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QUASI-INJECTIVE NEAR-RING GROUPS

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Abstract: We extend the concepts of quasi-injective modules and their endomorphism rings to near-ring groups. We attempt to derive the near-ring character of the set of endomorphism of quasi-injective N-groups under certain conditions and this leads us to a near-ring group structure which motivates us to study various characteristics of the structure. If E is a quasi-injective N-group and $S = End(injective\, hull\, of\, E)$ then we study the structure ES and various properties of ES. It is proved that ES is a minimal quasi-injective extension of E and any two minimal quasi-injective extensions are equivalent. This structure motivates to study the Jacobson radical of endomorphism near-ring of quasi-injective N-group E. It is established that the near-ring modulo the Jacobson radical is a regular near-ring. Some properties of quasi-injective N-groups relating essentially closed N-subgroups and complement N-subgroups are established.

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1. Prerequisites

All basic concepts used in this paper are available in Pilz [4].In this section we define the basic terms and results that are needed for the sequel.

Definition 1.1. For a right near-ring $(N, +, .)$ and a corresponding N-group E, suppose there is an $x \in E$ such that $\{nx|n \in N\} = E$. Then E is a monogenic N-group and x is a generator.

Definition 1.2. An N-subgroup B of E is called fully invariant if for each N-homomorphism $f : E \to E$, $f(B) \subset B$.

Definition 1.3. A left ideal A of N is called small (strictly small) if $N = B$ for each left ideal (N-subgroup) B such that $N = A + B$.

Since every left ideal is a left N-subgroup, a strictly small left ideal of N is also a small left ideal of N.

Definition 1.4. The intersection of all maximal ideals maximal as N-subgroups of N-group E is called radical of E and is denoted by J(E).

Lemma 1.1. *[3]: If the radical ideal J(N) is strictly small in N then the following conditions are equivalent- (i) $Y \in J(N)$ (ii) 1-xy is left invertble for all $x \in N$ (iii) yM = 0 for any irreducible left N-group M.*

Definition 1.5. An N-subgroup (ideal) I of E is said to be a essentially closed N-subgroup (ideal) of E if I has no proper essential extension in E.

Definition 1.6. An N-subgroup (ideal) I of E is said to be a essentially closed N-subgroup (ideal) of E if I has no proper essential extension in E.

Theorem 1.1. *An N-group E is quasi-injective if and only if E is fully invariant N-subgroup of its injective hull.*

Theorem 1.2. *If E is quasi-injective then its direct summands are also quasi-injective.*

Proof of Theorem 1.1, Theorem 1.2 are given in K. R. Goodearl [6].

Theorem 1.3. *[Clay]: For a near-ring (N, +, .) with identity 1, suppose E is a monogenic unitary N-group with generator x and suppose that $T = \{m \in N/Ann(x)m \in Ann(x)\}$ is a subgroup of (N, +, .). Then the N-endomorphisms E of N-group E forms a right near-ring where $(f \oplus g)$ (x) = f(x) + g(x) and (f.g)(x) = f(g(x)). Also E is an End_N E-group defined by $\phi : E \times End_N E \to E$ by $\phi(m.f) = m.f = f(m)$.*

2. Endomorphism Near Ring of Quasi-Injective N-Groups

In this section we investigate various characteristics of endomorphism near-ring of quasi-injective N-groups. We also study Jacobson radical of endomorphism near-ring of quasi-injective N-groups. Throughout this section unless and otherwise mention we assume E satisfies the condition of theorem 1.3 and N is a dgnr.

If $\hat{E}$ is injective hull of E and $S = End_N\hat{E}$, $\phi : \hat{E} \times S \to \hat{E}$ by $\phi(m, f) = m.f = f(m)$, $m \in \hat{E}$, $f \in$ S, then $\hat{E}$ is an S-group.

