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Singular and semi-simple character in E-injective N-groups with weakly descending chain conditions

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Abstract Injective modules and near-ring groups have been studied by several researchers like Mason et al., Faith et al., Jara, Harada, Cheng. Of these Oswald and Mason have studied injective and projective near-ring modules. Mason studied injective near-ring modules and defined the concepts like n-injective, loosely injective and almost injective near-ring modules. In Hazarika and Saikia (Int J Math Sci 33(2), 2013) we extended the notion of relative injectivity of modules to near-ring groups. Here E-injective N-groups with descending chain conditions are studied. It is shown that the singular and semi-simple characters play a vital role in characterization of E-injective N-groups with weakly descending chain conditions.

Keywords Near-ring group · E-injective N-group · Weakly Noetherian N-group · Singular N-group · Semi-simple N-group

Mathematics Subject Classification 16Y30

1 Prerequisites

All basic concepts used in this paper are available in Pilz [4]. In this section we define the basic terms and results that are needed in the sequel. Throughout the paper we consider N as a zero symmetric right near-ring and E as a left N-group.

Definition 1.1 If A, B are two N-subgroups of E such that $A \subseteq B$ then A is essential (weakly essential) in B when any non-zero N-subgroup (ideal) C of E contained in B has a

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nonzero intersection with A. In such case B is an essential(weakly essential) extension of A and is denoted by $A \leq_e B$ ($A \leq_{we} B$).

If A is an N-subgroup (ideal) of E and A is essential in E (when $B = E$) then we say that A is an essential N-subgroup (ideal) of E

Using the definition of essential N-subgroup easily we can prove the following proposition:

Proposition 1.1 *If A, B, C are N-subgroups of E, with* $A \subseteq B \subseteq C$, *then* $A \leq_e C$, *if and only if* $A \leq_e B \leq_e C$.

Definition 1.2 The singular subset of E, denoted by $Z(E)$, is defined as the set
$Z(E) = \{x \in E| \quad Ix = 0$ for some essential N-subgroup I of $N\}$.
E is called a singular N-group if $Z(E) = E$.
E is called a non-singular N-group if $Z(E) = 0$.

Definition 1.3 The set $(B : a)$ is defined as $(B : a) = \{n \in N|na \in B\}$.

It can be shown easily that if B is an essential N-subgroup of E and $a \in E$ then $(B : a)$ is an N-subgroup of N and if B is an essential N-subgroup, $(B : a)$ is also an essential N-subgroup of N.

Proposition 1.2 *If proper essential N-subgroups of N are distributively generated, then Z(E) is an N-subgroup of E.*

Proof Let $e_1, e_2 \in Z(E)$. Then there exist essential N-subgroups I_1, I_2 of N such that $(I_1 \cap I_2)(e_1 - e_2) = 0$ as $(I_1 \cap I_2)$ is distributively generated. So $e_1 - e_2 \in Z(E)$. Again let $e \in Z(E)$. For $a \in N$ and essential N-subgroup I of N, $(I : a)$ is an essential N-subgroup of N. So for $z \in (I : a)$ we get $(za)e = 0 \Rightarrow z(ae) = 0 \Rightarrow ae \in Z(E)$. Thus Z(E) is an N-subgroup of E. □

Definition 1.4 The weak singular subset of E, denoted by $Z_w(E)$, is defined as the set
$Z_w(E) = \{x \in E| \quad Ix = 0$ for some essential ideal I of $N\}$.
E is called a weak singular N-group if $Z_w(E) = E$.
E is called a weak non-singular N-group if $Z_w(E) = 0$.

Definition 1.5 An N-monomorphism $f : A \rightarrow B$ is said to be an essential N-monomorphism if $f(A) \leq_e B$.

