Advances in Algebra.
ISSN 0973-6964 Volume 5, Number 1 (2012), pp. 31-41

http://www.ripublication.com/aa.htm

Super–honest N-subgroups

Helen K. Saikia[1] and Navalakhi Hazarika[2]

[1]Department of Mathematics, Gauhati University,
Guwahati–781014, Assam, India
E-mail: hsaikia@yahoo.com
[2]Department of Mathematics,
Royal School of Engineering and Technology,
Guwahati-781035, Assam, India
E-mail: navalakhmi@gmail.com

Abstract

Extending the notion of super honesty in modules to near-ring groups, super-honest N-subgroups are defined. Various characteristics of these N-subgroups are investigated. The necessary and sufficient conditions for super honest N-subgroups are also established. The torsion and closure of a quasi injective N-group E with respect to a class of essential N-subgroups exhibit the super honesty character of certain N-subgroups of E.

Keywords: near-ring groups, super honest N-subgroups, closed N-subgroups.

Mathematics subject classification(AMS) 2010: 16Y30

Introduction

The theory of honest subgroups was developed by Abian and Rinehart in [1]. The concepts of isolated submodules, honest submodules are studied by Fay and Joebert, Jara in [3, 7]. For a skew field, the notions of isolated submodules and honest submodules coincide. The honest submodules lead to a new characterization of ore domain. Moreover following the theory developed by Fay and Joebert, Jara obatained the characterizations of rings of quotients in terms of honest operator. The concept of super honest submodules was introduced by Joubert and Schoeman [8]. Super honest submodules of quasi injective modules are studied by Cheng [2].

In this paper we attempt to extend the notion of super-honesty in modules to near-ring groups. We define super-honest N-subgroups and investigate various characteristics of these N- subgroups. Necessary and sufficient conditions for these N-subgroups are established. The torsion and closure of a quasi injective N-group E with respect to a class of essential N-subgroups lead to the super honesty character of certain N-subgroups.

Definitions and Notations

All basic concepts used in this paper are available in Pilz [8]. Throughout the paper N will mean a zero symmetric right near-ring with unity 1 and E a left N- group. Also if A is a subnear-ring of N and A is distributively generated then we assume for $a(x_1+x_2)=\sum_{i=1}^{k} s_i(x_1+x_2) =\sum_{i=1}^{k}(s_i x_1+s_i x_2)$, where $a =\sum_{i=1}^{k} s_i \in A$, s_i's are distributive and $x_1,x_2 \in E$.

If H_1, H_2 are two N-subgroups (ideals) of E such that $H_1 \subseteq H_2$ then H_1 is *essential* in H_2 when any non-zero N-subgroup (ideal) C of E contained in H_2 has a nonzero intersection with H_1. In such cases H_2 is an *essential extension* of H_1. If H_1 is essential in E (when $H_2 = E$) then we say that H_1 is an *essential N-subgroup (ideal)* of E.

If H_1, H_2 are two N-subgroups of E such that $H_1 \subseteq H_2$ then H_1 is weakly *essential* in H_2 when any non-zero ideal C of E contained in H_2 has a nonzero intersection with H_1.

It is easy to note that if A, B, C are three N-subgroups (ideals) of E such that $A \subseteq B \subseteq C$ then A is essential (weakly essential)in C if and only if A is essential (weakly essential) in B and B is essential (weakly essential) in C.

An N-subgroup (ideal) I of E is said to be a essentially closed N-subgroup (ideal) of E if I has no proper essential extension in E. Similarly we can define weakly essentially closed N-subgroup(ideal) of E.

An N-subgroup M of E is a *complement* of some N-subgroup C of E if M is maximal among the set of all N-subgroups D of E such that $D \cap C=0$. For every N-subgroup M of E there exists a complement of M and we denote it by M^c

Let χ be a non empty set of N-subgroups of N. Let $K \subseteq E$ be an N-subgroup of E. We say K is *χ -closed N- subgroup* or K is χ-closed in E, if for any $I \in \chi$ and any $x \in E$, if $Ix \subseteq K$, then $x \in K$.

