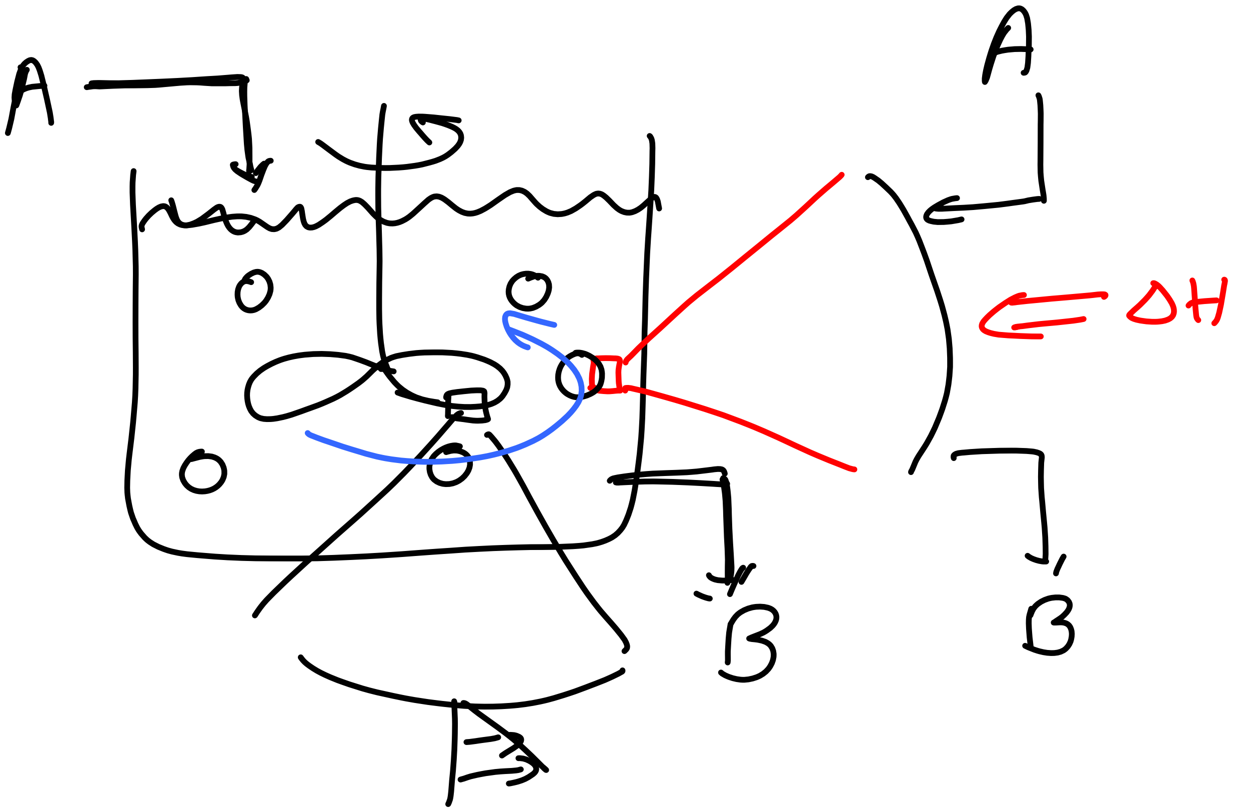
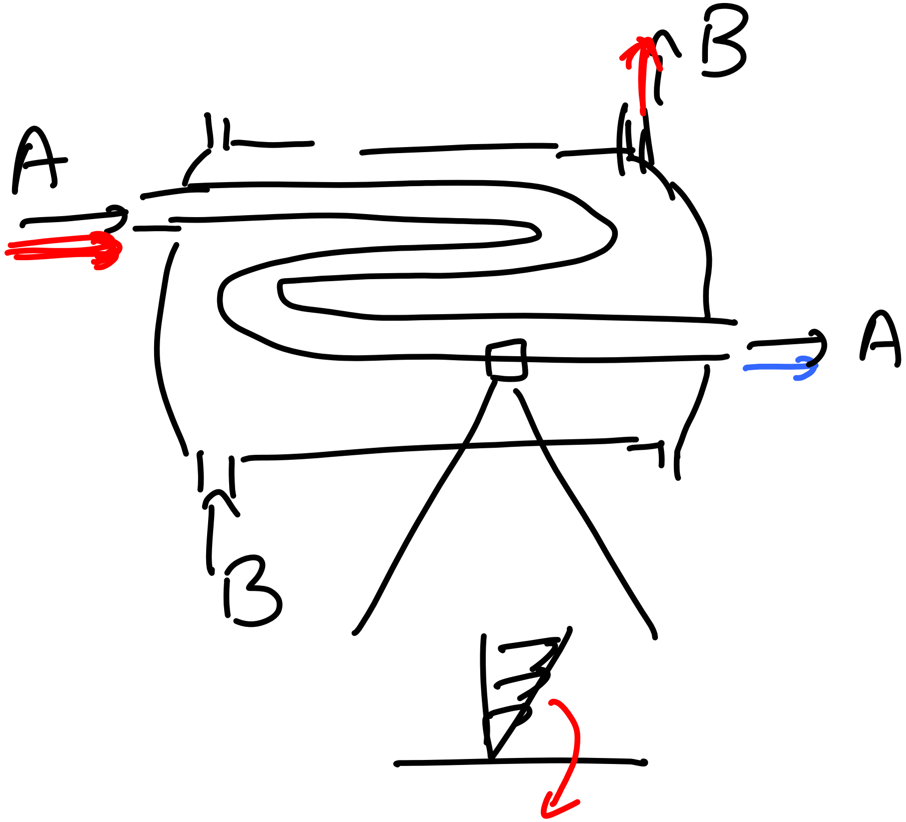
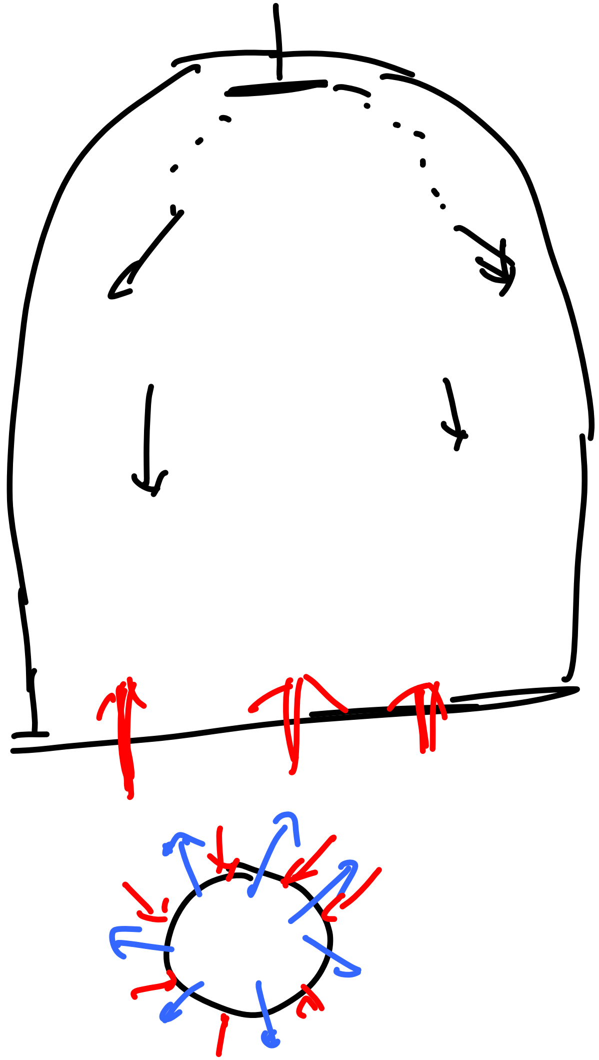
FUNDAMENTALS OF TRANSPORT PROCESSES - II

$$A \xrightarrow{\Delta H} B$$

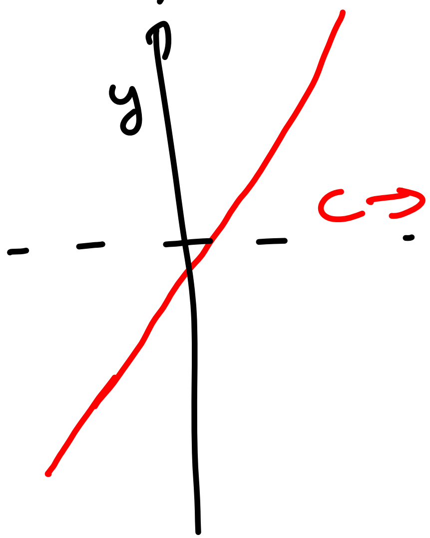


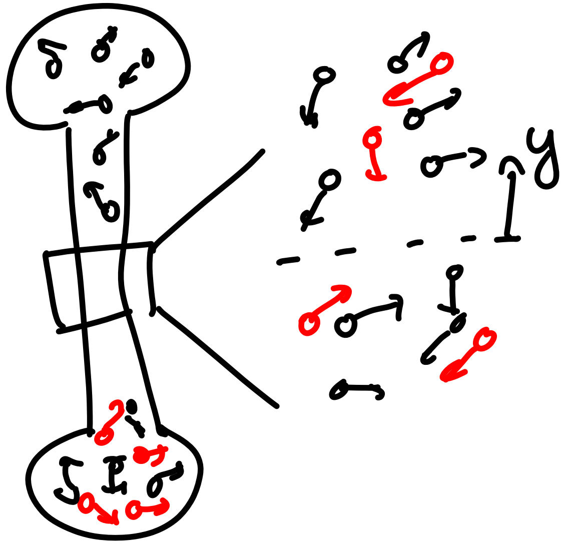

A
B
A
B



CONVECTION

Due to mean flow of fluid carrying mass/energy





DIFFUSION

Molecular motion

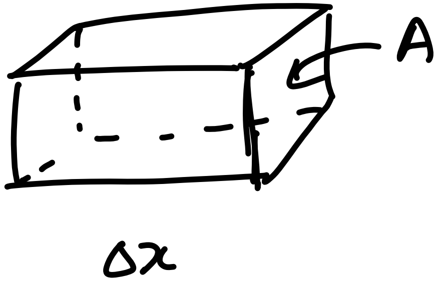
$$\frac{1}{2}mv^2 = \frac{3}{2}kT$$

$$j = \frac{\text{Mass transported}}{\text{Area time}}$$

$$j_x = -D \frac{\Delta c}{\Delta x}$$

$\longleftarrow \Delta c \longrightarrow$

$$\frac{\text{Mass transferred}}{\text{Area} \times \text{Time}} = -D \left[\frac{\text{Change in mass density}}{\text{Distance}} \right]$$

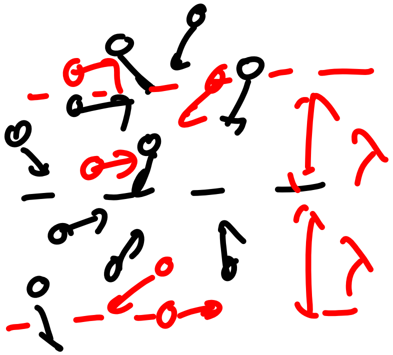


$$D = L^2 T^{-1}$$

$$\frac{\text{Heat transferred}}{\text{Area} \times \text{Time}} = -\alpha \left[\frac{\text{Change in energy density}}{\text{Distance}} \right]$$

$$\alpha = \left(\frac{k}{\rho C_p} \right)$$

$D \propto \boxed{v_{rms}\,\lambda}$

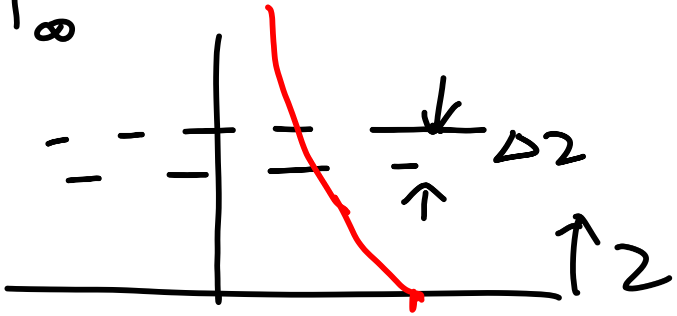


$\Delta c \propto \lambda \dfrac{\Delta c}{\Delta x}$

$$\begin{pmatrix}\text{Change in} \\ \text{mass/momentum/} \\ \text{energy} \\ \text{in time } \Delta t\end{pmatrix} = \begin{pmatrix}\text{Mass/momentum/} \\ \text{energy} \\ \text{IN}\end{pmatrix} - \begin{pmatrix}\text{Mass/momentum/} \\ \text{energy} \\ \text{OUT}\end{pmatrix}$$

$$+ \left(\text{Accumulation}\right)$$

$T = T_\infty$



At $t=0$ $T = T_0$

$$\frac{\Delta T}{\Delta t} = -\frac{\Delta q_z}{\Delta z}$$

Take limit $\Delta t \to 0$ & $\Delta z \to 0$

$$\frac{\partial T}{\partial t} = -\frac{\partial q_z}{\partial z}$$

$$q_z = -\alpha \frac{\Delta T}{\Delta z} = -\alpha \frac{\partial T}{\partial z}$$