Proposition 1.3 *An N-group C is singular if there exists a short exact sequence* $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ *such that f is an essential N-monomorphism.*

Proof For any $b \in B$, we have a map $k : N \rightarrow B$ defined by $k(n) = nb$. Then $k^{-1}(fA) \leq_e N$, which gives the N-subgroup $I = \{n \in N|nb \in fA\}$ is an essential N-subgroup of N. Now $Ib \leq fA = Kerg$. Hence $I(gb) = 0$ and so $gb \in Z(C)$. Since g is an N-epimorphism, $Z(C) = C$. □

Proposition 1.4 *If A, B are N-groups such that B is non-singular and B/A is singular then* $A \leq_{we} B$. *i.e A is weakly essential in B.*

Proof If B/A is singular and $x(\neq 0) \in B$, then $I\bar{x} = \bar{0}$ for some essential N-subgroup I of $N \Rightarrow Ix \leq A$. As B is non-singular, we have $Ix \neq 0$ and thus $Nx \cap A \neq 0$. Therefore $A \leq_{we} B$. □

Proposition 1.5 [12] *The following are equivalent*

(a) *Every normal N-subgroup of E is a direct summand.*
(b) *E is a sum of simple normal N-subgroups.*
(c) *E is a direct sum of simple normal N-subgroups.*

Definition 1.6 The strict socle of E, denoted by s-$SocE$, is defined as the direct sum of simple normal N-subgroups. E is called a strictly semisimple if s-$Soc(E) = E$. In other words E is strictly semisimple if one of the conditions of Proposition 1.5 holds.

Definition 1.7 The socle of E, denoted by $Soc(E)$, is defined as the sum of simple ideals of E. Equivalently, $Soc(E)$ = the direct sum of simple ideals of E.
E is called semisimple if $Soc(E) = E$

We observe that every semisimple N-group is strictly semisimple but the converse is not true. If N is a distributively generated near ring(dgnr) then every strictly semisimple N-group is semisimple.

The following is an example of strictly semisimple N-group which is not semisimple.

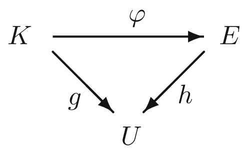
Example 1.1 We consider the near-ring $N = \{0, a, b, c, x, y\}$ under the addition and multiplication defined in the following table:

+	0	a	b	c	x	y
0	0	a	b	c	x	y
a	a	0	y	x	c	b
b	b	x	0	y	a	c
c	c	y	x	0	b	a
x	x	b	c	a	y	0
y	y	c	a	b	0	x

.	0	a	b	c	x	y
0	0	0	0	0	0	0
a	0	a	b	c	0	0
b	0	a	b	c	0	0
c	0	a	b	c	0	0
x	0	0	0	0	0	0
y	0	0	0	0	0	0

Here $\{0, a\}$, $\{0, b\}$, $\{0, c\}$, $\{0, x, y\}$ are simple left normal N-subgroups of N. And $N = \{0, a\} + \{0, b\} + \{0, c\} + \{0, x, y\}$. So N is strictly semisimple. But N is not semisimple.

Definition 1.8 Let E and U be N-groups. U is called E-injective or U is injective relative to E if for each N-monomorphism $\varphi : K \to E$, every N-homomorphism g from K into U can be extended to an N-homomorphism h from E into U. i.e. the diagram



commutes. i.e. $g = h\varphi$.

An N-group A is injective if it is E-injective for every N-group E of N. So if an N-group A is injective it is E-injective for any N-group E.

Definition 1.9 E is called a V N-group if every simple N-group is E-injective.
E is called a V_c N-group if every simple abelian N-group is E-injective.
E is called a GV N-group if every simple singular N-group is E-injective.
E is called a S^3 I N-group if every strictly semi-simple singular N-group is E-injective.
E is called a S^2S_wI N-group if every strictly semi-simple weak singular N-group is E-injective.

Definition 1.10 N is called a V near-ring if ${}_NN$ is a VN-group and a GV near-ring if ${}_NN$ is a GV N-group.
N is called a V_c near-ring if ${}_NN$ is a V_c N-group.

