollary 1.5: If $E = E_1 \oplus E_2 \oplus \ldots \ldots \ldots \oplus E_n$ i.e E is finite direct sum of ideals of N-group E then E is weakly Noetherian if and only if $E_1, E_2, \ldots \ldots \ldots, E_n$ are weakly Noetherian.

Definitions 1.6: An N-group E is said to have finite Goldie dimension if it does not contain an infinite direct sum of non-zero ideals of E. For an N-group E if there exists an integer n such that E has an independent family of n non-zero ideals, but no independent families of more than n non-zero ideals, then integer n is called the Goldie dimension of E.

Definitions 1.7: An N-group E is said to be uniform if intersection of two non-zero N-subgroups is non-zero.

ASCENDING CHAIN CONDITIONS IN N-GROUPS:

In this section we establish relations between weakly Noetherian, almost weakly Noetherian N-group E and A.C.C. on essential ideals of E.

Proposition 2.1: Let E be an N-group. Then E/M is weakly Noetherian for every essential ideal M of E if and only if E has A.C.C. on essential ideals.

Proof: Let M be an essential ideal of E. Then E/M is weakly Noetherian. We show E has A.C.C. on essential ideals. Let $M_1 \subset M_2 \subset M_3 \subset \ldots \ldots \ldots \to$ (1) be a chain of ideals of E where $M_i \leq_e E$. Considering an essential N-subgroup $M \subseteq M_i$ $\forall$ i, we can construct another chain $M_1/M \subset M_2/M \subset M_3/M \subset \ldots \ldots$ of E/M. Since E/M is weakly Noetherian we get $M_i/M = M_{i+1}/M$ for some i. Now $M_i \subset M_{i+1}$. Our aim is to show $M_{i+1} \subset M_i$. Let $x_{i+1} \in M_{i+1}$ but $x_{i+1} \notin M$. Then $x_{i+1} + M \in M_{i+1}/M \Rightarrow x_{i+1} + M \in M_i/M$ $\Rightarrow x_{i+1} \in M_i$ (since $x_{i+1} \notin M$). So $M_i = M_{i+1} \Rightarrow$ E has A.C.C. on essential ideals.

Converse is clear.

Definition2.2: N-group E is called almost weakly Noetherian if $\frac{E}{SocE}$ is weakly Noetherian.

Proposition 2.3: N-group E is almost weakly Noetherian if and only if E/M is weakly Noetherian for every essential ideal M of E.

Proof: Let $E/SocE$ be weakly Noetherian. We know if N ideal of M, M weakly Noetherian $\Leftrightarrow$ N & M/N weakly Noetherian, by proposition 1.4. M is essential ideal of E and SocE is the intersection of all essential ideals $\Rightarrow SocE \subseteq M \Rightarrow E/SocE$ is weakly Noetherian $\Leftrightarrow M/SocE$ and $\frac{E/SocE}{M/SocE} \cong E/M$ weakly Noetherian.

Conversely, E/M is weakly Noetherian for every essential ideal M of E.We show $\frac{E}{SocE}$ is weakly Noetherian. It is enough to show that every essential ideal of $\frac{E}{SocE}$ is finitely generated. Let $\frac{M}{SocE}$ be an essential ideal of $\frac{E}{SocE}$. Let K be an ideal of M maximal with respect to $K \cap SocE = 0$. Then $K \oplus SocE$ is essential in M and hence essential in E.