Let χ be a non empty set of N-subgroups of N such that $0 \notin \chi$. Let K be an N-subgroup of E. We say K is *χ-honest N-subgroup* or K is *χ -honest in E*, if for any $I \in \chi$ and any $x \in E$, if

$Ix(\neq 0) \subseteq K$, then $x \in K$.

So if K is χ-closed in E, then K is χ-honest in E.

We define the χ -torsion of E as the subset

$T_{\chi}(E) = \{x \in E \mid \text{there is } I \in \chi, \text{ such that } Ix = 0\}$

If $T_{\chi}(E) = 0$, E is χ -torsion free and E is χ -torsion if $T_{\chi}(E) = E$.

Let $M \subseteq E$ be an N-subgroup of E, we define the χ -closure of M in E as

$Cl_{\chi}^{E}(M)= \{x \in E : Ix \subseteq M, \text{ for some } I \in \chi\}$.

We have $T_{\chi}(E) = Cl_{\chi}^{E}(0)$.

If B is any non empty subset of E (or N), we define the set $(B:a)=\{ n\in N \mid na\in B\}$ **.** If B is an (left) N-subgroup of E(or N) then (B:a) is a left N-subgroup of N.

χ is *left closed* if for any $n \in N$ and any $I \in \chi$, there is $J\in \chi$ such that $J n\in I$

This means for any element $n\in N$ and any N-subgroup $I \in \chi$ we have $(I: n) \in \chi$.

If every proper essential N-subgroup of N is distributively generated and then as in [10], $Cl_{\chi}(M)$ is an N- subgroup of E. If $Cl_{\chi}(M) = M$ then M is called a *closed* N-subgroup of E.

We define *the set of torsion elements* of E as $T_N(E)=\{e \in E: ne = 0, \text{ for some } n(\neq 0) \in N\}$

It is clear that for each N-group E $T_{\chi}(E) \subseteq T_N(E)$ It is seen that $T_{\chi}(E)$ is an invariant subset of E.

An N-group E is called a *quasi-injective N-group* if every N-homomorphism of any N-subgroup A of E into E can be extended to an N- homomorphism of E into E.

An N-subgroup (ideal) M is super-honest in E means $x\in E \setminus M$ for $n\in N$, $nx\in M$ implies $n = 0$. If B is a left N-subgroup (ideal) of N then B is called a *super-honest N-subgroup (ideal)* of N if B is super-honest N-subgroup (ideal) of N considered N as left N-group ${}_N N$.

It is clear that N-group E is a super-honest N-subgroup of E itself. Again every super-honest N-subgroup contains $T_N(E)$. If E is a torsion N-group, then E is the only super-honest N-subgroup of E. N-group E is called *strictly uniform* if intersection of two non-zero N-subgroups of E is non-zero.

If A and B are N-subgroups of E then we say $A\beta B$ if and only if $A \cap B$ is essential in A and $A \cap B$ is essential in B. This is equivalent to $A \cap X =0$ if and only if $B \cap X = 0$, for any N-subgroup X of E**.**

Throughout the remaining section we assume that every proper essential N-subgroup of N is distributively generated.

Preliminaries

Here we prove some preliminary results needed for the sequel

Lemma 2.1: Let M be an N-subgroup of E. If M is a complement N-subgroup of some N-subgroup of E, then M is an essentially closed N-subgroup of E.

Proof: Let M be a complement N-subgroup of an N-subgroup C of E. If there exists an N-subgroup D of E such that $M \subset D$ and M is an essential N-subgroup of D, then $D \cap C$ is a non zero N- subgroup of D. But $(D\cap C) \cap M \subset C \cap M = 0$, contrary to the fact that M is an essential N-subgroup of D. So M is an essentially closed N-subgroup of E.

Lemma 2.2: If M is a weakly essentially closed N-subgroup of E then M is a complement N-subgroup of N-subgroup M^c of E.