Definition 1.11 E is said to be weakly Noetherian (Noetherian) if every strict ascending chain of ideals or normal N-subgroups(N-subgroups) $A_1 \subset A_2 \subset \cdots$ of E terminates after finitely many steps or equivalently for each chain $A_1 \subseteq A_2 \subseteq \cdots$ of E , $\exists n \in N$ such that $A_n = A_{n+1} = \cdots$.

2 Strictly semi-simple character of E-injective N-groups with weakly descending chain conditions

Proposition 2.1 *If $\{Ne\}_{e\in E}$ is an independent family of normal N-subgroups of E in a dgnr N and direct sum of E-injective N-groups is an abelian N-group then E is Noetherian V N-group(V_c N-group) implies every strictly semi-simple N-group is E-injective.*

Proof E is Noetherian VN-group $\Rightarrow$ E is Noetherian and every simple N-group is E-injective. Again any direct sum of E-injective N-groups is E-injective as E is Noetherian [5,Theorem 4.12]. Let K be any strictly semi simple N-group $\Rightarrow$ K is direct sum of simple normal N-subgroups. So K is E-injective. □

Proposition 2.2 *For a finitely generated N-group E every countably generated strictly semi-simple N-group is E-injective implies E is weakly Noetherian V_c N-group.*

Proof Suppose $\{A_\alpha\}_{\alpha\in J}$ is a family of N-groups such that for every countable subset K of J, $\bigoplus_{\alpha\in K} A_\alpha$ is E-injective. Then by [5, Theorem 4.11] $\bigoplus_{\alpha\in J} A_\alpha$ itself E-injective. Now given that every countably generated strictly semi simple N-group is E-injective. To show E is weakly Noetherian and every simple abelian N-group is E-injective, let U be a countably generated strictly semi-simple N-group. Then $U = \bigoplus U_\alpha$, where U_α is simple normal

N-subgroup, so U_α 's can be taken as abelian N-groups and $\alpha \in K$, K is countable subset of J (as U countably generated).Given U is E-injective. So we have $\bigoplus U_\alpha, \alpha \in J$ is also E-injective [5,Theorem 4.11]. So by [5,Theorem 4.8], we get every U_α is E-injective $\Rightarrow E$ is $V_c N$-group.

Next to show E is weakly Noetherian. Given E is finitely generated and W countably generated semi-simple N-group and W is E-injective.

Let $N_1 \subset N_2 \subset N_3 \subset \ldots$ be an ascending chain of distinct ideals of E. Let $f_K : N_K \to W(k = 1, 2, 3, \ldots)$. As W is E-injective, for inclusion map $i_K : N_K \to E$, $\exists$ a map $\gamma_K : E \to W$ such that $f_K = \gamma_K i_K$. Let $N^/ = \sum_{(k=1)}^{\infty} N_k$. Define the map $f : N^/ \to W$ by $f(x) = \sum_{(k=1)}^{\infty} f_k(x) = \sum_{(k=1)}^{\infty} \gamma_K i_K(x)$. f is well defined. $\because$ W is E-injective, $\exists$ a map $g : E \to W$ extending f. But E is finitely generated and $g(E) \subset W$, w countably generated. So g can be defined as $g(x) = \sum_{(k=1)}^{m} \gamma_K i_K(x)$ for some positive integer m, which gives chain of ideals must be finite. □

Corollary 2.1 *For a finitely generated N-group E, every strictly semi-simple N-group is E-injective implies E is weakly Noetherian V_c N-group.*

Proposition 2.3 *For dgnr N, if E is a finitely generated S^3I N-group, then $E/(Soc(E))$ is a weakly Noetherian V_c N-group.*