[$K \oplus SocE$ ideal of M. let M' ideal of M such that $M' \cap (K \oplus SocE) = 0$.Then $M' \oplus (K \oplus SocE)$ is a direct sum $\Rightarrow M' \oplus K \oplus SocE$ is a direct sum. Whence $(M' \oplus K) \cap SocE = 0$. By maximalilty of K, $(M' \oplus K) = K$, i.e. $M' = 0$.] Then $\frac{E}{K \oplus SocE}$ is weakly Noetherian. So $\frac{M}{K \oplus SocE}$ is finitely generated. From the exactness of the sequence $0 \to K \to \frac{M}{SocE} \to \frac{M}{K \oplus SocE} \to 0$, it suffices to show K is finitely generated. We claim that K is finite dimensional. For, if not $\exists$ an infinite direct sum of non-zero ideals $\oplus_{i \in I} K_i$ which is essential in K. Since $K_i \cap Soc\, E = 0$, each K_i has a proper essential ideal T_i. [since $K_i \cap SocE = Soc\, K_i = 0$]. Let $T = \oplus_{i \in I} T_i$. Then T is an essential ideal of K. Let K' be an ideal of K, $T = \oplus_{i\in I} T_i$, where T_i are essential ideals of K_i. Now $K^{/} = \oplus_{i\in I}K_i^{/}$, $K_i^{/}\subseteq K_i$. Then $T_i\cap K_i^{/}\neq 0 \Rightarrow \oplus_{i\in I}T_i\cap K_i^{/}\neq 0 \Rightarrow T \cap\oplus_{i\in I} K_i^{/}\neq 0 \Rightarrow T \cap K^{/}\neq 0$ Again SocE is an essential ideal of SocE and $T \cap SocE = 0$. So $T\oplus SocE\leq_e K \oplus SocE\Rightarrow T \oplus SocE$ is an essential ideal of E. Hence $E/_{T \oplus SocE}$ is weakly Noetherian. As ideal of a weakly Noetherian N-group is weakly Noetherian, $\frac{\oplus_{i\in I}K_i}{T\oplus SocE}$ is weakly Noetherian $\Rightarrow\frac{\oplus_{i\in I}T_i}{T\oplus SocE}$ is weakly Noetherian. $\frac{\oplus_{i\in I}T_i}{T\oplus SocE}\subseteq\frac{\oplus_{i\in I}K_i}{T\oplus SocE}$ and $\frac{\oplus_{i\in I}K_i}{T\oplus SocE}$ weakly Noetherian imply $\frac{\frac{\oplus_{i\in I}K_i}{T\oplus SocE}}{\frac{\oplus_{i\in I}T_i}{T\oplus SocE}} \cong \frac{\oplus_{i\in I}K_i}{\oplus_{i\in I}T_i} \cong \oplus_{i\in I}\frac{K_i}{T_i}$ is weakly Noetherian, a contradiction, since it is an infinite direct sum of non zero N-groups. Thus K is finite dimensional. Let $(K_i)_{i=1}^{n}$ be a family of non-zero ideals of K such that $\oplus_{i=1}^{n}K_i$ is essential in $K \Rightarrow\oplus_{i=1}^{n} K_i\leq_e K$, so $\oplus_{i=1}^{n}K_i\oplus SocE\leq_e K\oplus SocE\leq_e E\Rightarrow\oplus_{i=1}^{n}K_i\oplus SocE\leq_e E\Rightarrow\frac{E}{\oplus_{i=1}^{n}K_i\oplus Soc\,E}$ is weakly Noetherian. We define $f :\frac{K}{\oplus_{i=1}^{n}K_i}\rightarrow\frac{K}{\oplus_{i=1}^{n}K_i\oplus Soc\,E}$ by $f (k + \oplus_{i=1}^{n} K_i) = f (k + \oplus_{i=1}^{n} K_i \oplus SocE)$. Now $f (k_1 + \oplus_{i=1}^{n} K_1) \neq f (k_2 + \oplus_{i=1}^{n} K_i)\Rightarrow (k_1 + \oplus_{i=1}^{n} K_i\oplus SocE) \neq (k_2 + \oplus_{i=1}^{n} K_i\oplus SocE)$. Next, let $\bar{k}\in\frac{K}{\oplus_{i=1}^{n}K_i\oplus Soc\,E}$. If $\bar{k}= k_1 + \oplus_{i=1}^{n}K_i\oplus Soc\,E$, $\exists\ k_1 + (\oplus_{i=1}^{n}K_i\ ) \in\frac{K}{\oplus_{i=1}^{n}K_i}$ such that $f (k_1 + (\oplus_{i=1}^{n}K_i)) = k_1 + (\oplus_{i=1}^{n}K_i\oplus L)$.So f is onto, that is f is isomorphism. Thus $\frac{K}{\oplus_{i=1}^{n}K_i}$ is isomorphic to the ideal $\frac{K}{\oplus_{i=1}^{n}K_i\oplus Soc\,E}$ of weakly noetherian N-group $\frac{E}{\oplus_{i=1}^{n}K_i\oplus Soc\,E}$. So we have that $\frac{K}{\oplus_{i=1}^{n}K_i}$ is finitely generated, whence K is finitely generated. Thus $\frac{E}{SocE}$ is weakly Noetherian.