For this S-group we get the following:

Proposition 2.1. *ES is an N-subgroup of $\hat{E}$.*

Proof. Let $a, b \in ES$,

$$a = \sum x_i f_i,\ b = \sum y_j f_j, \quad a - b = \sum x_i f_i - \sum y_j f_j \in ES.$$

Let $n \in N, a \in ES$ to show $na \in ES$.

$$\begin{aligned} na &= n \sum x_i f_i = n \sum f_i(x_i) \\ &= (s_1 + s_2 + s_3 + \cdots + s_n) \sum f_i(x_i) \\ &= s_1 \sum f_i(x_i) + s_2 \sum f_i(x_i) + s_3 \sum f_i(x_i) + \cdots + s_n \sum f_i(x_i) \\ &= \sum s_1 f_i(x_i) + \sum s_2 f_i(x_i) + \sum s_3 f_i(x_i) + \cdots + \sum s_n f_i(x_i) \\ &= \sum f_i(s_1 x_i) + \sum f_i(s_2 x_i) + \sum f_i(s_3 x_i) + \cdots + \sum f_i(s_n x_i) \\ &= \sum (s_1 x_i) f_i + \sum (s_2 x_i) f_i + \sum (s_3 x_i) f_i + \cdots + \sum (s_n x_i) f_i \in ES, \end{aligned}$$

because $(s_j x_i) \in E$). □

Proposition 2.2. *(a) ES is quasi-injective.*

(b) ES is the intersection of all quasi-injective N-subgroups of $\hat{E}$ containing E. So ES is the smallest N-subgroup of $\hat{E}$ containing E.

(c) E is quasi-injective if and only if $E = ES$.

Proof. (a) Let M be an N-subgroup of ES and $f : M \to ES$. We take the inclusion map $i : ES \to \hat{E}$. Then the composite map $h = if : M \to \hat{E}$. Since $\hat{E}$ is injective , so h can be extended by some $\lambda : \hat{E} \to \hat{E}$ such that for $x \in M$, $x.\lambda = \lambda(x) = x.h = x.(if) = (if)(x) = i(f(x)) = f(x) = x.f$ where

$x.f = f(x) \in ES$. Thus f is induced by $\lambda \in S$. Now let $g \in S$. Then for $y = \sum x_i g_i \in ES$, $\sum(x_i g_i)\lambda = \sum x_i(g_i\lambda) \in ES$, since $g_i\lambda \in S$.

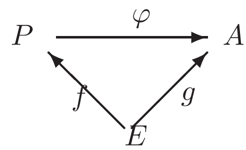
Therefore $(ES)\lambda \subseteq ES$. λ induces $\bar{\lambda} : ES \to ES$. i.e. λ can be restricted by some $\bar{\lambda} : ES \to ES$ such that $x\bar{\lambda} = x.\lambda$ for $x \in ES. Therefore x\bar{\lambda} = x.f$ for $x \in M\,[\because x\lambda = x.f$ for $x \in M$ and $M \subseteq ES]$. i.e. f is induced by $\bar{\lambda} : ES \to ES \Rightarrow ES$ is quasi-injective.