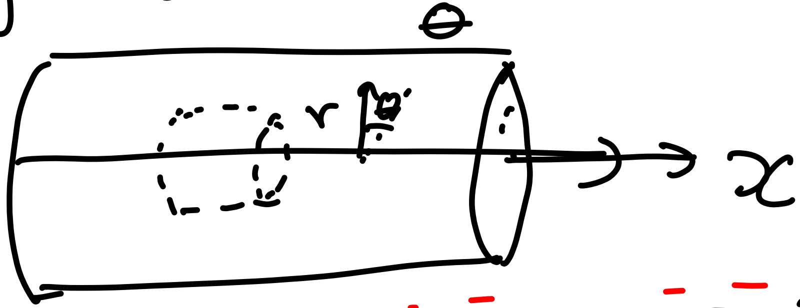
$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2}$$

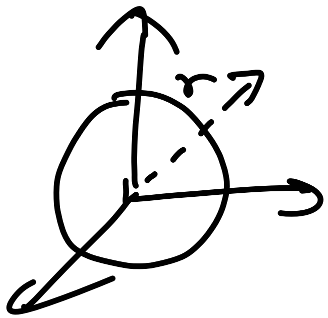
$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial z^2}$$

$$\frac{\partial T}{\partial t} = \lim_{\Delta t \to 0} \frac{T(z, t+\Delta t) - T(z,t)}{\Delta t}$$

$$\frac{\partial T}{\partial z} = \lim_{\Delta z \to 0} \frac{T(z+\Delta z, t) - T(z,t)}{\Delta z}$$

Cylindrical co-ordinate system



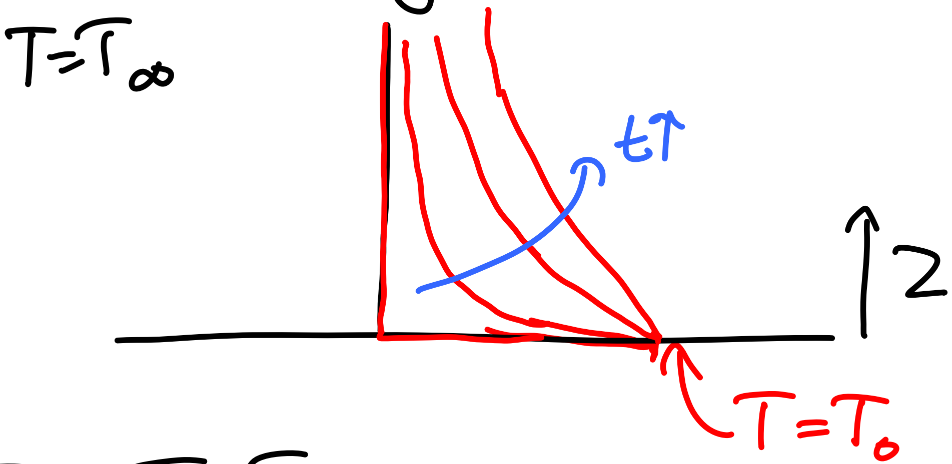


$$\frac{\partial T}{\partial t} = \alpha \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \right)$$

$$\frac{\partial T}{\partial t} = \alpha \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) \right]$$

$$\frac{\partial c}{\partial t} = D \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \right]$$

① Similarity solutions

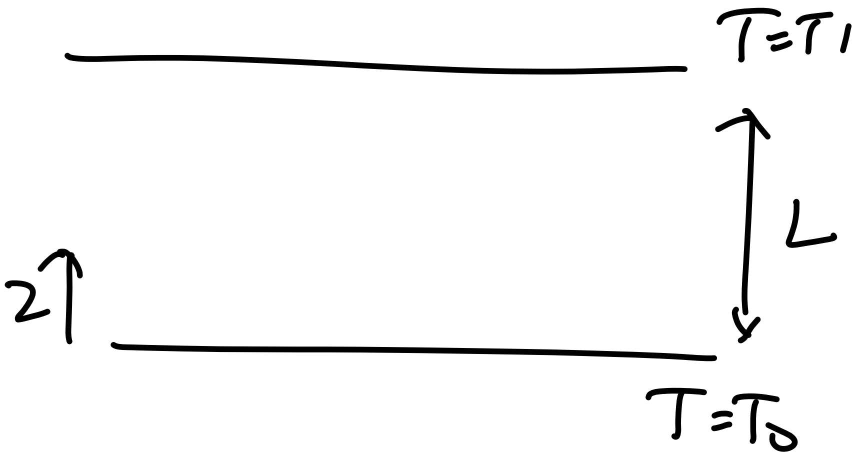


$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2}$$

$$T = \frac{T-T_\infty}{T_0-T_\infty}$$

$$\frac{T-T_\infty}{T_0-T_\infty} = 1 - \frac{1}{\sqrt{\pi}}\int_0^{z/\sqrt{\alpha t}} d\xi\, e^{-\xi^2/4}$$

$$\eta = \frac{z}{\sqrt{\alpha t}}$$



$$\frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial z^{*2}}$$

$$\frac{1}{F}\frac{dF}{dt^*} = \frac{1}{Z}\frac{d^2 Z}{dz^{*2}} = -(n\pi)^2$$

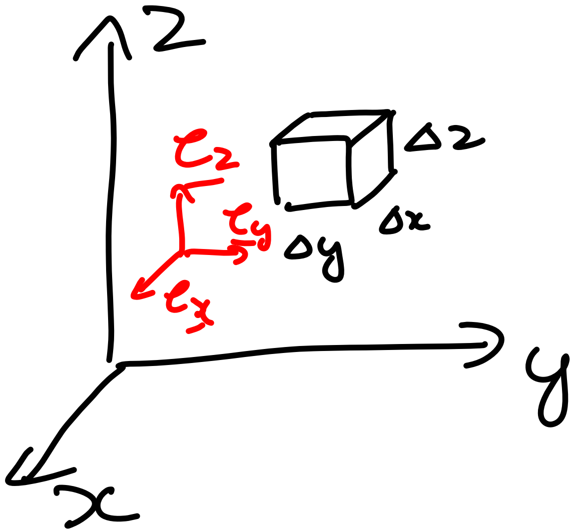
$$T^* = \sum_{n=1}^{\infty} C_n \sin(n\pi z^*) e^{-n^2\pi^2 t^*}$$

$$T^* = \frac{T - T_0}{T_1 - T_0} \qquad z^* = \frac{z}{L}$$

$$T^*(z^*, t^*) \qquad t^* = \frac{t\alpha}{L^2}$$

$$T^* = Z(z^*) F(t^*)$$

$$T^* = (1 - z^*) - \sum_n \left(\frac{2}{n\pi}\right) \sin(n\pi z^*) e^{-n^2\pi^2 t^*}$$

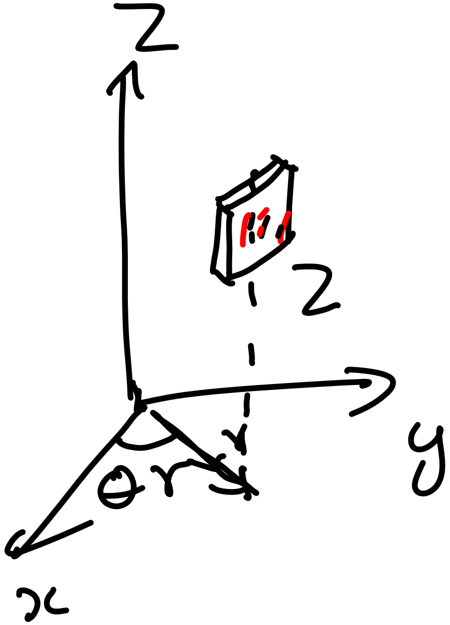


$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x}(u_x c) + \frac{\partial}{\partial y}(u_y c) + \frac{\partial}{\partial z}(u_z c)$$
$$= -\frac{\partial j_x}{\partial x} - \frac{\partial j_y}{\partial y} - \frac{\partial j_z}{\partial z}$$

$$\frac{\partial c}{\partial t} + \nabla \cdot (\underline{u} c) = -\nabla \cdot \underline{j}$$
$$= D\nabla^2 c$$

$$\underline{j} = j_x \underline{e_x} + j_y \underline{e_y} + j_z \underline{e_z}$$
$$\underline{u} = u_x \underline{e_x} + u_y \underline{e_y} + u_z \underline{e_z}$$
$$\nabla = \underline{e_x} \frac{\partial}{\partial x} + \underline{e_y} \frac{\partial}{\partial y} + \underline{e_z} \frac{\partial}{\partial z}$$

$$\underline{j} = -D \nabla \underline{c}$$
$$= -D\left(\underline{e_x} \frac{\partial c}{\partial x} + \underline{e_y} \frac{\partial c}{\partial y} + \underline{e_z} \frac{\partial c}{\partial z}\right)$$



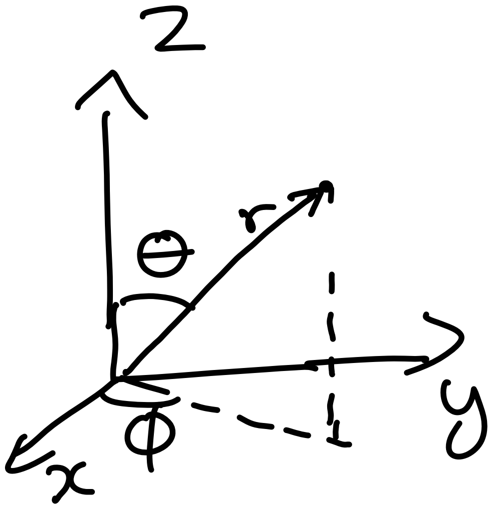
$$\frac{\partial c}{\partial t} + \nabla\cdot(\underline{u}c) = D\nabla^2 c$$

$$\nabla\cdot(\underline{u}c) = \frac{1}{r}\frac{\partial}{\partial r}\left(r u_r c\right) + \frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta c) + \frac{\partial}{\partial z}(u_z c)$$

$$\nabla^2 c = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 c}{\partial \theta^2} + \frac{\partial^2 c}{\partial z^2}$$

$$\nabla\cdot(\underline{u}c) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r c\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, u_\theta c\right)$$

$$+ \frac{1}{r^2}\frac{\partial^2 c}{\partial \phi^2}$$



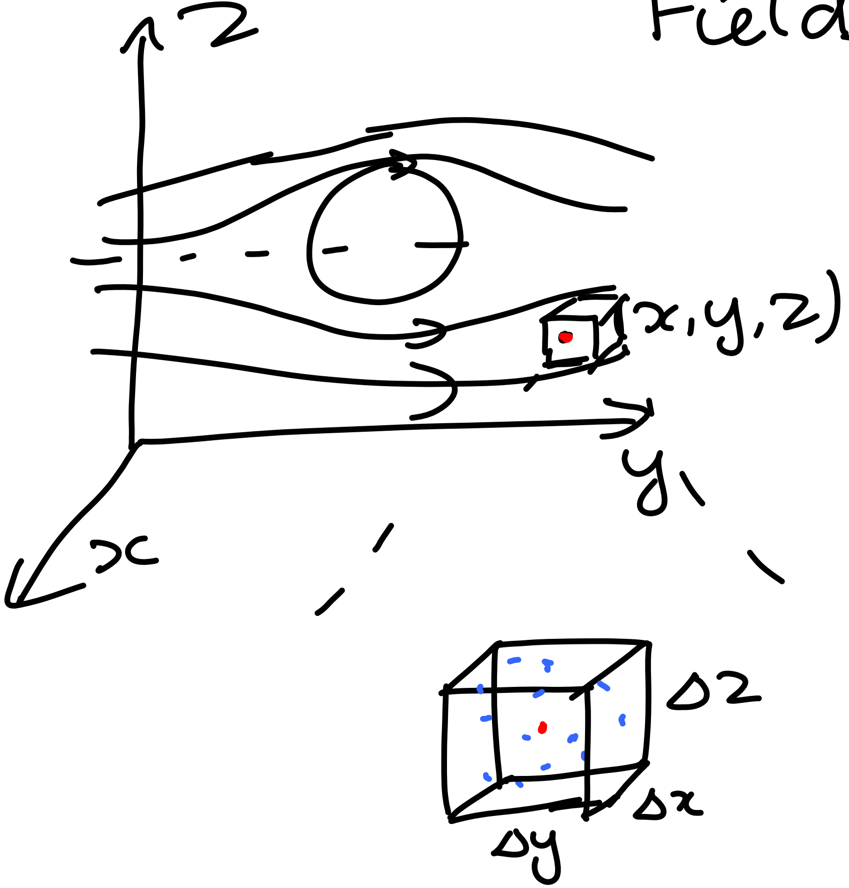
$$\nabla^2 c = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial c}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial c}{\partial \theta}\right)$$