Proof From the above Corollary 2.1, it is enough to show that every strictly semi-simple N-group is $E/(Soc(E))$-injective. Let L be a strictly semi-simple N-group. So as N dgnr, L is a semi-simple N-group. Let M be an ideal of E such that $M/(Soc(E))$ is an ideal of $E/(Soc(E))$ and $f : M/(Soc(E)) \to L$ be a non-zero N-homomorphism. Let $K/(Soc(E)) = Kerf$. We claim K is essential ideal in M. For if $K \cap I = 0$ for some non-zero ideal I of M then $I \cong (I + K)/K$ and since the latter is isomorphic to an ideal of L, it follows that for some ideal $I_1 \neq 0$ and contained in I that $I_1 \subset L$, hence $I_1 \subseteq Soc(E) \subseteq K$, a contradiction. Now M/K singular, we may take L singular, since $f(M/K) \subseteq Z(L)$. Let $\eta : M \to M/SocE$ denote the quotient map and consider the map $f.\eta : M \to L$. $\because$ L is E-injective $f.\eta$ extends to a map of E into L. $\because$ $Soc(E) \subseteq K$. This yields a map of $E/(Soc(E))$ into L by [5, Proposition2.6]. □

3 Semi-simple and singular character of E-injective N-groups with weakly descending chain conditions

Proposition 3.1 *Let N be a dgnr. If E is an N-group satisfying the following conditions*

1. *$\{Ne\}_{e \in E}$ is an independent family of normal N-subgroups of E,*
2. *the direct sum of E-injective N-groups is an abelian N-group*
3. *no non-zero homomorphic image of Nx, $\forall x (\neq 0) \in Soc(E)$, is semi-simple, singular*
4. *$E/(Soc(E))$ is a Noetherian V N-group, then E is an S^3I N-group.*

Proof Let L be a strictly semi-simple singular N-group. Let M be an N-subgroup of E. $f : M \to L$ a non-zero map with $kerf = K$. Then by given condition $Soc(E) \cap M$ is contained in K. [For $x \in Soc(E) \cap M \Rightarrow x \in Soc(E), x \in M \Rightarrow Nx \subseteq Soc(E), Nx \subseteq M \Rightarrow Nx \in Soc(E) \cap M$]. So by [5, Proposition2.6], $\exists$ an N-homomorphism $f^/ : M/(Soc(E) \cap M) \to L$. Since $M/(Soc(E) \cap M) \cong (Soc(E)+M)/(Soc(E))$, so $f^/ : (Soc(E)+M)/(Soc(E) \to L$. As $E/(Soc(E))$ is Noetherian VN-group and L semi-simple singular by Proposition 2.1, L is $E/(Soc(E))$-injective, that is $f^/$ is extended to $g^/ : E/(Soc(E)) \to L$. If we define $g : E \to L$ by $g(e) = g^/(\bar{e} + Soc(E))$. g is extension of f. □

Definition 3.1 N-group E is called almost weakly Noetherian if $E/SocE$ is weakly Noetherian.

Proposition 3.2 *If E is non-singular and every singular homomorphic image of E is weakly Noetherian then E is almost weakly Noetherian.*

Proof Let M be an essential ideal of E and E is non-singular. Then E/M is singular. Again E/M is homomorphic image of E, by given condition E/M is weakly Noetherian. □

Proposition 3.3 *If E is non-singular and almost weakly Noetherian and in E every weakly essential N-subgroup is essential then every singular homomorphic image of E is weakly Noetherian.*

Proof Let $f : E \to L$ be an N-epimorphism and L is singular. Now E is non-singular and $kerf \subseteq E$, $L \cong E/kerf$ singular, so $kerf \leq_{we} E$ by Proposition 1.4. Then $Soc(E) \subseteq kerf$. So by [5, Proposition2.6] we get $L \cong E/(Soc(E))$. As E is almost weakly Noetherian, L is weakly Noetherian. □

Definition 3.2 An N-subgroup U of an N-group E is called pure in E if $IU = U \cap IE$ for each ideal I of N.