(b) Let P be any quasi-injective N-subgroup of $\hat{E}$ containing E. We wish to show $ES = \cap P$. Since by (a) ES is quasi-injective. So $\cap P \subseteq ES$. Now to show $ES \subseteq \cap P$. We will show $ES \subseteq P$, so it is sufficient to show that $P\alpha \subseteq P \quad \forall \alpha \in S$. Since if $\forall \alpha \in S, P\alpha \subseteq P$ then $PS \subseteq P$. But $E \subseteq P \Rightarrow ES \subseteq PS \quad [\because E \subseteq P \Rightarrow E\lambda \subseteq P\lambda] \Rightarrow ES \subseteq P$. To prove this we see that $Q(\alpha) = \{x \in P/x\alpha \in P\}$ is an N-subgroup of P. Let $x, y \in Q(\alpha) \Rightarrow x\alpha \in P$, $y\alpha \in P$. $x\alpha - y\alpha \in P \Rightarrow \alpha(x) - \alpha(y) \in P \Rightarrow \alpha(x-y) \in P \Rightarrow (x-y) \in Q(\alpha)$. Next to show for $n \in N$, $x \in Q(\alpha)$, $nx \in Q(\alpha)$. $x \in Q(\alpha) \Rightarrow x \in P$ such that $x.\alpha \in P$. $\because x \in P, n \in N \Rightarrow nx \in P(\because NP \subseteq P)$. $(nx).\alpha = \alpha(nx) = n\alpha(x) = n(x.\alpha) \in P(NP \subseteq P) \Rightarrow nx \in Q(\alpha)$. Therefore $Q(\alpha)$ is an N-subgroup of P. We have only to show that $Q(\alpha) = P \quad \forall \alpha \in S$, since then $y \in P \Rightarrow y \in Q(\alpha) \Rightarrow y.\alpha \in P \Rightarrow P\alpha \subseteq P$. Since $q \to q\alpha, q \in Q(\alpha) = Q$ a map of Q into P and since P is quasi-injective, so there exists $\alpha_1 : P \to P$ such that $q\alpha_1 = q\alpha \quad \forall q \in Q$. Since $\hat{E}$ is injective, $\exists \alpha' \in S$ such that $x\alpha' = x\alpha_1 \quad \forall x \in P$. Since $P\alpha' \subseteq P$. If $P(\alpha' - \alpha) = 0$ then $P\alpha' = P\alpha$. So $P\alpha \subseteq P$. So if $Q(\alpha) \neq P$ then $P(\alpha' - \alpha) \neq 0$. As we know $E \leq_e \hat{E} \Rightarrow P \leq_e \hat{E} \quad (\because$ if $A(\neq 0) \leq \hat{E}\,\&\, P \cap A = 0$ then $E \cap A = 0$ contradicts $E \leq_e \hat{E})$. Now $P(\alpha' - \alpha)$ is N-subgroup of $\hat{E}$. $a, b \in P(\alpha' - \alpha)$. Then let $a = p_1(\alpha' - \alpha), b = p_2(\alpha' - \alpha) \quad a - b = p_1(\alpha' - \alpha) - p_2(\alpha' - \alpha) = (p_1 - p_2)(\alpha' - \alpha) \in P(\alpha' - \alpha) \quad (\because (\alpha' - \alpha) \in S)$. For $n \in N, x \in P(\alpha' - \alpha)$, let $x = p_1(\alpha' - \alpha)$. Now $np_1(\alpha' - \alpha) = n(\alpha' - \alpha)p_1 = n\alpha'(p_1) - n\alpha(p_1) = \alpha'(np_1) - \alpha(np_1) = (\alpha' - \alpha)(np_1) = (np_1)(\alpha' - \alpha) \in P(\alpha' - \alpha)$, Therefore $P(\alpha' - \alpha)$ N-subgroup of $\hat{E}$. Consequently we have $P(\alpha' - \alpha) \cap P \neq 0$. But if $x, 0 \neq y \in P$ are such that $y = x(\alpha' - \alpha) \in P(\alpha' - \alpha) \cap P$. Then since $x\alpha' = x\alpha_1[\because x \in P \quad y = x(\alpha' - \alpha) = (\alpha' - \alpha)(x) = (\alpha' x - \alpha x) = x\alpha' - x\alpha]$ $x\alpha = x\alpha_1 - y \in P$. Then $x \in Q(\alpha)$ so that $x\alpha = x\alpha'$ and so $y = 0$, a contradiction. Which establishes (b).

(c) Since ES is the intersection of all quasi-injective N-subgroups of $\hat{E}$, containing E. E is quasi-injective $\Rightarrow ES \subseteq E$. And $E \subseteq ES$ is obvious by inclusion map: $ES = E$ □

Definition 2.1. (P, E, f) denotes a N-monomorphism $f : E \to P$ and is called an extension of E. An extension (P, E, f) of an N-group E is a minimal quasi-injective extension in case P is quasi-injective and the following condition is satisfied:

If (A, E, g) is any quasi-injective extension of E, then there exists a monomorphism $\phi : P \to A$ such that:



commutes i.e. $g = \phi f$

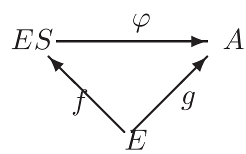
Proposition 2.3. *ES is minimal quasi-injective extension of E. Any Two minimal quasi-injective extensions are equivalent.*

Proof. Let (A, E, g) be any quasi-injective extension of E. Let $\hat{A} = E(A)$ & $\Omega = Hom_N(\hat{A}, \hat{A})$. Then by proposition 2.2, $A\Omega \subseteq A$. Since ES is an essential extension of E, the N-monomorphism $g : E \to \hat{A}$ can be extended to a monomorphism (also denoted by g) of ES in $\hat{A}$. [Since if $f : A \xrightarrow{mono} E$, E injective, $A \leq_e B$, then f extends to $f' : B \xrightarrow{mono} E$] Since $g(ES)$ is quasi-injective. [$\because g(ES) \cong ES, \because Kerg = 0(f : A \xrightarrow{mono} B, A \cong f(A))$].