$$+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2 c}{\partial \phi^2}$$

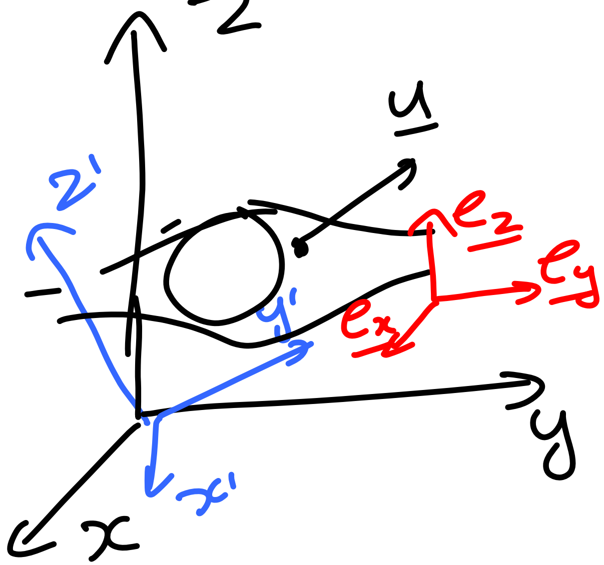
FLUID MECHANICS:

CONTINUUM APPROXIMATION

Fields Density $\rho = \lim_{\Delta V \to 0} \frac{m n}{\Delta V}$



$$\rho \underline{u} = \lim_{\Delta V \to 0} \frac{\sum_{i=1}^{n} m \underline{u}_i}{\Delta V}$$



$$\underline{u} = u_x \underline{e_x} + u_y \underline{e_y} + u_z \underline{e_z}$$
$$= u_x' \underline{e_x'} + u_y' \underline{e_y'} + u_z' \underline{e_z'}$$

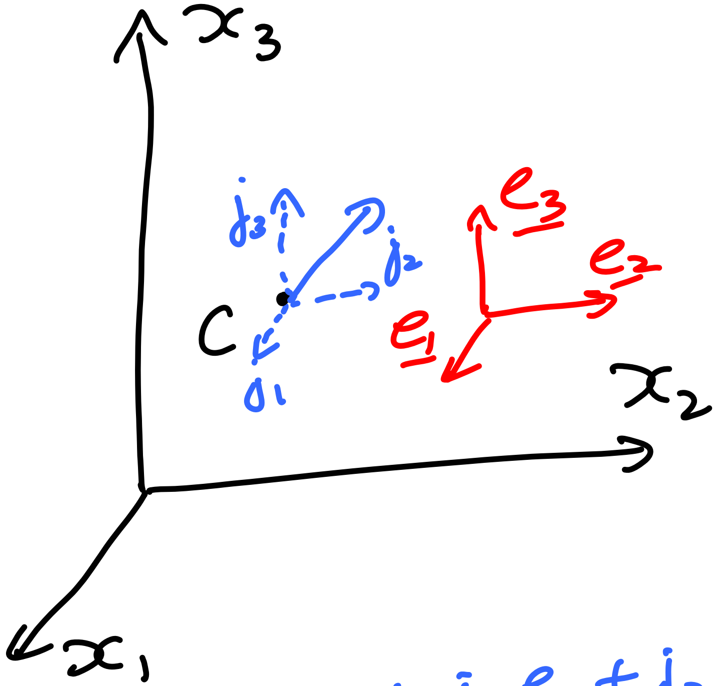
$$|\underline{u}| = \sqrt{u_x^2 + u_y^2 + u_z^2}$$
$$= \sqrt{u_x'^2 + u_y'^2 + u_z'^2}$$

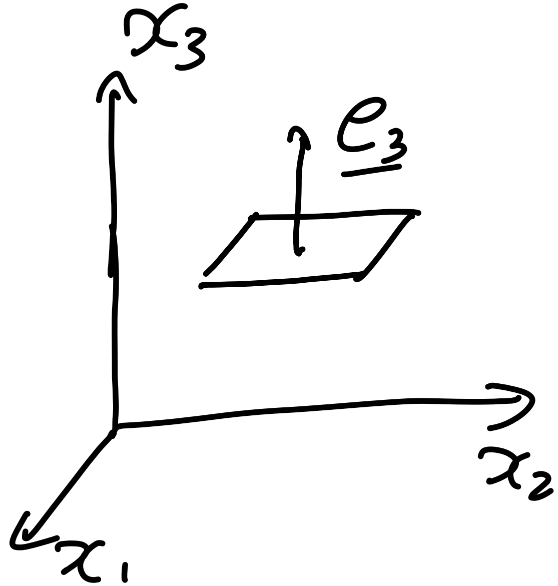
$$\underline{A} \cdot \underline{B} = A_x B_x + A_y B_y + A_z B_z$$
$$= A_x' B_x' + A_y' B_y' + A_z' B_z'$$

$$\underline{j} = -D \nabla c$$
$$\underline{q} = -k \nabla T$$

$$\underline{q} = q_x \underline{e_x} + q_y \underline{e_y} + q_z \underline{e_z}$$

$$\nabla c = \underline{e_x} \frac{\partial c}{\partial x} + \underline{e_y} \frac{\partial c}{\partial y} + \underline{e_z} \frac{\partial c}{\partial z}$$



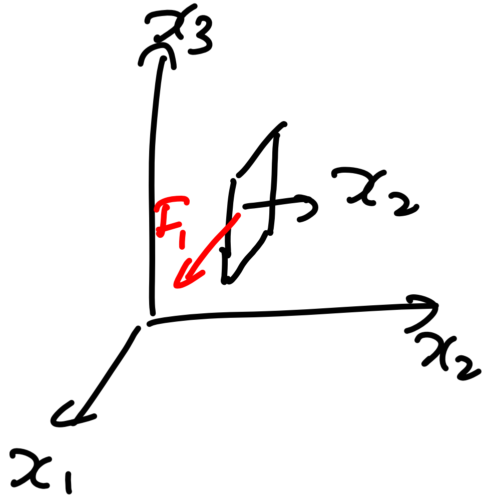


$$\underline{j} = j_1 \underline{e}_1 + j_2 \underline{e}_2 + j_3 \underline{e}_3$$

$\underline{\underline{\tau}}$ = Force / unit area at a surface with unit normal

$$= \tau_{11} \underline{e}_1 \underline{e}_1 + \tau_{12} \underline{e}_1 \underline{e}_2 + \tau_{13} \underline{e}_1 \underline{e}_3 + \cdots + \tau_{33} \underline{e}_3 \underline{e}_3$$

τ_{11} = Force / Area in x_1 direction acting at a surface with 'outward' unit normal in x_1 direction

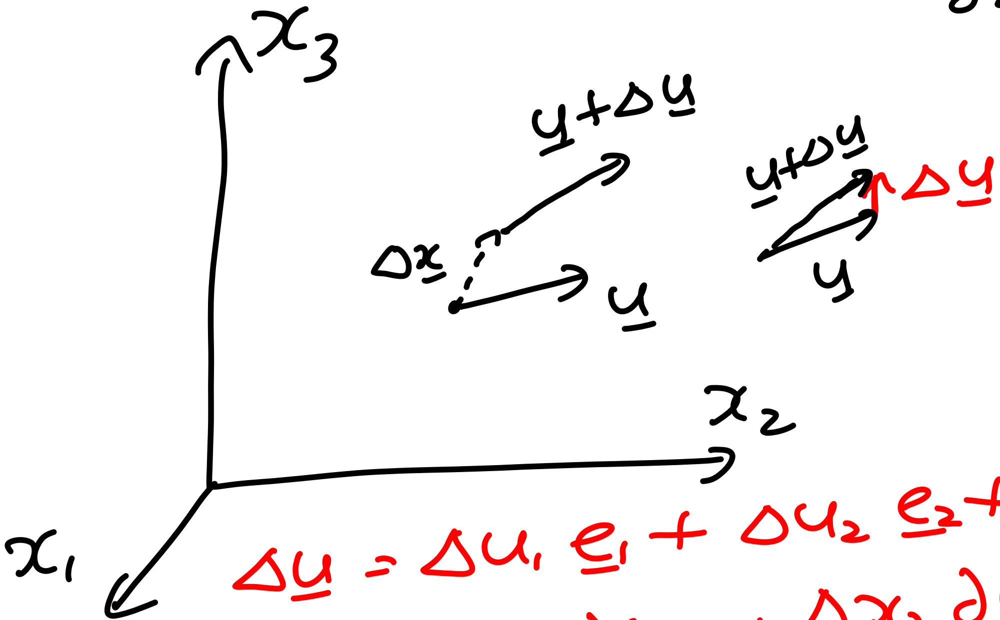


τ_{12} = Force / Area in x_1 direction acting at surface with unit normal in x_2 direction

τ_{ij} = Force / Area in x_i direction acting at a surface with unit normal in x_j direction
'outward'

$$\nabla \underline{u} = \underline{e}_1 \underline{e}_1 \frac{\partial u_1}{\partial x_1} + \underline{e}_1 \underline{e}_2 \frac{\partial u_2}{\partial x_1} + \underline{e}_1 \underline{e}_3 \frac{\partial u_3}{\partial x_1} + \cdots$$

$$+ \underline{e}_3 \underline{e}_3 \frac{\partial u_3}{\partial x_3}$$



$$\Delta\underline{u} = \Delta u_1 \underline{e}_1 + \Delta u_2 \underline{e}_2 + \Delta u_3 \underline{e}_3$$

$$\Delta u_1 = \Delta x_1 \frac{\partial u_1}{\partial x_1} + \Delta x_2 \frac{\partial u_1}{\partial x_2} + \Delta x_3 \frac{\partial u_1}{\partial x_3}$$

$$= \Delta\underline{x} \cdot \nabla u_1$$

$$\Delta u_2 = \Delta\underline{x} \cdot \nabla u_2$$

$$\Delta u_3 = \Delta\underline{x} \cdot \nabla u_3$$

$$\Delta\underline{u} = \Delta\underline{x} \cdot \nabla(u_1 \underline{e}_1 + u_2 \underline{e}_2 + u_3 \underline{e}_3) = \Delta\underline{x} \cdot \nabla\underline{u}$$

$$\nabla \underline{u} = e_1 e_1 \frac{\partial u_1}{\partial x_1} + e_1 e_2 \frac{\partial u_2}{\partial x_1} + e_1 e_3 \frac{\partial u_3}{\partial x_1} + \cdots + e_3 e_3 \frac{\partial u_3}{\partial x_3}$$

$$\frac{\partial c}{\partial t} + \nabla \cdot (\underline{u} c) = D \nabla^2 c$$