Definition 3.3 E is an injective hull of its N-subgroup (ideal) K if E is injective and $K \subseteq L \subseteq E$, where L is injective N-subgroup (ideal) $\Rightarrow L = E$. Equivalently, E is an injective hull of its N-subgroup (ideal) K if E is injective and E is an essential extension of K.

Proposition 3.4 *If N is non-singular, SocN is pure and every injective right N-group is injective as an N/K-group for ideal K of N then the direct sum of (countably many) injective hulls of simple weak singular left N-groups is injective implies N is an almost weakly Noetherian near-ring.*

Proof Let $\{S_i\}_{i\in I}$ be a family of simple weak singular $N/Soc(N)$-groups. Since a simple N-group is weak singular if and only if it is annihilated by $Soc(N)$. For let E is simple and weak singular. So $Z_w(E) = x \in E| \quad Ix = 0, I \leq_{ei} N = E$. So $x \in E \Rightarrow I \leq_{ei} N$ such that $Ix = 0 \Rightarrow Soc(N)x = 0$. Thus E is annihilated by $Soc(N)$. Again let E be annihilated by $Soc(N)$, we get $Soc(N)E = 0 \Rightarrow Soc(N) \subseteq Ann(E)$. Now we show $Ann(E) = \{x \in N| \quad xE = 0\}$ is essential ideal in N. If possible $Ann(E)$ is not essential ideal in N. Then $Ann(E) \cap J = 0$ for some non-zero ideal J of N. If $\forall x \in E, \quad f : J \to Jx$, defined by $f(j) = jx$, it is a well defined N-homomorphism. $f(j_1) \neq f(j_2) \Rightarrow (j_1x) \neq (j_2x) \Rightarrow (j_1 - j_2)x \neq 0 \Rightarrow (j_1 - j_2) \neq 0 \Rightarrow j_1 \neq j_2$. So f is well-defined. Next let $j_1 \neq j_2 \Rightarrow (j_1 - j_2) \neq 0 \Rightarrow (j_1 - j_2)x \neq 0 \Rightarrow (j_1x) \neq (j_2x) \Rightarrow f(j_1) \neq f(j_2)$. So f is one-one. Again for every $jx \in Jx, \quad \exists j \in J$ such that $f(j) = jx$. So f is onto. $f(j_1 + j_2) = (j_1 + j_2)x = (j_1x + j_2x) = f(j_1) + f(j_2) \quad \& \quad f(nj) = (nj)x = n(jx) = nf(j)$. So f is N-isomorphism $\Rightarrow \forall x \in E, J \cong Jx$. Again $Z(N) = 0 \Rightarrow Z(J) = 0 \Rightarrow Z(Jx) = 0 \Rightarrow \forall I \leq_{ei} N, I(Jx) \neq 0 \Rightarrow SocN.(Jx) \neq 0$. But $Jx \subseteq E$ and $SocN.E = 0 \Rightarrow SocN.(Jx) = 0$, a contradiction. So $Ann(E)$ is essential ideal of N, so E is weak singular. It follows that each ${}_NS_i$ is weak singular as an N-group. Since $SocN$ is pure we get $Soc({}_NN).E({}_NS_i) \cap {}_NS_i = SocN.S_i, \quad \forall i \in I$. As each $({}_NS_i)$ is annihilated by $Soc(N)$, $SocN.S_i = 0$. So $Soc({}_NN).E({}_NS_i) \cap {}_NS_i = 0$. i.e. $\forall x \in E({}_NS_i), \quad Soc({}_NN).x \cap ({}_NS_i) = 0$. $E({}_NS_i)$ is an essential extension of ${}_NS_i$, and since $Soc({}_NN).x$ is N-subgroup of $E({}_NS_i)$ we get $\forall x \in E({}_NS_i), \quad Soc({}_NN).x = 0$. Thus $E({}_NS_i)$ is annihilated by $Soc(N), \quad \forall i \in I$. We claim that $\forall i \in I, \quad E({}_NS_i)$ is weak singular as N-group. For $x \in E(_N S_i)$ with $x \notin Z_w E(_N S_i)$ then $\forall I \leq_{ie} N$, $Ix \neq 0 \Rightarrow Ann_N(x)$ is not essential in N. So $Ann_N(x) \cap J = 0$ for some non-zero ideal J of N. Since $J \cong Jx$ and $Z(N) = 0$, we infer that $Z(Jx) = 0$, whence $Jx \cap S_i = 0$. [Let $Jx \cap S_i \neq 0$. $Z(Jx \cap S_i) = 0 \Rightarrow \forall I \leq_{ie} N$, $I(Jx \cap S_i) \neq 0 \Rightarrow SocN(Jx \cap S_i) \neq 0$. But $(Jx \cap S_i) \subseteq E(_N S_i)$ and $SocN.E(_N S_i) = 0$, a contradiction]. This implies that $Jx = 0$. So $J \subseteq Ann_N(x)$, a contradiction. Now $E_{(N/Soc(N)} S_i) = \{x \in E(_N S_i) | Soc(N)x = 0\} = E(_N S_i)$ is injective as N-group. By given condition $\oplus_{i \in I} E_i$ is injective as an N-group and hence injective as $N/Soc(N)$-group. This implies that $N/Soc(N)$ is weakly Noetherian by [7, Proposition 2.8]. □