Then $(g(ES))\Omega \subseteq g(ES)$ and we conclude that $(B)\Omega \subseteq B$ where $B = g(ES) \cap A \subseteq g(ES)$, so $g^{-1}(B) \subseteq (ES)$ [$\because AB \subseteq B, AC \subseteq C, A(B \cap C) = AB \cap AC \subseteq B \cap C$].

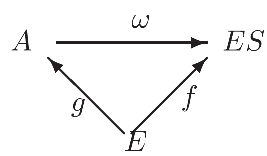
Since $B \subseteq (B)\Omega$ is obvious. Therefore by Proposition 2.2 B is quasi-injective. It follows that $g^{-1}(B)$ is a quasi-injective extension of $E \subseteq ES$. Since ES is the smallest quasi-injective extension of E contained in $\hat{E}$, we conclude that $g^{-1}(B) = (ES)$. So $B = g(ES) \subseteq A$. This establishes that ES is a minimal quasi-injective extension.

Next if (A, E, g) is also a minimal quasi-injective extension of E, then (A, E, g) is also equivalent to ES. ES minimal quasi-injective extension of E. (A, E, g) is also quasi-injective extension of E. By definition for $E \xrightarrow{mono\, \phi} ES$, $E \xrightarrow{mono\, f} A$, there exists $ES \xrightarrow{mono\, \phi} A$ such that the diagram



commutes i.e. $g = \phi f$

Again (A, E, g) is minimal quasi-injective extension of E. ES is also quasi-injective extension of E. By definition for $E \xrightarrow{mono\, g} A$, $E \xrightarrow{mono\, f} ES$ there exists $A \xrightarrow{mono\, \omega} ES$ such that the diagram



commutes i.e. $f = \omega g$.

Now $f = \omega g \Rightarrow f = \omega\phi f$. So $I = \omega\phi$. Again $g = \phi f \Rightarrow g = \phi\omega g$. So $I = \phi\omega$. Thus ω and ϕ both are invertible which implies both ω and ϕ are isomorphic. Hence $ES \cong A$. □

Definition 2.2. A near-ring N is said to be a regular near-ring if for every element $x \in N$, there exists an element $y \in N$ such that $xyx = x$.

Theorem 2.1. *Let E be quasi-injective N-group let $\Lambda = Hom(E, E)$ and let $J = J(\Lambda)$ denote the Jacobson radical of Λ and is strictly small in Λ. Then $J = \{\lambda \in \Lambda/E$ essential extension of $Ker\lambda\}$. If for $\gamma \in J$, $\lambda \in \Lambda$, $\gamma\lambda \in J$ then Λ/J is a regular near-ring. Where addition of two N-subgroups is again N-subgroup of E and N need not be dgnr.*

Proof. Let $I = \{\lambda \in \Lambda/E \,\text{essential extension of}\, Ker\lambda\}$.

If $\lambda \in \Lambda$, $\mu, \gamma \in I$, then $Ker(\mu + \gamma) \supseteq Ker\mu \cap Ker\gamma$. Since $x \in Ker\mu \cap Ker\gamma \Rightarrow x \in Ker\mu$ & $x \in Ker\gamma \Rightarrow \mu(x) = 0 \,\&\, \gamma(x) = 0 \Rightarrow (\mu + \gamma)(x) = 0 \Rightarrow x \in Ker(\mu+\gamma)$. Since $Ker\mu \cap Ker\gamma$ is an essential N-subgroup, therefore $Ker(\mu + \gamma)$ is an essential N-subgroup of E $x \in Ker\gamma \Rightarrow \gamma(x) = 0$. Now for $\mu, \lambda \in \Lambda, \gamma \in I, (\mu(\lambda + \gamma) - \mu\lambda)(x) = (\mu(\lambda + \gamma)(x) - (\mu\lambda)(x) = (\mu\lambda)(x) + 0 - (\mu\lambda)(x) = 0$, since $\gamma(x) = 0$. Therefore $x \in Ker(\mu(\lambda + \gamma) - \mu\lambda)$. And so $Ker\lambda \subseteq Ker(\mu(\lambda + \gamma) - \mu\lambda) \Rightarrow (\mu(\lambda + \gamma) - \mu\lambda) \in I$. Therefore I is left ideal of Λ.