Diffusion dominated

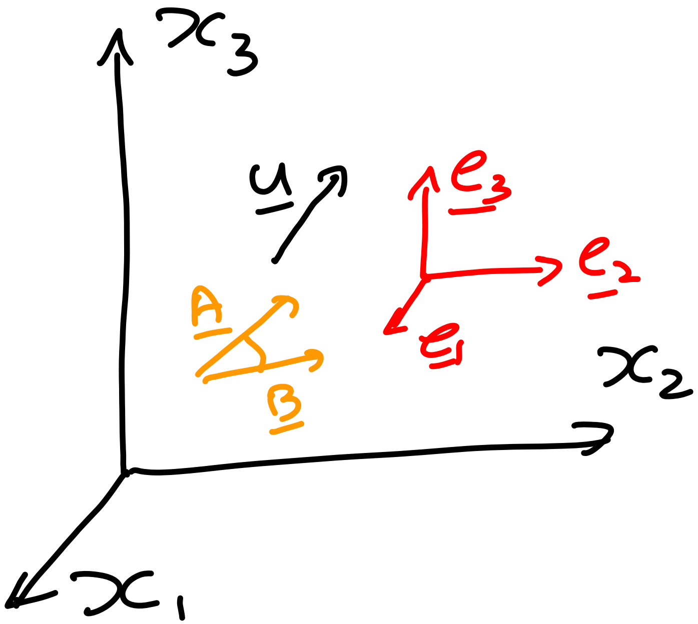
Convection dominated

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0$$

$$\rho \left[\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} \right] = -\nabla p + \mu \nabla \left[\nabla u + \nabla u^T - \frac{2}{3} \underline{\underline{I}} \nabla \cdot \underline{u} \right]$$

Reynolds number $= \dfrac{\rho U D}{\mu} = \dfrac{U D}{\nu}$

VECTORS & TENSORS:



$$\underline{u} = u_1 \underline{e}_1 + u_2 \underline{e}_2 + u_3 \underline{e}_3$$

$$= \sum_{i=1}^{3} u_i \underline{e}_i$$

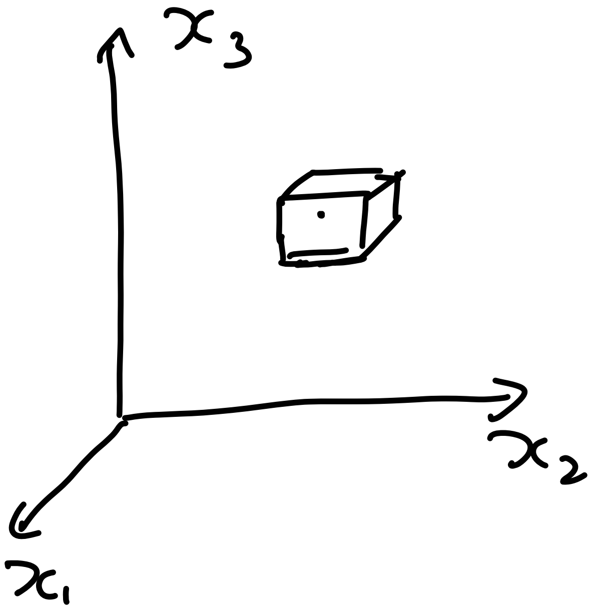
$$= u_i$$

$$\underline{A} = A_1 \underline{e}_1 + A_2 \underline{e}_2 + A_3 \underline{e}_3 = \sum_{i=1}^{3} A_i \underline{e}_i = A_i$$

$$\underline{B} = B_1 \underline{e}_1 + B_2 \underline{e}_2 + B_3 \underline{e}_3 = \sum_{i=1}^{3} B_i \underline{e}_i = B_i$$

$$\underline{A} \cdot \underline{B} = |A||B| \cos\theta$$

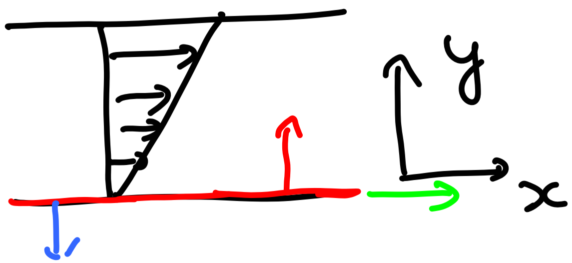
$$= A_1 B_1 + A_2 B_2 + A_3 B_3 = \sum_{i=1}^{3} A_i B_i = A_i B_i$$



$$\tau_{xy} = \mu \frac{du_x}{dy}$$

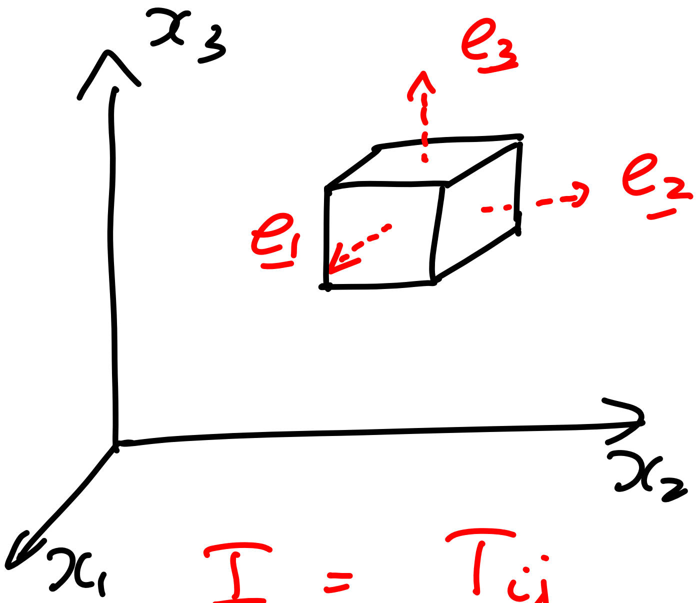
= Force/unit area in x direction at a surface with unit normal (outward) in y dir.

τ_{ij} = Force/Area in the x_i direction acting at a surface with outward unit normal in x_j dir.



$$\underline{u} = u_1 \underline{e}_1 + u_2 \underline{e}_2 + u_3 \underline{e}_3 = \sum_{i=1}^{3} u_i \underline{e}_i$$

$$\underline{\underline{T}} = T_{11} \underline{e}_1 \underline{e}_1 + T_{12} \underline{e}_1 \underline{e}_2 + T_{13} \underline{e}_1 \underline{e}_3 + T_{21} \underline{e}_2 \underline{e}_1 + T_{22} \underline{e}_2 \underline{e}_2 + T_{23} \underline{e}_2 \underline{e}_3 + T_{31} \underline{e}_3 \underline{e}_1 + T_{32} \underline{e}_3 \underline{e}_2 + T_{33} \underline{e}_3 \underline{e}_3 = \sum_{i=1}^{3}\sum_{j=1}^{3} T_{ij}\, \underline{e}_i \underline{e}_j = T_{ij}$$



$$= \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

$$\underline{\underline{T}} = T_{ij}$$

$$\underline{A} \cdot \underline{B} = \left[\sum_{i=1}^{3} A_i \underline{e}_i\right] \cdot \left[\sum_{j=1}^{3} B_j \underline{e}_j\right]$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \left[A_i B_j \, \underline{e}_i \cdot \underline{e}_j\right]$$

$$\underline{e}_i \cdot \underline{e}_j = 1 \text{ if } i = j$$

$$= 0 \text{ if } i \neq j$$

$$= \delta_{ij}$$

$$\underline{A} \cdot \underline{B} = \sum_{i=1}^{3} \sum_{j=1}^{3} A_i B_j \delta_{ij}$$

$$= \sum_{i=1}^{3} A_i B_i$$

$$\delta_{ij} = \sum_{i=1}^{3} \sum_{j=1}^{3} \delta_{ij} \, \underline{e}_i \, \underline{e}_j$$

'Identity tensor'

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{A} \times \underline{B} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \underline{e}_1 (A_2 B_3 - A_3 B_2) + \underline{e}_2 (A_3 B_1 - A_1 B_3) + \underline{e}_3 (A_1 B_2 - A_2 B_1)$$

$$\underline{A} \times \underline{B} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \underline{e}_i \, \epsilon_{ijk} (A_j B_k)$$

$$= \underline{e}_1 (A_2 B_3 - A_3 B_2) + \underline{e}_2 (A_3 B_1 - A_1 B_3) + \underline{e}_3 (A_1 B_2 - A_2 B_1)$$

$$\underline{A} \times \underline{B} = \epsilon_{ijk} A_j B_k$$

Antisymmetric tensor

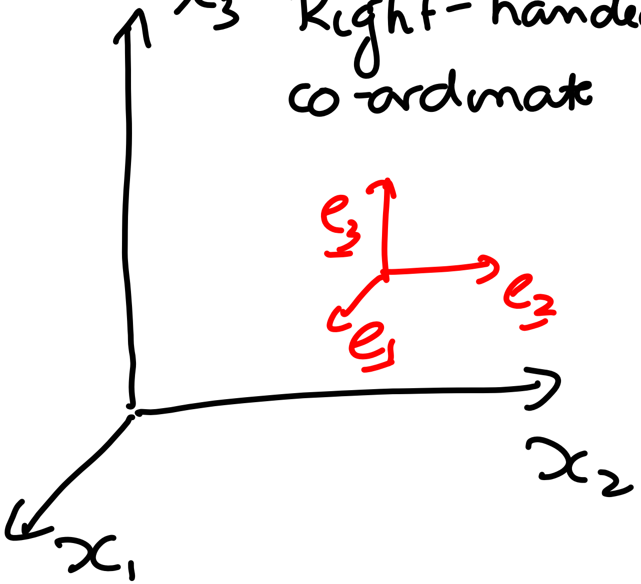
$$\epsilon_{ijk} = 1 \text{ if } (ijk) = (1\,2\,3), (3\,1\,2), (2\,3\,1)$$

$$= -1 \text{ if } (ijk) = (1\,3\,2), (3\,2\,1), (2\,1\,3)$$

$$= 0 \text{ otherwise}$$

$$\epsilon_{ijk} = -\epsilon_{ikj}$$

Right-handed co-ordinate



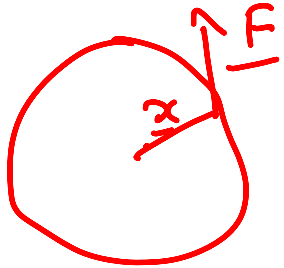
$$\underline{e}_1 \times \underline{e}_2 = \underline{e}_3$$

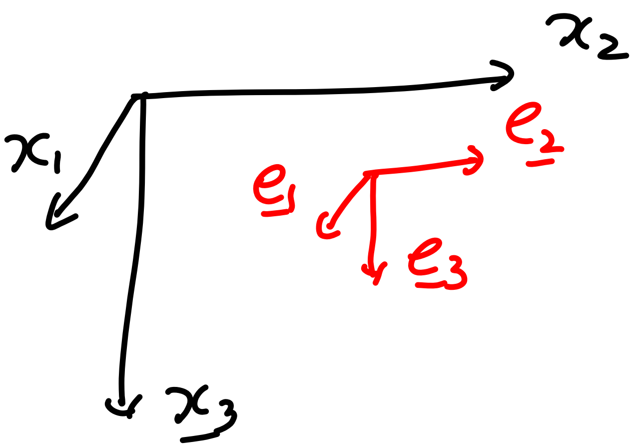
$$\underline{T} = \underline{x} \times \underline{F}$$

$$= \epsilon_{ijk} x_j F_k$$

Pseudo-vector

$$\underline{v} = \underline{r} \times \underline{\omega}$$





$$\underline{e}_1 \times \underline{e}_2 = \underline{e}_3$$

Vectors & Tensors:

$$\underline{u} = \sum_{i=1}^{3} u_i \underline{e_i}$$

$$\underline{\underline{T}} = \sum_{i=1}^{3} \sum_{j=1}^{3} T_{ij} \underline{e_i}\, \underline{e_j}$$

$$\underline{A} \cdot \underline{B} = \sum_{i=1}^{3} \sum_{j=1}^{3} A_i B_j \delta_{ij}$$

$$\underline{A} \times \underline{B} = \epsilon_{ijk} A_j B_k$$

$$= -\epsilon_{ikj} A_j B_k$$

$$= -(\underline{B} \times \underline{A})$$

$$T_{ii} = \sum_{i=1}^{3} T_{ii} = T_{11} + T_{22} + T_{33}$$

$$\underline{A}\,\underline{B} = \sum_{i=1}^{3} \sum_{j=1}^{3} A_i B_j \underline{e_i}\, \underline{e_j} = A_i B_j$$

Rules:

1. Unrepeated index – fundamental direction
   Summation + Unit vector
2. Index repeated two times – dot product
   Summation
3. Index repeated three times
   MISTAKE !!!