For a distributively generated near-ring we get the following:

Definition 3.4 [Pilz] The Jacobson-radical of N-group E is the intersection of maximal ideals of E which is maximal as N-subgroup. We denote it by $J_2(E)$.

Note3.1[Pilz]:The Jacobson-radical, $J_2(E)$ of N-group E contains all nilpotent N-subgroups of E.

Lemma 3.1 *Let N be a GV- near-ring, then* $Z(E) \cap J_2(E) = 0$, *for every N-group E.*

Proof If $Z(E) = 0$, we are done. Otherwise let $(0 \neq) x \in Z(E)$. By Zorn's lemma , the set of all ideals M of E with $x \notin M$, has a maximal member L. The quotient N-group $S = (Nx + L)/L$ is simple and singular, therefore E-injective.

$[Z((Nx + L)/L) = \{\bar{x} \in ((Nx + L)/L) | \; I\bar{x} = (\bar{0})$ for essential N-subgroup I of $N\}$. Let $\bar{y} \in (Nx + L)/L$ such that $\bar{y} = nx + l + L$. Now for essential N-subgroup I in N, $I\bar{y} = \{n^{/}\bar{y} | \; n^{/} \in I\} = \{(\sum_{(i=1)}^{k} s_i))(nx + L) | \; n^{/} = (\sum_{(i=1)}^{k} s_i) \in I\} = \{s_1(nx + L) + s_2(nx + L) + \cdots + s_k(nx + L) | \; n^{/} \in I\} = \{s_1 nx + L + s_2 nx + L + \cdots + s_k nx + L | \; n^{/} \in I\} = \{(s_1 nx + s_2 nx + \cdots + s_k nx) + L | \; n^{/} \in I\}$ [since $s_i \in I$ and $nx \in Z(E)] = \{L\} = \bar{0}$. So $\bar{y} \in Z((Nx + L)/L)]$

This means that the natural map of Nx onto S extends to all of E. The kernel of this extension map is a maximal ideal of E which does not contain x. Whence x can not be in $J_2 E$). So $Z(E) \cap J_2(E) = 0$. □

Definition 3.5 If $B \subseteq E$, then the annihilator of B in N is defined as the set $\{n \in N | nx = 0, \forall x \in B\}$ and is denoted by $Ann_N(B)$, which is a left ideal of N.