However if $\lambda \in I, Ker(1+\mu\lambda) = 0$ for $Ker\lambda \cap Ker(1+\mu\lambda) = 0$. For if, $\lambda \in I$ we have E essential extension of $Ker\lambda$. $x \in Ker\lambda \cap Ker(1 + \mu\lambda) \Rightarrow \lambda(x) = 0$ and $(1+\mu\lambda)(x) = 0 \Rightarrow$ x $+\mu(\lambda)(x) = 0 \Rightarrow x + \mu(0) \Rightarrow x = 0$. Again $\lambda \in I \Rightarrow E$ essential extension of $Ker\lambda \Rightarrow Ker(1 + \mu\lambda) = 0$.

$(1+\mu\lambda) : E \to (1+\mu\lambda)E$ is an isomorphism $\Rightarrow \exists g \in \Lambda$ such that $g(1+\mu\lambda) = i$, so $(1 + \mu\lambda)$ has a left inverse $\forall\lambda \in I$ & $\forall\mu \in \Lambda$. So, $\lambda \in J$ [by lemma 1.1]. This establishes that $I \subseteq J$.

Next let λ be arbitrary element of Λ, let L be a complement N-subgroup of E corresponding to $K = Ker(\lambda)$ and consider the correspondence $\lambda x \to x$ $\forall\, x \in L$. If $\lambda x = \lambda y$ with $x, y \in L$, then $\lambda(x - y) = 0$ and then $(x - y) \in K \cap L$ $= 0$. Since E is quasi-injective, the map $\lambda x \to x$ of λL in L is induced by some $\theta \in \Lambda$. If $u = x + y \in L + K, x \in L, y \in K$, then

$$\begin{aligned}(\lambda - \lambda\theta\lambda)(u) =& \lambda(x) - \lambda\theta\lambda(x) = \lambda(x) - \lambda(x) = 0 \\ \Rightarrow& \lambda - \lambda\theta\lambda = 0 \end{aligned} \tag{1}$$

$[\lambda\theta\lambda(x) = \lambda\theta(x) = \lambda(x) = x$, as for $x \in L\theta(x) = \lambda(x) = x]$.

Since $E \supseteq_e L + K$ as$K \subseteq L + K$ and since $Ker(\lambda - \lambda\theta\lambda) \supseteq L + K$, we conclude that $\lambda - \lambda\theta\lambda \in I$. Now to show $J = I$. If $\lambda \in J$ and $\theta \in \Lambda$ is chosen so that $u = (\lambda - \lambda\theta\lambda) \in I, (1 - \theta\lambda)^{-1}$ exists.(Since J is Jacobson Redical. Therefore $(1-\theta\lambda)^{-1}(u) = (1-\theta\lambda)^{-1}(\lambda - \lambda\theta\lambda) = (1-\theta\lambda)^{-1}(1-\theta\lambda)\lambda = \lambda$ and $\lambda \in I[\because I$ is a left ideal]. Thus $J = I$ is asserted. Also I is an ideal by given condition. Thus Λ is a regular modulo I.

From (1) $\lambda\bar{\theta}\lambda = \bar{\lambda}$ in Λ/I. Therefore Λ/J is regular ring as $J = I$. □

3. Some Properties of Quasi-Injective N-Groups

This section contains some properties of quasi-injective N-groups related to essentially closed N-subgroups and complement N-subgroups. Let M be an N-subgroup of E. We consider $F = \{P/P \quad N\text{-subgroup of} E, P \cap M = 0\}.F \neq \Phi, (0) \in F.C = \{P_i/P_i \in F\}$ is a chain in F. Let $K = \cup P_i.[x, y \in \cup P_i \Rightarrow x \in P_i, y \in P_j.Ifi > J, x, y \in P_j$. Therefore $(x - y) \in P_j \Rightarrow (x - y) \in \cup P_i$. Again $n \in N, x \in \cup P_i \Rightarrow x \in P_j$ for some j, then $nx \in P_j \Rightarrow nx \in \cup P_i]$ Since $P_i \cap M = 0 \forall i.(\cup_i P_i) \cap M = \cup_i(P_i \cap M) = 0 \& \cup_i P_i \leq E$. Therefore $\cup_i(P_i \in C$. So by Zorn's Lemma the N-subgroup K is maximal in the set of those N-subgroups P satisfying $P \cap N = 0$. Then K is said to be complement of M in E.