Proof: Let there exist an N-subgroup D of E such that $D \supset M$ and $D \cap M^c = 0$.By given condition M is not weakly essential N-subgroup of D. So there exists a non zero ideal D' of D such that $D' \cap M = 0$.Then $D \cap (M^c + D') = D' + (M^c \cap D) = D'$. Now $M \cap D' = M \cap D \cap (M^c + D') \Rightarrow 0 = M \cap (M^c + D')$. So, $M \cap (M^c + D') = 0$,which contradicts to the fact that M^c is a complement N-subgroup of M in E. So M is a complement N-subgroup of M^c in E.

If every weakly essential N-subgroup is essential then from lemma 2.1 and lemma 2.2 we get the following:

Lemma 2.3: If M is an N-subgroup of E and M^c is a complement N-subgroup of B in E, then the following statements are equivalent.

i. M is essentially closed N-subgroup of E.
ii. M is a complement N- subgroup of M^c in E
iii. M is a complement N- subgroup of some N-subgroup of E.

Lemma 2.4: If M is an N-subgroup of E such that $T_\chi(E) \subseteq M$, then M is an essential N-subgroup of $Cl_\chi(M)$.

Proof: Let A be N-subgroup of $Cl_\chi(M)$.We assume $A \cap M = 0$. Let $x \in A$, then $x \in Cl_\chi(M)$ implies $Ix \subseteq M$, for some I of χ . Also $Ix \subseteq A$ implies $Ix \subseteq M \cap A = 0$. This gives $Ix = 0$. Thus $x \in T_\chi(E) \subseteq M$. So, $x \in A \cap M$. implies $x = 0$.This gives $A = 0$.Thus M is an essential N-subgroup of $Cl_\chi(M)$.

If M is an essentially closed N-subgroup of E such that $T_\chi(E) \subseteq M$, then by lemma 2.4, we get $M = Cl_\chi(M)$. On the other hand if M is an χ-closed N-subgroup of E then M is essentially closed N-subgroup of E and $T_\chi(E) \subseteq M$. If M is an essential N-subgroup of C where C is N-subgroup of E then for each $x \in C$, $(M: x)$ is an essential left N-subgroup of N. So $x \in Cl_\chi(M) = M$. Hence $C = M$. Again $T_\chi(E) = Cl_\chi(0) \subseteq Cl_\chi(M) = M$ Hence we get the following lemma:

Lemma 2.5 : Let M be an N-subgroup of E. Then M is essentially closed N-subgroup of E satisfying $T_\chi(E) \subseteq M$ if and only if $Cl_\chi(M) = M$.

Note: If every weakly essential N-subgroup is essential in E, then the following are equivalent

1. M is essentially closed and $T_\chi(E) \subseteq M$,
2. M is a complement N-subgroup of some ideal M^c and $T_\chi(E) \subseteq M$,
3. M is a complement N-subgroup of some ideal of E and $T_\chi(E) \subseteq M$,
4. M is χ-closed.

Using the equivalent conditions of the above note and following similar method as in [4] we get the following

Lemma 2.6: If E is a quasi-injective N-group and every weakly essential N-subgroup is essential in E then every χ- closed N-subgroup (ideal) M is a semi-direct (direct) summand of E.

Lemma 2.7: If M and P are N-subgroups of E then $Cl_{\chi}(M)\beta(M + Cl_{\chi}(0))$ and if Pβ M then P $\subseteq Cl_{\chi}$ (M)

Using the above lemma in a similar way as in [5] we get

Lemma 2.8: If every proper essential left N-subgroups are distributively generated and χ is left closed, then $Cl_{\chi}\ T_{\chi}$(E) is χ –closed N-subgroup of E.

Super honest N-subgroups

In this section we investigate various characteristics of super honest N-subgroups.

Lemma 3.1: Let M be an N-subgroup (ideal) of E. Then M is a super-honest N-subgroup (ideal) of E if and only if for each a$\in$E, (M: a) is a super-honest left N-subgroup (ideal) of N.