Proposition 2.4: If N-group E is almost weakly Noetherian then E has A.C.C. on essential ideals.

Proof: Given $\frac{E}{SocE}$ is weakly Noetherian. To show E has A.C.C. on essential ideals. Soc E is the intersection of all essential ideals of E. Hence if $\frac{E}{SocE}$ is weakly Noetherian, E has A.C.C. on essential ideals.

Proposition 2.5: Let N be a dgnr. If N-group E has A.C.C. on essential ideals then E is almost weakly Noetherian.

Proof: We assume that E has A.C.C. on essential ideals.

Let $A \subseteq B$ be ideals of M such that A is essential in B. By Zorn's lemma there is a maximal ideal L of E such that $L \cap A= 0$. And $A\oplus L$ is essential in E. Since $A + L = A \oplus L$, so that $A \oplus L$ is an ideal of E. Let C ideal of E with $C \cap (A \oplus L) =0$. Then $(A \oplus L) \oplus) C$ is direct $\Rightarrow (A \oplus L) + C = (A \oplus L \oplus C)$ whence $A \cap (L\oplus C) = 0$. By maximality of L we obtain $L \oplus C = L$ Thus $C = 0$. $\therefore A\oplus L$ essential ideal of E. Hence $E/ (A \oplus L)$ satisfies ACC on its ideals.

We consider the map $\phi : B \oplus L \longrightarrow B/A$ by $b + l \longrightarrow b + A$. [N dgnr]

Now $\phi (b_1 + l_1+ b_2 + l_2) = \phi (b_1+ b_2 + l_1+l_2) = (b_1+ b_2)+A = b_1 + A + b_2 + A= \phi (b_1 + l_1) + \phi (b_2 + l_2)$

Again, $\phi n(b + l) = \phi(n_1 + n_2 + n_3 + \dots \dots + n_k)(b + l)$
$= \phi\{ n_1(b + l) + n_2(b + l) + \dots \dots + n_k(b + l)\}$
$= \phi\{ (n_1b + n_1l) + (n_2b + n_2l) + \dots \dots + (n_kb + n_kl)\}$
$= (n_1b + A) + (n_2b + A) + \dots \dots + (n_kb + A) = (n_1b + n_2b + \dots \dots + n_kb) + A$
$= nb + A = n(b + A) = n\phi(b + l)$
So ϕ is an N-homomorphism.
$\text{Ker}\phi = \{ \bar{x} / \phi(\bar{x}) = A\} = \{ a + l / \phi(a + l) = A\} = A + L$
As $A \leq B$ and $B \cap L = 0$, $A \cap L = 0$. $\therefore$ $\text{Ker}\phi = A \oplus L$
So $B/A \cong (B \oplus L)/(A \oplus L)$.Hence we get B/A also satisfies acc on its ideals.
In particular, every uniform ideal of E satisfies acc on its ideals.

Since if I is uniform ideal of E and $J_1 \subseteq J_2 \subseteq \dots \dots \dots$ an ascending chain of ideals of I. As I is uniform, each $J_i \leq_e I \Longrightarrow I/J_i$ satisfies acc on its ideals $\Longrightarrow$ I satisfies acc on essential ideals (by proposition 2.1). As each $J_i \leq_e I$, $\exists t$ such that $J_t = J_{t+1} \Longrightarrow$ I satisfies acc on its ideals.