Definition 3.1. The N-subgroup K is maximal in the set of those N-subgroups P satisfying $P \cap M = 0$ is said to be complement of M in E. A complement N-subgroup(ideal) of E is an N-subgroup A which is a complement in E of some N-subgroup(ideal) B.

If sum of two N-subgroups is again an N-subgroup of an N-group we get the following:

Lemma 3.1. *If M is an N-subgroup of E and if K is any complement of M in E, then there exists a complement Q of K in E such that $Q \supseteq M$. Furthermore any such Q is a maximal essential extension of M in E.*

Proof. Let $F = \{I/I \cap K = 0, M \subseteq I\}$. Since $M \in F, F \neq \Phi$. Let $C = \{C_i/i \in \lambda, \lambda index, C_i \in F\}$ be a chain. $Q = \cup C_i$. Now $(\cup_{i\in\lambda} C_i) \cap K = \cup_{i\in\lambda}(C_i \cap K) = 0 \quad \forall i. \quad [\because C_i \cap K = 0 \quad \forall i] \quad \& \quad M \subset \cup_{i\in\lambda} C_i \quad \forall i, M \subseteq C_i$. So by Zorn's Lemma $Q \in F$, maximal element exists.Thus Q in the first sentence exists.

Now to prove the second part.

Let T be any non-zero N-subgroup of Q and assume that $T \cap M = 0$. Since $T \cap K = 0 \quad [Q \leq_c K, T \leq_s Q]$. Therefore the sum $K_1 = T + K$ is direct and K_1 properly contains K. $\because K_1 \cap M = 0$. [If possible let $K_1 \cap M \neq 0$. $K_1 \cap M = (T + K) \cap M$. Let $t + k = n \in (T + K) \cap M \Rightarrow k \in K \cap (M + T) \subseteq K \cap Q \Rightarrow k = 0 \Rightarrow n = t \in M \cap T$ contradiction to $T \cap M = 0$. Therefore $K_1 \cap M = 0$.] This contradicts the definition of K. This proves that Q is an essential extension of M. If P is an N-subgroup of E properly containing Q , then $P \cap K \neq 0$ and $(P \cap K) \cap M = P \cap (K \cap M) = P \cap 0 = 0$. Thus P is not essential extension of M, completing the proof. □

Lemma 3.2. *The essentially closed N-subgroups of an N-group E coincide with the complement N-subgroup of E. If M and K are complement N-subgroups and if K is a complement of M in E then M is a complement of K in E.*

Proof. Let M be a essentially closed N-subgroup and K is any complement of M. Then by lemma 3.1 there exists a complement Q of K such that $M \subseteq Q$. This Q is maximal essential extension of M in E. But M is essentially closed, so it has no proper essential extension. Therefore $M = Q$ is a complement N-subgroup.

Next let M be complement of an N-subgroup P. Then $\exists$ a complement K of M which contains P.

If possible let $M^/ \leq E$ such that $M \subseteq M^/ \quad \& \quad K \cap M^/ = 0$. Then $P \cap M^/ = 0. \quad \because P \subset K$, which contradicts (1). Therefore M is also maximal such that $K \cap M = 0$. Therefore M is complement of K. Then M is essentially closed by lemma3.1. This also proves the last statement.