Proof: Let M be a super-honest N-subgroup of E. If n $\in$ N is such that n $\notin$(M: a) with $n^{/}$ n $\in$ (M:a) for some $n^{/}$ $\in$ N then $n^{/}$ n a $\in$ M. Since M is a super-honest N-subgroup of E, we have $n^{/}$ = 0. Hence (M:a) is a super-honest N-subgroup of N.

Let (M: a) be a super-honest left N-subgroup of N. If a $\in$ E is such that a $\notin$ M with na $\in$ M for some n $\in$ N then 1 $\notin$ (M : a). This implies n.1 = n $\in$ (M : a). Since (M:a) is a super-honest left N-subgroup of N, so n= 0. Hence M is a super-honest N-subgroup of E.

Lemma 3.2: Let M be an N-subgroup (ideal) of E. Then M is a super-honest N-subgroup (ideal) of E if and only if (M: a) = 0 for each a$\in$E– M.

Proof: Let M be a super-honest N-subgroup (ideal) in E. Then for each x$\in$E– M, n$\in$N, nx$\in$ M implies n=0.this gives (M: x) =0, for each x$\in$ E– M.

On the other hand, let (M: x) = 0 for each x$\in$E– M. If for some n$\in$N, nx$\in$ M then n$\in$(M: x). This implies n = 0. Thus M is super-honest in E.

Lemma 3.3: {0} is a super-honest left N-subgroup of N if and only if N has no left zero divisors.

Proof: If N has no left zero divisors then it is obvious that {0} is a super-honest left N-subgroup (ideal) 0f N.

Let {0} be a super-honest left N-subgroup of N. If $n(\neq 0)\in N$ satisfying $n'n = 0$ for some $n' \in N$ then $n' = 0$. Thus N has no left zero divisors.

Lemma 3.4: If E has a proper super-honest N-subgroup (ideal) M, then N has no left zero divisors.

Proof: M is super-honest N-subgroup (ideal) in E

$\Leftrightarrow$ (M:a) = 0 for $a\in E-M$, by lemma 3.2

$\Leftrightarrow$ 0 = (M:a) is super-honest N-subgroup (ideal) of N for $a\in E-M$,by lemma 3.1

$\Leftrightarrow$ N has no left zero divisors.

Theorem 3.1: If M is an ideal of E then M is a super-honest in E if and only if M is essentially closed in E , $T_\chi(E) \subseteq M$ and $T_\chi(E/M) \supseteq T_N(E/M)$.

Proof: Let C be an N-subgroup of E such that M is an essential N-subgroup of C. Then there exists $a\in C-M$ such that Na is a non zero N-subgroup of C. Since $Na\cap M\neq 0$, so $(M:a) \neq 0$, this contradicts that M is a super-honest N-subgroup of E. Thus M = C. This implies M is essentially closed.

Again $a \in T_N(E) \Rightarrow (0:a)\neq 0 \Rightarrow x(\neq 0) \in (0:a)$ so xa = 0.

If $a\in M$ then it is done. If $a\notin M$ then. $a \in E\setminus M$. This gives x = 0, as M is super-honest in E.

Hence it contradicts the fact that $x \neq 0$. So $a\in M$ .Thus $T_N(E) \subseteq M$. And so $T_\chi(E) \subseteq M$, because $T_\chi(E) \subseteq T_N(E)$.

Let $\bar{a} \in T_N(E/M)$, $a\notin M \Rightarrow (\bar{0}:\bar{a}) \neq 0 \Rightarrow \exists x(\neq 0)$ such that $x\in(\bar{0}:\bar{a}) \Rightarrow x\bar{a} = \bar{0} \Rightarrow xa\in M$ where $a\notin M \Rightarrow x=0$, as M is super-honest in E. Therefore $\forall\, a \in E\setminus M$, $(\bar{0}:\bar{a}) = 0$ and so $T_N(E/M) = \bar{0} = M$.

$T_\chi(E/M) \supseteq T_N(E/M)$ holds trivially.