Dyadic product

$$A_i S_{jk} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} A_i S_{jk} \underline{e_i}\, \underline{e_j}\, \underline{e_k}$$

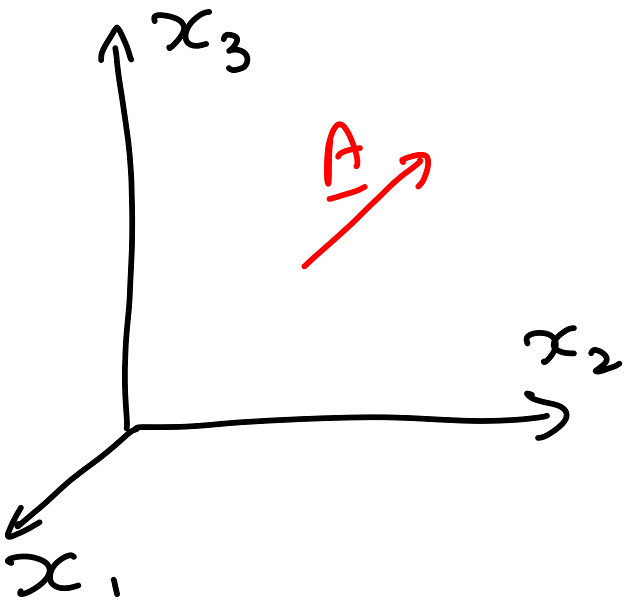
$$A_i B_{kk} + S_{lm} H_{lmi} = C_i$$

$$A_i S_{ik} = \sum_{i=1}^{3} \sum_{k=1}^{3} A_i S_{ik} \underline{e_k}$$

4. In equations, the order and unrepeated indices of all terms are the same.

5. All terms have same parity when going from right - left handed co-ordinate system

Vectors & Tensors:



$$\underline{A} = A_1 \underline{e}_1 + A_2 \underline{e}_2 + A_3 \underline{e}_3$$

$$= A_i$$

$$\underline{\underline{T}} = T_{ij}$$

T_{ij} = Force/Area in i direction at surface with outward unit normal in j direction

$$\underline{A} \cdot \underline{B} = A_i B_i$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} A_i B_j (\underline{e}_i \cdot \underline{e}_j)$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} A_i B_j \delta_{ij}$$

$\delta_{ij} = 1$ if $i = j$

$= 0$ if $i \neq j$

$$\underline{A} \cdot \underline{B} = \sum_{i=1}^{3} \sum_{j=1}^{3} A_i B_j \delta_{ij}$$

$$\underline{A} \times \underline{B} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= \epsilon_{ijk} A_j B_k$$

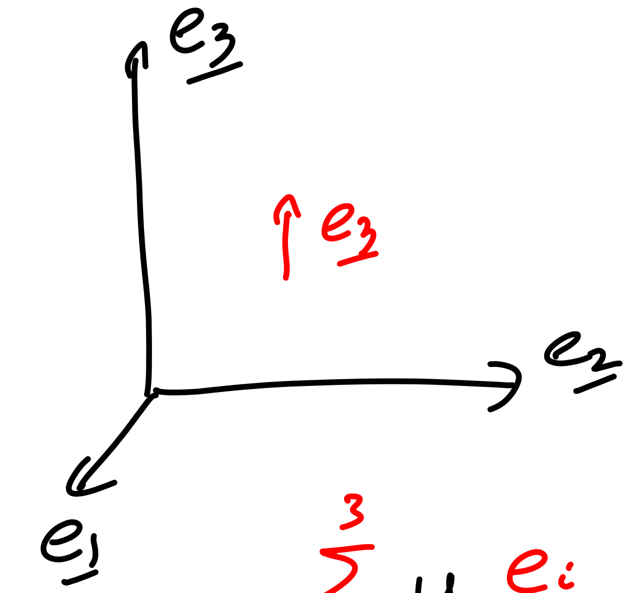
ϵ_{ijk} = Antisymmetric tensor

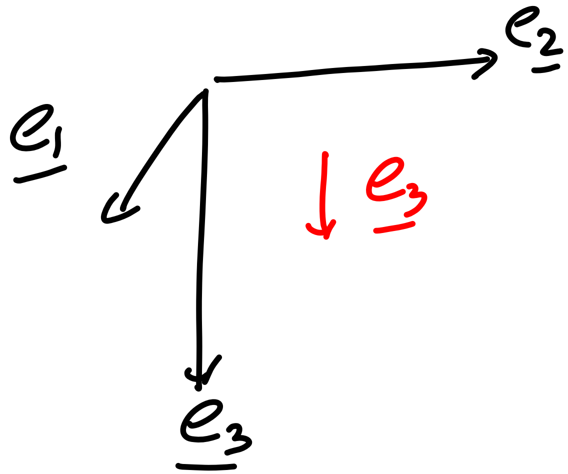
$= 1$ for $(ijk) = (123), (312), (231)$

$= -1$ for $(ijk) = (132), (321), (213)$

$= 0$ otherwise

$$\underline{e}_1 \times \underline{e}_2 = \underline{e}_3$$





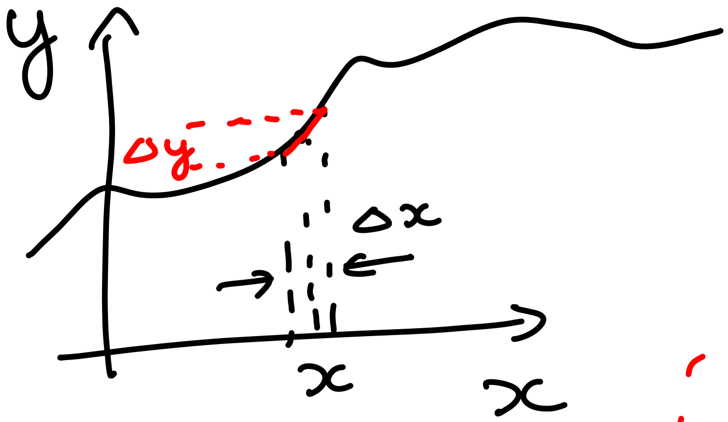
$$\underline{u} = \sum_{i=1}^{3} u_i \, \underline{e}_i$$

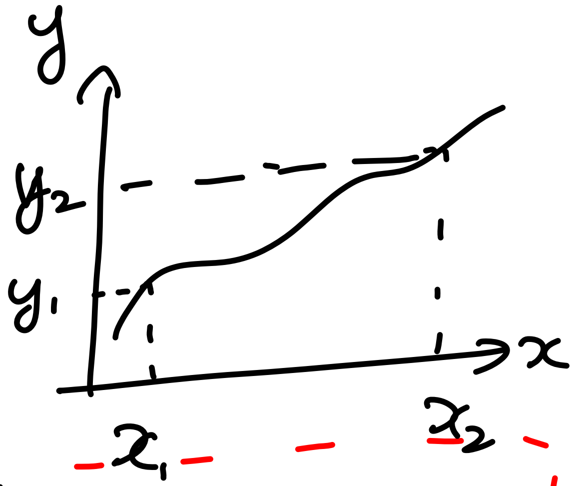
$$\underline{\underline{T}} = \sum_{i=1}^{3} \sum_{j=1}^{3} T_{ij} \, \underline{e}_i \, \underline{e}_j$$

$$\underline{A} \cdot \underline{B} = \sum_{i=1}^{3} A_i B_i$$

$$\underline{A} \times \underline{B} = \epsilon_{ijk} A_j B_k$$

Vector calculus:

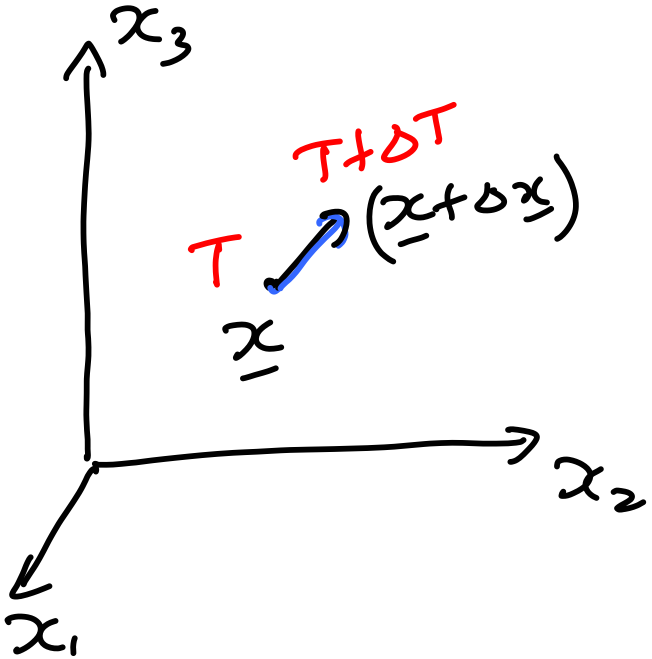




$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\lim_{\Delta x \to 0} \Delta x \left(\frac{dy}{dx} \right) = \Delta y$$

$$\int_{x_1}^{x_2} dx \left(\frac{dy}{dx} \right) = y_2 - y_1$$



$$\Delta T = \frac{\partial T}{\partial x_1}\Delta x_1 + \frac{\partial T}{\partial x_2}\Delta x_2 + \frac{\partial T}{\partial x_3}\Delta x_3$$

$$= \left(\underline{e}_1 \frac{\partial T}{\partial x_1} + \underline{e}_2 \frac{\partial T}{\partial x_2} + \underline{e}_3 \frac{\partial T}{\partial x_3}\right) \cdot \left(\Delta x_1 \underline{e}_1 + \Delta x_2 \underline{e}_2 + \Delta x_3 \underline{e}_3\right)$$

$$\Delta T = \nabla T \cdot \Delta \underline{x}$$

$$\text{grad}\, T \cdot \Delta \underline{x}$$

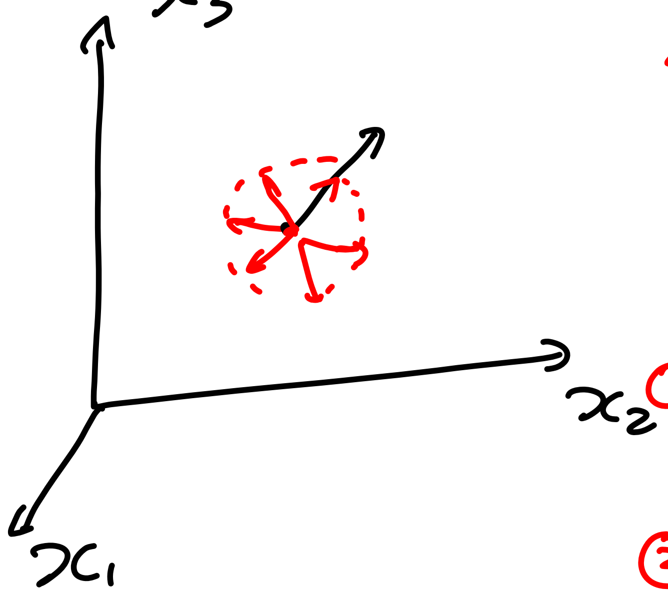
$$\underline{x} = (x_1, x_2, x_3)$$

$$(\underline{x}+\Delta \underline{x}) = (x_1+\Delta x_1,\ x_2+\Delta x_2,\ x_3+\Delta x_3)$$

$$\frac{\partial T}{\partial x_1} = \lim_{\Delta x_1 \to 0}\left(\frac{T(x_1+\Delta x_1, x_2, x_3) - T(x_1, x_2, x_3)}{\Delta x_1}\right)$$

$$|\text{grad}\, T| = \left[\left(\frac{\partial T}{\partial x_1}\right)^2 + \left(\frac{\partial T}{\partial x_2}\right)^2 + \left(\frac{\partial T}{\partial x_3}\right)^2\right]^{1/2}$$

$(\text{grad } T) \cdot \Delta \underline{x} = \Delta T$ in the limit $\Delta \underline{x} \to 0$



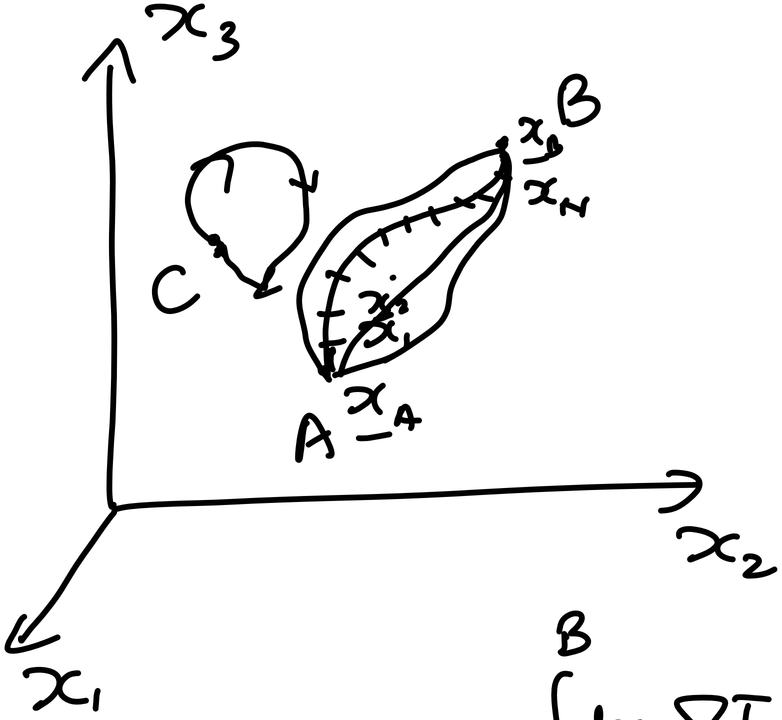
ΔT for equal $|\Delta \underline{x}|$ is maximum when (grad T) is parallel to $\Delta \underline{x}$.