Theorem 3.1. *Let E be quasi-injective and let M be a essentially closed N-subgroup, then for each N-subgroup K of E, N-homomorphism $w : K \Rightarrow M$ can be extended to N-homomorphism $u : E \rightarrow M$*

Proof. Let $F = \{L/w$ is extended to a map of T into M for N-subgroup T of E containing $L\}$ By Zorn's lemma we can assume that K is such that w cannot be extended to a map of T into M for any N-subgroup T of E which properly contains K. Since, E is quasi-injective, w is induced by a map $u : E \to E\&$ let L complement of M in E. Suppose $u(E) \subsetneq M$. Since M is essentially closed. M is a complement of L. Therefore, since $u(E) + M \supset M$, we see that $(u(E) + M) \cap L \neq 0$. Let $0 \neq x = a + b \in (u(E) + M) \cap L \Rightarrow a \in u(E), b \in M$. If $a \in M$ then $x \in M \cap L = 0$, a contradiction. Therefore $a \notin M$ and $a = x - b \in L + M$. Now $T = y \in E/u(y) \in L + M$ is an N-subgroup of E containing K. $\because x \in K \Rightarrow w(k) \in M \Rightarrow u(k) \in M \quad \forall k \in K$. Therefore T contains K. If $y \in E$ is such that $u(y) = a$ then $y \in T$, but $y \notin K$ since $a \notin M$. $[\because y \in T \Rightarrow u(y) = a \in L \& y \in K \Rightarrow w(y) \in M \Rightarrow u(y) \in M \quad \forall y \in K$, contradiction to $a \notin M]$.

Let π denote the projection of $L+M$ on M. Then πu is a map of T in M and $\pi u(y) = u(y) = w(y) \quad \forall y \in K$. $[\because y \in K \Rightarrow w(y) \in M \Rightarrow u(y) \in M \quad \forall y \in K]$. Thus πu is a proper extension of w, a contradiction. Therefore $u(E) \subseteq M$, so u is the desired extension.

Corollary 3.1. *For quasi-injective N-group E.*

(1) If M is essentially closed N-subgroup of E, then M is a direct summand of E and M is quasi-injective. Also M has a complement in E.

(2) If P is any N-subgroup of E, then there exists a quasi-injective essential extension of P contained in E.

(3) Each minimal quasi-injective extension of an N-group K is an essential extension of K.

Proof. (1) If $e : E \to M$ is the extension given by theorem 3.1 of the injection map $M \to M$ then $E = M \bigoplus Ker(e)$ where $e(m) = \begin{cases} m, & m \in M \\ 0, & m \notin M \end{cases}$ So that M is a direct summand of E. Therefore M is quasi-injective by theorem 1.2 Moreover $Ker(e)$ is complement of M. Since

$$M \cap Ker(e) = (0) \tag{2}$$

M essentially closed $\Rightarrow M$ complement of some N-subgroup $K \Rightarrow K$ is complement of M, i.e.

$$\overset{max}{M} \bigcap \overset{max}{K} = (0) \tag{3}$$

(2) & (3) $\Rightarrow Ker(e) \subset K$. Let $(0 \neq)x \in K \Rightarrow x \notin M \Rightarrow e(x) = 0$ [by definition of e] $\Rightarrow x \in Ker(e)$.

Therefore $K \subset Ker(e)$. Therefore $K = Ker(e) \Rightarrow Ker(e)$ complement of M.

(2) Let $F = \{I/P \subseteq I, P \leq_e I\}.P \in F$. Therefore $F \neq \phi$.

Let $\{C_i/C_i \in F\}$ be a chain in F. $M = \cup_i C_i, P \subseteq C_i \quad \forall I \Rightarrow P \subseteq \cup_i C_i, P \leq_e C_i \quad \forall I \Rightarrow P \leq_e \cup_i C_i$ [$P \cap A_i \neq 0 \quad \forall i, A_i \leq C_i$. Since $P \cap (\cup_i A_i) = \cup_i (P \cap A_i) \neq 0. \cup A_i \leq \cup C_i$]

If possible $M = \cup_i C_i \leq_e K$. Therefore $P \leq_e M \leq_e K \Rightarrow P \leq_e K$, contradicts maximality of M. So by Zorn's Lemma P is contained in essentially closed N-subgroup M which is essential extension of P and M is quasi-injective by (1).

(3) Let A be any minimal quasi-injective extension of an N-group K. Let K is contained in quasi-injective essential extension B by (2). i.e.B essentially closed. So as B is essential extension of K, A is also essential extension of K. Thus every minimal quasi-injective extension of an N-group K is an essential extension of K. □

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