Conversely let $a \in E\setminus M$ with $na \in M$ for some $n\in N$. If $n \neq 0$, then $\bar{a} = a+M \in T_N(E/M)$. as $n\bar{a} = na+M \in M=\bar{0} \Rightarrow n\in(\bar{0}:\bar{a}) \Rightarrow \bar{a} \in T_N(E/M)$. So $\bar{a} \in T_\chi(E/M)$. Thus $(\bar{0}:\bar{a}) = (M:a)$ belongs to χ.

So $a \in Cl_\chi(M)$. By lemma 2.5, $Cl_\chi(M) = M$, a contradiction. Thus n = o, giving thereby the super honesty character of M.

Using lemma 2.5, we get the following:

Corollary 3.1: If M is an ideal of E such that M is super-honest in E then M is χ-honest.

Theorem 3.2: Let M be an ideal of E. If M is χ-closed in E and $T_\chi(E/M) \supseteq T_N(E/M)$, then

M is a complement N-subgroup of some torsion-free N-subgroup of E.

Proof: M is χ-closed N-subgroup of E $\Longrightarrow$ M essentially closed N-subgroup of E. So by Lemma 2.2, M is complement of some ideal M^c in E. It remains to show M^c is a torsion-free N-subgroup of E. Suppose there exists $0 \neq a \in M^c$ such that na=0 for some $0 \neq n \in N$. Then $\bar{a}$ = a+M $\in T_N$ (E /M) and so $\bar{a} \in T_\chi$(E/M), which implies that $(\bar{0}: \bar{a})$ = (M:a) belongs to χ.

Thus a $\in Cl_\chi$(M) = M. But then a $\in M \cap M^c = 0$, contradiction to $0 \neq$ a. therefore M^c is torsion-free N-subgroup of E.

Corollary 3.2 : If M is an ideal of E and every weakly essential N-subgroup of E is essential then the following statements are equivalent

1. M is super-honest in E
2. M is complement N-subgroup of some torsion-free N-subgroup of E and T_χ(E) $\subseteq$ M and T_χ(E/M) $\supseteq T_N$(E/M)
3. M is $\chi -$ closed and T_χ(E/M) $\supseteq T_N$(E/M)
4. M is an essentially closed in E, T_χ(E) $\subseteq$ M and T_χ(E/M) $\supseteq T_N$(E/M).

It is obvious that T_χ(E/M) $\subseteq T_N$(E/M).

Thus T_χ(E/M) $\supseteq T_N$(E/M) if and only if T_χ(E/M) = T_N(E/M).

Also T_χ(E/M) $\supseteq T_N$(E/M) implies (M:a) is an essential N-subgroup of N if (M:a) $\neq$ 0 for some a $\in$ E. On the other hand, if (M:a) is an essential N-subgroup of N for some a $\in$ E then T_χ(E/M) $\supseteq T_N$(E/M) .

Hence we get

Corollary 3.3: If $_N$N is strictly uniform and if M is an ideal of E and every weakly essential N-subgroup of E is essential then the following are equivalent.

1. M is super-honest in E
2. M is complement N-subgroup of some torsion-free N-subgroup of E and T_χ(E) $\subseteq$ M
3. M is $\chi -$ closed
4. M is an essentially closed in E, T_χ(E) $\subseteq$ M.

The following can be proved easily:

Theorem 3.3: The intersection of super-honest N-subgroups (ideals) of E is also super-honest N-subgroup (ideal) of E.

From above result we see the existence of smallest super-honest N-subgroup (ideal) of E

The intersection of all super-honest N-subgroups (ideals) of E is the smallest super-honest N-subgroup (ideal) of E. We denote it by S. If $S \subsetneq E$, then E has proper super-honest N-subgroups, otherwise E is the only super-honest N-subgroup of E itself.

Theorem 3.4: If E and E′ are N groups, f is an N-homomorphism from E to E′, then for each super-honest N-subgroup B′ of E′, $f^{-1}(B')$ is a super-honest N-subgroup of E.