Theorem 3.1 *If N is a GV near-ring with A.C.C. on essential ideals and if finite intersection of essential N-subgroups of N is distributively generated, then* $Z(N) = 0$. *In particular, if N is* $S^3 I$ *near-ring with unity then it is non-singular.*

Proof Let $x \in Z(N)$. Then $Ann_N(x) \subseteq Ann_N(x^2) \subseteq \cdots$ is an ascending chain of essential left ideals in N, since $Ann_N(x) \leq_e N$. So for some $t \in I^+$, $Ann_N(x^{t+1}) \leq_e N$ by Proposition 1.2. We claim $x^t = 0$. Suppose $x^t \neq 0$. Then we get $Ann_N(x^{t+1}) \cap Nx^t \neq 0$. As N has A.C.C. on essential left ideals $\exists t \in I^+$ such that $Ann_N(x^t) = Ann_N(x^{t+1})$, whence we get $Ann_N(x^{t+k}) = Ann_N(x^t)$ for all $k \in I^+$. Let $y = nx^t (\neq 0) \in Ann_N(x^{t+1}) \cap Nx^t$ for $n \in N$. Now $y \in Ann_N(x^t) \Rightarrow yx^t = 0 \Rightarrow nx^{2t} = 0 \Rightarrow n \in Ann_N(x^{2t}) = Ann_N(x^t) \Rightarrow y = nx^t = 0$, a contradiction. i.e. $y \in Ann_N(x^{t+1}) \Rightarrow y \notin Ann_N(x^t) \Rightarrow Ann_N(x^t) \neq Ann_N(x^{t+1})$, a contradiction. Thus $Z(N)$ contains nilpotent elements. As finite intersection of essential N-subgroups of N is distributively generated, $Z(N)$ is N-subgroup of N [by Proposition 1.1]. So $J_2(N)$ contains $Z(N)$. By Lemma 3.1, $Z(N) = 0$. For $S^3 I$ near-ring N, $N/Soc(N)$ is weakly Noetherian by Proposition 2.3. Again by [7, Proposition 2.4], (considering N as N-group) it follows that N has acc on essential ideals when we get N is non singular. □

Theorem 3.2 *If* $\{N\bar{e}\}_{(\bar{e}\in N/Soc(N))}$ *is an independent family of normal N-subgroups of* $N/Soc(N)$*-group E, direct sum of E-injective* $N/Soc(N)$*-groups is abelian N-group, then* N/I *is weakly Noetherian* V_c *N-group for every essential ideal I of N implies* $N/(Soc(N))$ *is weakly Noetherian* V_c *near-ring.*

Proof N/I is weakly Noetherian for every essential ideal I of N implies $N/(Soc(N))$ is weakly Noetherian as [7, Proposition 2.3]. Let L be a strictly semi-simple $N/Soc(N)$-group.Then as N dgnr, L is a semi-simple $N/Soc(N)$-group. $I/Soc(N)$ is an ideal of $N/Soc(N)$ and $f : (I/Soc(N)) \to L$ is a non-zero N-homomorphism. Let $Kerf = (K/Soc(N))$. Now K is essential in N. For if $K \cap J = 0$ for some non-zero ideal J of N then $J \cong (J + K)/K$ and since the latter is isomorphic to an ideal of L, it follows that for some ideal $I_1 \neq 0$ and contained in J that $I_1 \subseteq L$, hence $I_1 \subseteq Soc(N) \subseteq K$, a contradiction. Thus N/K is a weakly Noetherian $V_c N$-group. If $N \to N/Soc(N)$ is canonical quotient map, then $(N/Soc(N))/(K/Soc(N))$ is a weakly Noetherian V_c N-group. Proposition 2.1, yields a map of $N/(Soc(N))$ into L. So, L is $N/(Soc(N))$-injective.Thus by Corollary 2.1, $N/(Soc(N))$ is weakly Noetherian V_c near-ring. □

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