$\Delta T = |\text{grad } T| \, |\Delta \underline{x}| \cos\theta$

1. (grad T) is in direction of maximum variation of T.
2. (grad T) is perpendicular to surfaces of constant T.

$\underline{j} = -D \nabla C$

$\underline{q} = -k \nabla T$



$$\int d\underline{x}\cdot\nabla T = T_B - T_A$$

$$\sum_{i=1}^{N} \nabla T\cdot\Delta x_i = \Delta\underline{x}_1\cdot(\nabla T)\big|_{\underline{x}_1} + \Delta\underline{x}_2(\nabla T)\big|_{\underline{x}_2} + \dots + \Delta\underline{x}_N\cdot\nabla T\big|_{\underline{x}_N}$$

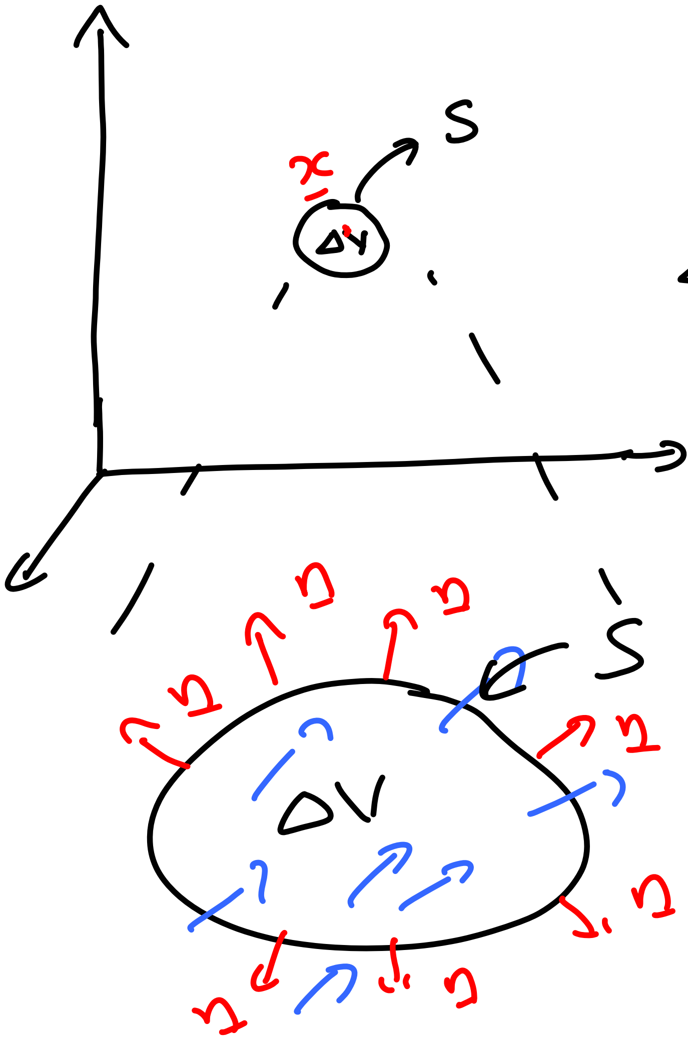
$$= \left[T(\underline{x}_1) - T(\underline{x}_A)\right] + \left[T(\underline{x}_2) - T(\underline{x}_1)\right] + \dots + \left[T(\underline{x}_N) - T(\underline{x}_{N-1})\right] + \left[T(\underline{x}_B) - T(\underline{x}_N)\right]$$

$$\int_A^B d\underline{x}\cdot\nabla T = T(\underline{x}_B) - T(\underline{x}_A)$$

$$\int_C^C d\underline{x}\cdot\nabla T = T(\underline{x}_c) - T(\underline{x}_c) = 0$$

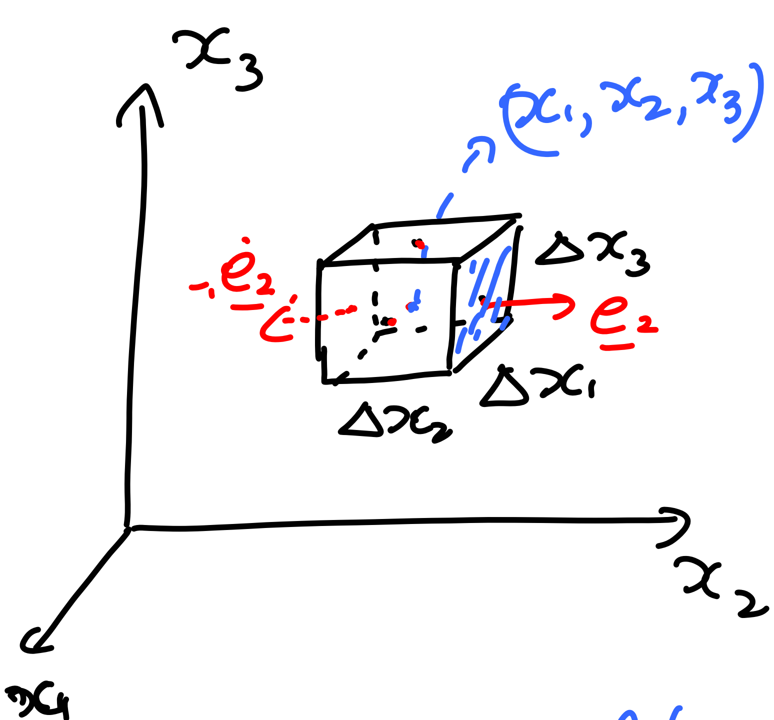
Divergence:

$$\operatorname{div} \underline{A} = \lim_{\Delta V \to 0} \frac{\int ds\, \underline{n}\cdot\underline{A}}{\Delta V}$$



$$\lim_{\Delta V \to 0} \left[\Delta V \operatorname{div} A = \int ds\, \underline{n}\cdot\underline{A} \right]$$

$$\lim_{\Delta V \to 0} \left[\Delta V \operatorname{div} \underline{q} = \int ds\, \underline{n}\cdot\underline{q} \right]$$



$$\operatorname{div} \underline{A} = \lim_{\Delta V \to 0} \frac{\int ds\, \underline{n} \cdot \underline{A}}{\Delta V}$$

$$\begin{aligned}
\int ds\, \underline{n} \cdot \underline{A} = \; & \underline{e}_2 \cdot \underline{A}\left(x_1, x_2 + \frac{\Delta x_2}{2}, x_3\right) \Delta x_1 \Delta x_3 \\
& + (-\underline{e}_2) \cdot \underline{A}\left(x_1, x_2 - \frac{\Delta x_2}{2}, x_3\right) \Delta x_1 \Delta x_3 \\
& + \underline{e}_1 \cdot \underline{A}\left(x_1 + \frac{\Delta x_1}{2}, x_2, x_3\right) \Delta x_2 \Delta x_3 \\
& + (-\underline{e}_1) \cdot \underline{A}\left(x_1 - \frac{\Delta x_1}{2}, x_2, x_3\right) \Delta x_2 \Delta x_3 \\
& + \underline{e}_3 \cdot \underline{A}\left(x_1, x_2, x_3 + \frac{\Delta x_3}{2}\right) \Delta x_1 \Delta x_2 \\
& + (-\underline{e}_1) \cdot \underline{A}\left(x_1, x_2, x_3 - \frac{\Delta x_3}{2}\right) \Delta x_1 \Delta x_2
\end{aligned}$$

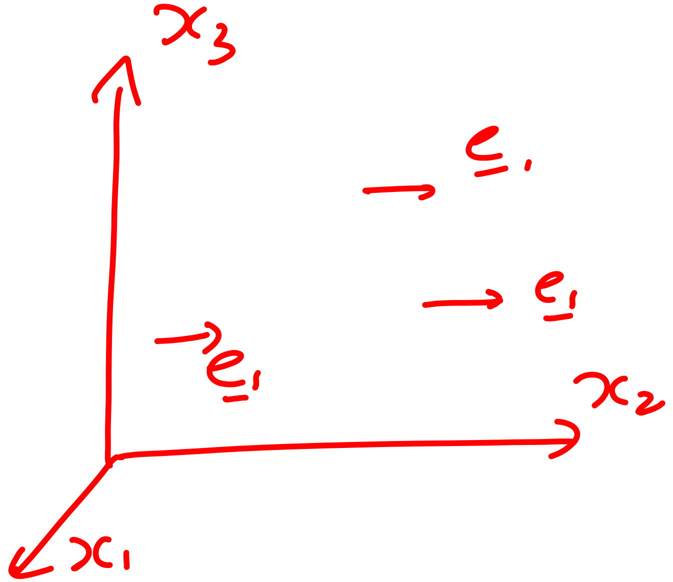
$$\int ds(\underline{n}.\underline{A}) = \Delta x_1 \Delta x_3 \left[A_2\left(x_1, x_2+\frac{\Delta x_2}{2}, x_3\right) - A_2\left(x_1, x_2-\frac{\Delta x_2}{2}, x_3\right)\right]$$
$$+ \Delta x_2 \Delta x_3 \left[A_1\left(x_1+\frac{\Delta x_1}{2}, x_2, x_3\right) - A_1\left(x_1-\frac{\Delta x_1}{2}, x_2, x_3\right)\right]$$
$$+ \Delta x_1 \Delta x_2 \left[A_3\left(x_1, x_2, x_3+\frac{\Delta x_3}{2}\right) - A_3\left(x_1, x_2, x_3-\frac{\Delta x_3}{2}\right)\right]$$

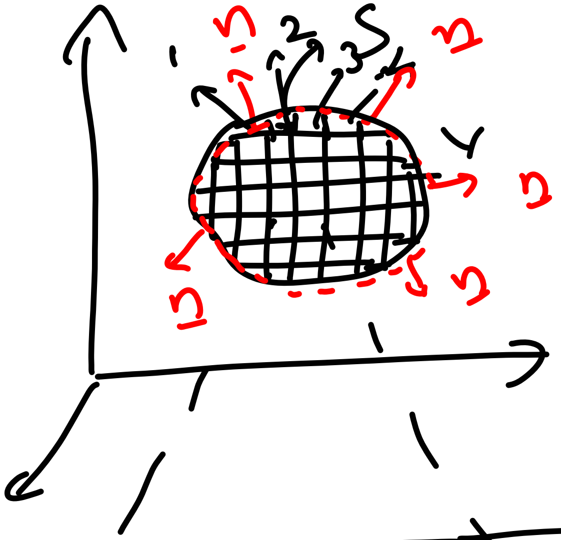
$$\frac{\int ds\, \underline{n}.\underline{A}}{\Delta V} = \frac{A_2\left(x_1, x_2+\frac{\Delta x_2}{2}, x_3\right) - A_2\left(x_1, x_2-\frac{\Delta x_2}{2}, x_3\right)}{\Delta x_2}$$
$$+ \frac{A_1\left(x_1+\frac{\Delta x_1}{2}, x_2, x_3\right) - A_1\left(x_1-\frac{\Delta x_1}{2}, x_2, x_3\right)}{\Delta x_1}$$
$$+ \frac{A_3\left(x_1, x_2, x_3+\frac{\Delta x_3}{2}\right) - A_3\left(x_1, x_2, x_3-\frac{\Delta x_3}{2}\right)}{\Delta x_3}$$
$$= \frac{\partial A_2}{\partial x_2} + \frac{\partial A_1}{\partial x_1} + \frac{\partial A_3}{\partial x_3}$$

$$\operatorname{div} \underline{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} \qquad \frac{\partial}{\partial x_1}(A_1 \underline{e}_1) = \underline{e}_1 \frac{\partial A_1}{\partial x_1}$$

$$= \left(\underline{e}_1 \frac{\partial}{\partial x_1} + \underline{e}_2 \frac{\partial}{\partial x_2} + \underline{e}_3 \frac{\partial}{\partial x_3}\right) \cdot \left(A_1 \underline{e}_1 + A_2 \underline{e}_2 + A_3 \underline{e}_3\right)$$

$$= \nabla \cdot \underline{A}$$





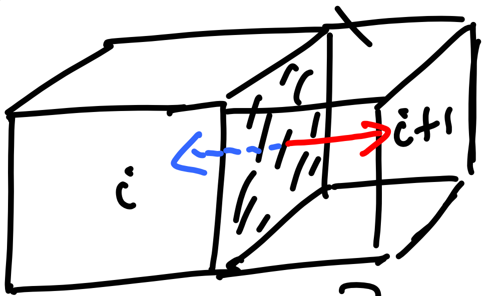
$$\int ds\, \underline{n}\cdot\underline{A} = \int dV (\mathrm{div}\, \underline{A})$$