Theorem 3.5: If S is the smallest super-honest N-subgroup of an N-group E, then for each N-endomorphism f of E, $f^{-1}(S) \supset S \supset f(S)$

Proof: Since $f^{-1}(S)$ is a super-honest N-subgroup of E, $f^{-1}(S) \supset S$. Hence $S \supset f(S)$.

As the smallest super-honest N-subgroup S of an N-group E, we know $S \supset Cl_{\chi}(D)$, where D is the N-subgroup of E generated by $T_N(E)$

Theorem 3. 6: If $_N N$ is a strictly uniform N-group and M is an ideal of E and every weakly essential N-subgroup of E is essential then every χ -closed N-subgroup of E is super-honest in E. In particular $Cl_{\chi}\ T_{\chi}(E)$ is a super-honest N-subgroup of E.

Proof: If $x \in T_N(E)$ then $(0 : x) \neq 0$. Since, $_N N$ is uniform so this gives $(0 : x) \cap A \neq 0$, for every non-zero N-subgroup A of $_N N$. This implies $x \in T_{\chi}(E)$. Thus $T_{\chi}(E) = T_N(E)$

Then every χ-closed N-subgroup of E is super-honest in E by corollary 3.3. In particular $Cl_{\chi}\ T_{\chi}(E)$ is closed N-subgroup of E, hence super-honest in E.

Theorem 3.7: If E has no proper super-honest N-subgroup, S' is the smallest super-honest N-subgroup of the N-group E', then for each N-homomorphism f from E into E', $f(E) \subset S'$.

Proof: By theorem 3.4, $f^{-1}(S')$ is a super-honest N-subgroup of E. But E has no proper super-honest N-subgroup and so $f^{-1}(S') = E$. Then $f(E) \subset S'$.

Corollary 3.4: If the N-group E has no proper super-honest N-subgroup, E' is a torsion free N-group then only N-homomorphism from E into E' is the zero homomorphism.

Proof: Since E' is a torsion free, 0 is the smallest super-honest N-subgroup of E'.

Theorem 3.8: If M is a χ- closed ideal of an quasi-injective N-group E and $T_{\chi}(E) \subseteq M$ then M is super-honest ideal of E.

Proof: Let $a \in E \setminus M$ with $na \in M$ for some $n \in N$. Since M is a χ-closed ideal of E, by lemma 2.6, $E = M + M^c$, where M^c is a complement N-subgroup of M in E. Then $a = m+m'$ for some $m \in M$ and $m' \in M^c$. Now $m' = -m + a$ implies $n(-m + a) = n(-m+a) - na + na = nm' \in M \cap M^c = 0$.But $0 \neq m'$, for otherwise $a = m \in M$ and $m' \notin T_N(E)$. Since $M \supset T_N(E)$ and $a \notin M$, so n=0.Hence M is super-honest in E.

Theorem 3.9: If E is a quasi-injective N-group and $Cl_\chi\, T_\chi(E)$ is a super-honest N-subgroup in E then $Cl_\chi\, T_\chi(E)$ is the smallest super-honest N-subgroup of E.

Proof: Since S contains $T_N(E)$, so it contains $T_\chi(E)$. But S is closed N-subgroup of E, therefore $S = Cl_\chi(S) \supset Cl_\chi(T_\chi(E))$. Also $Cl_\chi\, T_\chi(E) \supset S$, since S is the smallest super-honest N-subgroup of E. Hence $Cl_\chi\, T_\chi(E) = S$.

Theorem 3.10: If E is a quasi-injective N-group, $Cl_\chi\, T_\chi(E)$ is an ideal of E containing $T_N(E)$ then $Cl_\chi\, T_\chi(E)$ is super-honest in E .

Proof: Since $Cl_\chi\, T_\chi(E)$ is a χ-closed ideal of E and $Cl_\chi\, T_\chi(E) \supset T_N(E)$, $Cl_\chi\, T_\chi(E)$ is super-honest in E by theorem 3.9.

Theorem 3.11: Let E be a quasi-injective N-group. Then the smallest super-honest N-subgroup S of E is the closure of the N-subgroup D generated by $T_N(E)$.