Now, let H be an ideal of E which is maximal with respect to the condition $H \cap \text{Soc}(E) = 0$.

Then $H \oplus \text{Soc}(E)$ is essential in E and $E/_{H \oplus \text{Soc}(E)}$ satisfies acc on its ideals. Hence for proving that $E/\text{Soc}(E)$satisfies acc on its ideals it is enough to prove that H satisfies acc on its ideals. We first show that H has finite Goldie dimension. Assume that H contains an infinite direct sum $X = X_1 \oplus X_2 \oplus \dots \dots$ of non-zero ideals X_i. Since, $\text{Soc}(X_i) = X_i \cap \text{Soc}(E)$, each X_i contains a proper essential ideal Y_i and $Y = Y_1 \oplus Y_2 \oplus \dots \dots \dots$ is an essential ideal of X.By the above $X/_Y$ satisfies acc on its ideals. But this is impossible because $X/_Y = X_1/_{Y_1} \oplus X_2/_{Y_2} \oplus \dots \dots \dots$ with each $X_i/_{Y_i}$ non zero. This contradiction shows that H has finite Goldie dimension k (say).

Then H contains k independent uniform ideals U_i such that $U = U_1 \oplus U_2 \oplus \dots \dots \oplus U_k$ is essential in H. By the above U and H/U satisfies acc on ideals. Hence H satisfies acc on ideals (by proposition1.4).

Corollary 2.6: The following conditions on an N-group E of a dgnr near-ring N are equivalent:

(i) E is almost weakly Noetherian.
(ii) $E/_M$ is weakly Noetherian for every essential ideal M of E..
(iii) E has A.C.C. on essential ideals.

If N-group E contains an infinite direct sum of non zero independent family of ideals $H = \oplus_\lambda H_\lambda \Rightarrow$ the factor N-group E/H has infinite Goldie dimension we get the following theorem:

Theorem 2.7: N-group E with A.C.C. on essential ideals is weakly Noetherian.
Proof: Assume E has A.C.C. on essential ideals. Then by proposition 2.1 $\frac{E}{\text{Soc}(E)}$ is weakly Noetherian. So $\frac{E}{\text{Soc}(E)}$ cannot contain an infinite direct sum of ideals. i.e.

$\frac{E}{Soc(E)}$ has finite Goldie dimension. So by given condition, E cannot contain an infinite direct sum SocE = $\oplus_{\lambda} M_{\lambda}$. i.e. SocE is finite direct sum of simple ideals. Since every simple ideal is weakly Noetherian, by corollary 1.5 SocE is weakly Noetherian. Now if we consider the exact sequence 0 ⟶SocE⟶ E ⟶$\frac{E}{Soc(E)}$⟶ 0, SocE and $\frac{E}{Soc(E)}$ are weakly Noetherian, so by proposition 1.4 E is also weakly Noetherian.

Proposition 2.8: Near-ring N is weakly Noetherian if $\oplus_{i \in I} E_i$ of injective N-groups is injective.

Proof: Let $\oplus_{i \in I} E_i$ of commutative N-groups is injective and that $I_1 \leq I_2 \leq \ldots \ldots \ldots$ be an ascending chain of left ideals in N. Let I = $\cup_{i=1}^{\infty} I_i$. If a ϵ I, then a ϵ I_i for all but finitely many I ϵ N. So there is an f : I ⟶$\oplus_{i=1}^{\infty}$E (N/ I_i) defined as Π_i f (a) = a + I_i (a ϵ I). By theorem1.3, there is an x ϵ $\oplus_{i=1}^{\infty}$E (N/ I_i) such that f(a) = ax for all a ϵ I. Now choose *n* such that Π_{n+k}I(X)= 0, k = 0, 1, … … ... So I/ I_{n+k} = Π_{n+k}(f(I)) = $\Pi_{n+k}(I_x)$ = IΠ_{n+k}(x) = 0 or, equivalently, $I_n = I_{n+k}$ for all k = 0, 1, 2, … … ... . So, N is weakly Noetherian.

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