Divergence theorem

$$\int_V dV\, \mathrm{div}\, \underline{q} = \left(\mathrm{div}\,\underline{A}\big|_{\underline{x}_1} \Delta V_1\right) + \left(\mathrm{div}\,\underline{A}\big|_{\underline{x}_2} \Delta V_2\right) + \dots$$

$$+ \left(\mathrm{div}\,\underline{A}\big|_{\underline{x}_i} \Delta V_i\right) + \left(\mathrm{div}\,\underline{A}\big|_{\underline{x}_{i+1}} \Delta V_{i+1}\right) + \dots$$

$$= \int_S ds\, \underline{n}\cdot\underline{q}$$

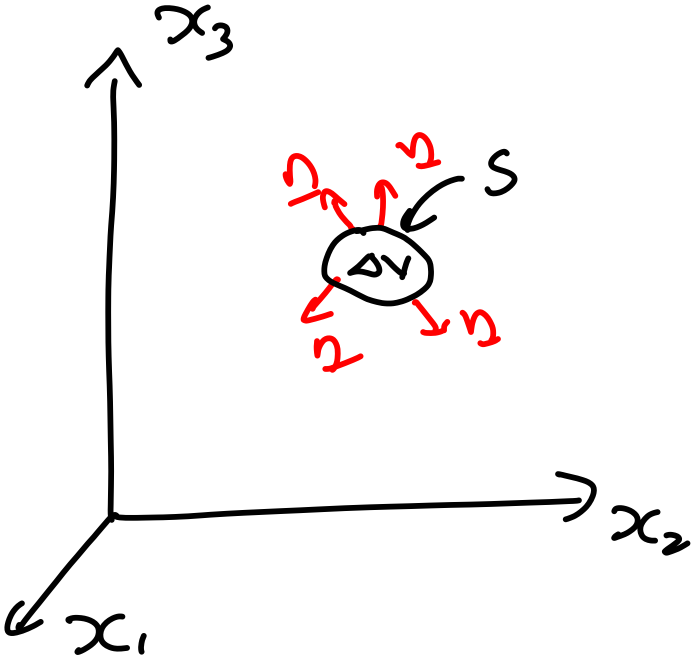


$$\left[\mathrm{div}\,\underline{A}\big|_{\underline{x}_i} \Delta V_i\right] = \int_{S_i} ds\, \underline{n}\cdot\underline{A}$$

$$\left[\mathrm{div}\,\underline{A}\big|_{(\underline{x}_{i+1})} \Delta V_{i+1}\right] = \int_{S_{i+1}} ds\, \underline{n}\cdot\underline{A}$$

$$\text{curl}\,\underline{A} = \lim_{\Delta V \to 0} \frac{\int ds\, \underline{n} \times \underline{A}}{\Delta V} = \lim_{\Delta V \to 0} \frac{\int ds\, \epsilon_{ijk} n_j A_k}{\Delta V}$$

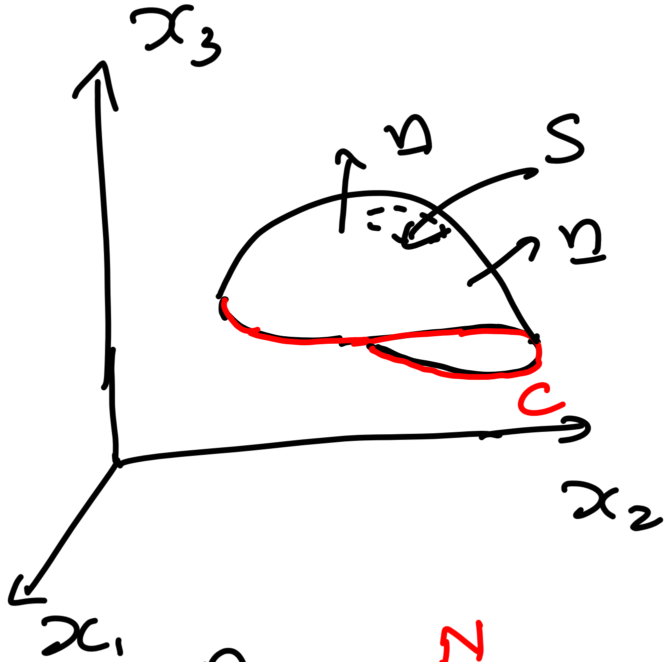
$$= \epsilon_{ijk} \lim_{\Delta V \to 0} \frac{\int ds\, n_j A_k}{\Delta V}$$



$$= \epsilon_{ijk} \frac{\partial A_k}{\partial x_j} = \nabla \times \underline{A}$$

$$= \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

Curl-Integral relation:



$$\int ds\, \underline{n}\cdot \text{curl}\,\underline{A} = \oint_C d\underline{x}\cdot\underline{A}$$

$$\underline{n}\cdot\text{curl}\,\underline{A} = \frac{1}{\Delta V}\int d\sigma\, \underline{n}\cdot(\underline{N}\times\underline{A})$$

$$= \frac{1}{\Delta V}\int d\sigma\, \underline{A}\cdot(\underline{n}\times\underline{N})$$

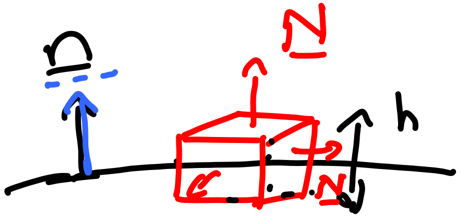
$$= \frac{1}{\Delta V}\int_{\text{sides}} d\sigma\, \underline{A}\cdot(\underline{n}\times\underline{N})$$

$$= \frac{1}{\Delta V}\int_{\text{sides}} d\sigma\, \underline{A}\cdot\underline{t}$$

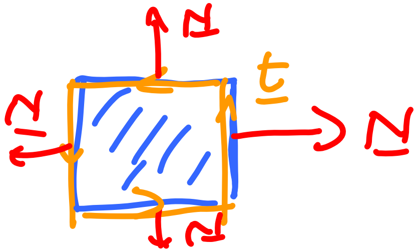
$$= \frac{1}{\Delta S\, h}\int_{\text{sides}} h\, dx\, \underline{A}\cdot\underline{t}$$

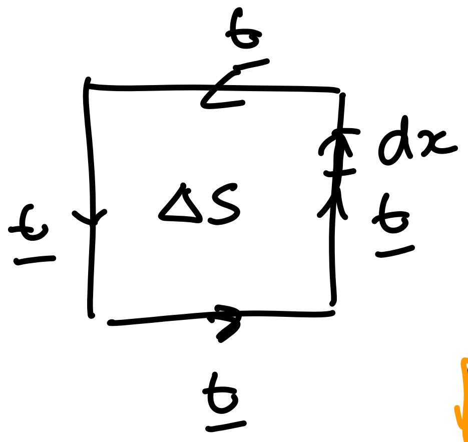
$$= \frac{1}{\Delta S}\int dx\, \underline{A}\cdot\underline{t} = \frac{1}{\Delta S}\int d\underline{x}\cdot\underline{A}$$

Side



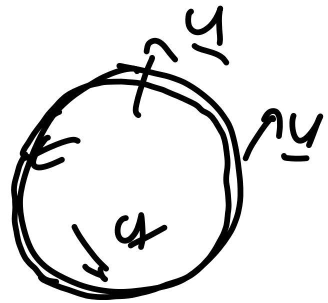
Top

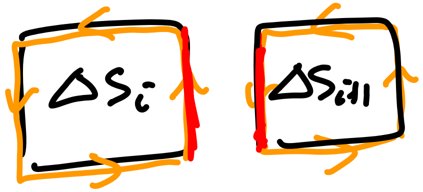


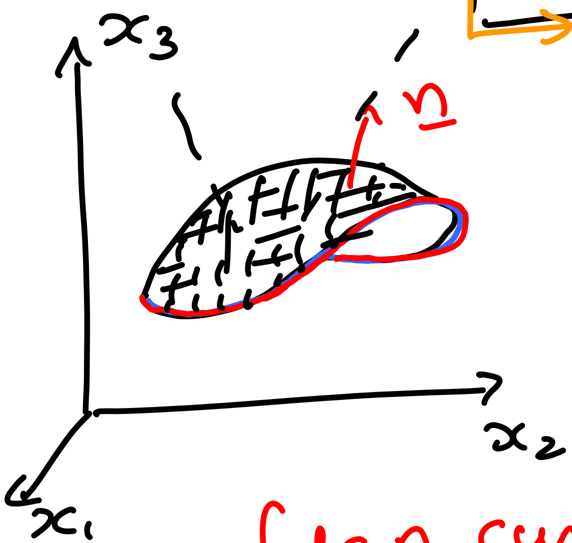


$$\underline{n}\cdot \text{curl}\,\underline{A} = \frac{1}{\Delta s}\oint d\underline{x}\cdot\underline{A}$$

$$\Delta s\, \underline{n}\cdot\text{curl}\,\underline{A} = \oint d\underline{x}\cdot\underline{A}$$







$$\int ds\, \underline{n}\cdot\text{curl}\,\underline{A} = \sum_i \Delta s\, \underline{n}\cdot\text{curl}\,\underline{A}$$

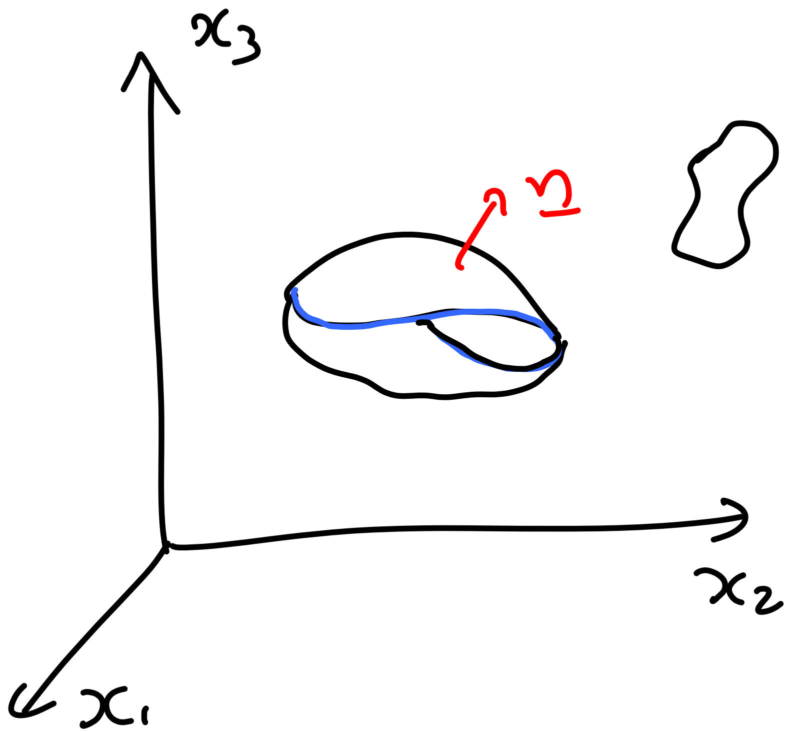
$$\Delta s_i\, \underline{n}\cdot\text{curl}\,\underline{A} = \int_{C_i} d\underline{x}\cdot\underline{A}$$

$$\Delta s_{i+1}\, \underline{n}\cdot\text{curl}\,\underline{A} = \int_{C_{i+1}} d\underline{x}\cdot\underline{A}$$

$$\int ds\, \underline{n}\cdot\text{curl}\,\underline{u} = \oint_C d\underline{x}\cdot\underline{u}$$