Proof: Since every super-honest N-subgroup of E contains $T_N(E)$ (hence contains D) and is a closed N-subgroup of E. We have $P = Cl_\chi(S) \supseteq Cl_\chi(D)$.On the other hand since $T_\chi(E) \subseteq T_N(E) \subseteq D \subseteq Cl_\chi(D)$. By lemma 2.4, $Cl_\chi(D)$ is an essential N-subgroup of $Cl_\chi\, Cl_\chi(D)$ and D is an essential N-subgroup of $Cl_\chi(D)$. Therefore D is an essential N-subgroup of $Cl_\chi\, Cl_\chi(D)$ .Then $Cl_\chi\, Cl_\chi(D) \subseteq Cl_\chi(D)$[Lemma3.8] and so $Cl_\chi\, Cl_\chi(D) = Cl_\chi(D)$ is a χ- closed N- subgroup of E. Since $Cl_\chi(D) \supseteq T_N(E)$. By theorem 3.10, $Cl_\chi(D)$ is super-honest in E. Hence, $Cl_\chi(D) \supseteq S$. This implies $Cl_\chi(D) = S$.

Examples

In this section we consider some examples of the concepts defined in the previous sections.

4.1 Example: Here N = {0, 1, 2, 3, 4, 5}, N_2 = {0, 1}, N_3= {0,1,2} are near-rings under the operation '+' as addition module 6, modulo 2 , modulo 3 respectively and the multiplication '*' defined as

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	2	2	2	2	2
3	0	3	3	3	3	3
4	0	4	4	4	4	4
5	0	5	5	5	5	5

Then $N_2 \oplus N_3 \oplus N$ is an N-group and $N_2 \oplus N_3$ is a superhonest N-subgroup of $N_2 \oplus N_3 \oplus N$

4.2 Example : If N = { 0,1,2,3,4,5}, then N is a near-ring under the operation '+' as addition module 6 and the multiplication '*' defined as the following table:

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	2	2	2	2	2
3	0	3	3	3	3	3
4	0	4	4	4	4	4
5	0	5	5	5	5	5

Here {0} is a superhonest N-subgroup of $_NN$ which has no zero divisors.

The following example shows the existence of near-ring group in which every weakly essential N-subgroup is essential.

4.3 Example: If N= {0,1,2,3,4,5,6,7} then it is a near-ring under addition modulo 8 and multiplication defined as follows:

*	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

Clearly, every weakly essential N-subgroup of $_NN$ is essential.

References

[1] A. Abian and D Reinhart: Honest subgroups of abelian groups, Rend. Circ Math. Palermo 12, (1963), 353-356.

[2] Mau-Hei-Chang: Note on Superhonest modules, Chinese journal of mathematics, 14, (1986), 109-119.

[3] T H Fay and S V Joubert: Isolated submodules and skew fields, Applied Categorical Structures, 8,(2000), 317-326.

[4] Carl Faith and Yuzo Utumi, Quasi-injective modules and their endomorphism rings. Arch Math.15,(1964) 166-174.

[5] A. W Goldie, torsion free modules and rings, Journal of Algebra, 1964, 268-287

[6] M Harada: Note on Quasi injective Modules, Osaka J Math, 2, (1965)351-356.

[7] P. Jara: Honest submodules, Czechoslovak Mathematical Journal,57,(2007) 225-241.

[8] S. V. Joubert and M. J. Schoeman: Super honestry for Modules and Abelian groups, Chinese J. of Math., 13,(1985) 1-13

[9] G. Mason, G. and A. Oswald : Injective and projective near-ring modules,

[10] Compositio Mathematica, l 33, Fasc 1.,(1976) 43-54.

[11] A. Masum, H K Saikia and K.C. Chowdhury : On left Goldie near-rings and its parts having minimum conditions, Indian Journal of pure and Appl. Maths, , 25(11), (1994) 1155-1162

[12] G. Pilz : Near-rings, North Holland publishing Company, Amsterdam, 1983.