Integral theorem for curl

$$\int ds\, \underline{n} \cdot \text{curl}\, \underline{A} = \oint_c d\underline{x} \cdot \underline{A}$$



Closed surface

$$\int ds\, \underline{n} \cdot \text{curl}\, \underline{A} = 0$$

Vector Calculus:

$\underline{u} = u_i$

$\underline{\underline{T}} = T_{ij}$

$\underline{A} \cdot \underline{B} = A_1 B_1 + A_2 B_2 + A_3 B_3$

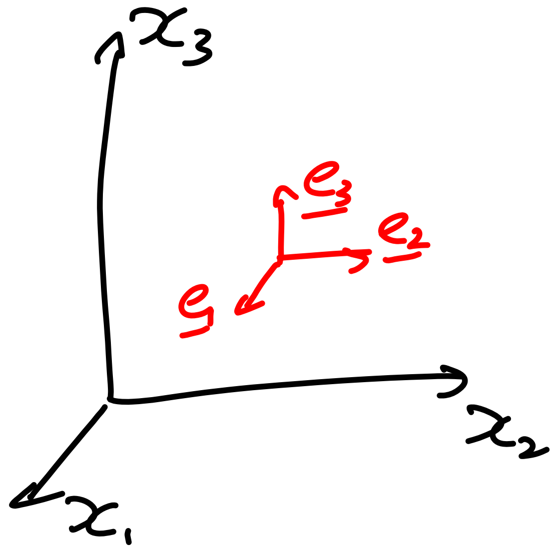
$= A_i B_i$

$\underline{A} \times \underline{B} = \epsilon_{ijk} A_j B_k$

$\delta_{ij} = 1$ for $i = j$ $= \underline{e}_i \cdot \underline{e}_j$

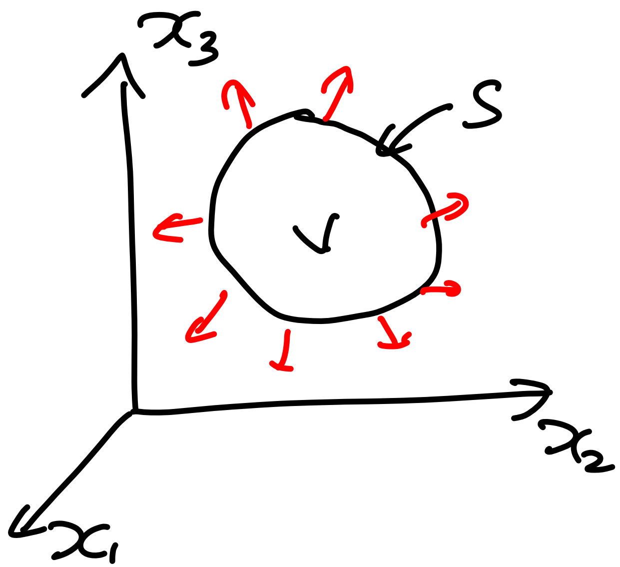
$= 0$ for $i \neq j$

$\underline{A} \cdot \underline{B} = A_i B_j \delta_{ij}$



Elements of vector calculus:

$$(\text{grad } T) \cdot \Delta \underline{x} = T(\underline{x} + \Delta \underline{x}) - T(\underline{x})$$



$$\text{grad } T = \underline{e}_1 \frac{\partial T}{\partial x_1} + \underline{e}_2 \frac{\partial T}{\partial x_2} + \underline{e}_3 \frac{\partial T}{\partial x_3}$$

$$\int_A^B d\underline{x} \cdot \text{grad } T = T_B - T_A$$

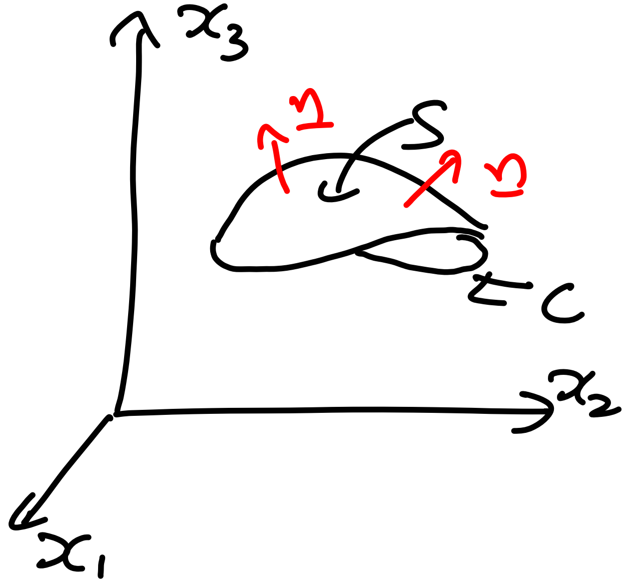
$$\int dS \, \underline{n} \cdot \underline{q} = \int_V dV \, \text{div}(\underline{q})$$

$$\text{div } \underline{A} = \lim_{\Delta V \to 0} \frac{\int dS \, \underline{A} \cdot \underline{n}}{\Delta V}$$

$$\text{div } \underline{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} = \nabla \cdot \underline{A}$$

$$\int_V dV \, \text{div} A = \int_S dS \, \underline{n} \cdot \underline{A}$$

$$\text{curl}\,\underline{A} = \lim_{\Delta V \to 0} \frac{\int ds\, \underline{n} \times \underline{A}}{\Delta V}$$



$$\text{curl}\, A = \epsilon_{ijk} \frac{\partial}{\partial x_j}(A_k) = \nabla \times \underline{A}$$

$$= \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$\int ds\, \underline{n} \cdot \text{curl}\, \underline{A} = \oint_C d\underline{x} \cdot \underline{A}$$

$$\nabla \underline{u} = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial u_i}{\partial x_j} \underline{e}_i\, e_j$$

$$\nabla \cdot \underline{u} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i}$$

$$\nabla \cdot \underline{\underline{T}} = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial T_{ij}}{\partial x_j} \underline{e}_i$$

$$\nabla \underline{\underline{T}} = \frac{\partial}{\partial x_k}(T_{ij})$$

$\nabla \times \nabla T = 0 \quad = \epsilon_{ijk} \frac{\partial}{\partial x_j}\left(\frac{\partial T}{\partial x_k}\right) = -\epsilon_{ikj} \frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial x_j} T\right) = -\nabla \times \nabla T$

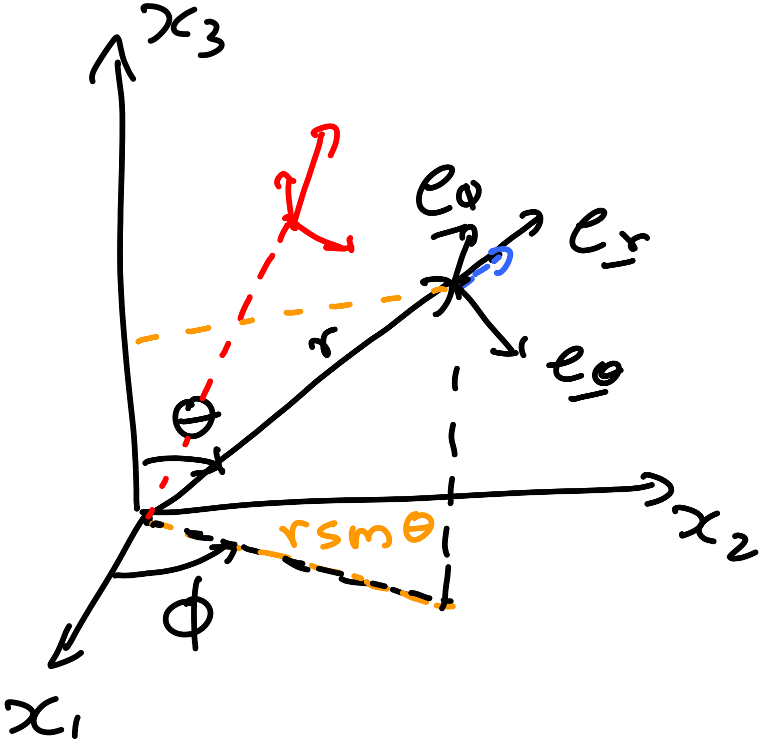
If $\nabla \times \underline{A} = 0$, then $\underline{A}$ can be written as $\nabla \phi$

$\nabla \cdot (\nabla \times \underline{A}) = 0$

If $\nabla \cdot \underline{B} = 0$, then $\underline{B}$ can be expressed as $\nabla \times \underline{A}$

$A \times \underline{B} \times \underline{C} = \epsilon_{ijk} A_j (\epsilon_{klm} B_l C_m)$

$\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$



(r,θ,ϕ)	Distance travelled
$(r \to r+\Delta r)$	Δr
$(\theta \to \theta+\Delta\theta)$	$r\Delta\theta$
$(\phi \to \phi+\Delta\phi)$	$r\sin\theta\,\Delta\phi$

$$\Delta \underline{x} = \Delta r\, \underline{e}_r + r\Delta\theta\, \underline{e}_\theta + r\sin\theta\,\Delta\phi\, \underline{e}_\phi$$
$$= S_a\, \underline{e}_a \Delta x_a + S_b\, \underline{e}_b \Delta x_b + S_c\, \underline{e}_c \Delta x_c$$

$$\Delta T = \Delta \underline{x} \cdot \nabla T$$

$$\nabla T = \underline{e}_1 \frac{\partial T}{\partial x_1} + \underline{e}_2 \frac{\partial T}{\partial x_2} + \underline{e}_3 \frac{\partial T}{\partial x_3}$$
$$= \underline{e}_r \frac{\partial T}{\partial r} + \frac{\underline{e}_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{\underline{e}_\phi}{r\sin\theta} \frac{\partial T}{\partial \phi}$$

$$\nabla T = \frac{\underline{e}_a}{S_a}\frac{\partial T}{\partial x_a} + \frac{\underline{e}_b}{S_b}\frac{\partial T}{\partial x_b} + \frac{\underline{e}_c}{S_c}\frac{\partial T}{\partial x_c}$$

$$\nabla \cdot \underline{A} = \left(\frac{\underline{e}_a}{S_a}\frac{\partial}{\partial x_a} + \frac{\underline{e}_b}{S_b}\frac{\partial}{\partial x_b} + \frac{\underline{e}_c}{S_c}\frac{\partial}{\partial x_c}\right) \cdot \left(A_a \underline{e}_a + A_b \underline{e}_b + A_c \underline{e}_c\right)$$

$$\Delta \underline{x} = \underline{e}_a S_a \Delta x_a + \underline{e}_b S_b \Delta x_b + \underline{e}_c S_c \Delta x_c$$

$$\frac{\partial \underline{x}}{\partial x_a} = \underline{e}_a S_a \qquad \frac{\partial \underline{x}}{\partial x_b} = \underline{e}_b S_b$$

$$\frac{\partial}{\partial x_b}\left(\frac{\partial \underline{x}}{\partial x_a}\right) = \boxed{S_a \frac{\partial \underline{e}_a}{\partial x_b}} + \underline{e}_a \frac{\partial S_a}{\partial x_b} \qquad \frac{\partial}{\partial x_a}\left(\frac{\partial \underline{x}}{\partial x_b}\right) = \frac{\partial \underline{e}_b}{\partial x_a} S_b + \boxed{\underline{e}_b \frac{\partial S_b}{\partial x_a}}$$

$$\frac{\partial \underline{e}_a}{\partial x_b} = \frac{\underline{e}_b}{S_a}\frac{\partial S_b}{\partial x_a}$$

Spherical co-ordinate system

$s_r = 1 \quad s_\theta = r \quad s_\phi = r\sin\theta$

$$\frac{\partial \underline{e}_a}{\partial x_b} = \frac{\underline{e}_b}{s_a}\frac{\partial s_b}{\partial x_a}$$

$$\frac{\partial \underline{e}_r}{\partial \theta} = \frac{e_\theta}{s_r}\frac{\partial s_\theta}{\partial r} = \underline{e}_\theta$$

$$\frac{\partial \underline{e}_r}{\partial \phi} = \frac{e_\phi}{s_r}\frac{\partial s_\phi}{\partial r} = \sin\theta\,\underline{e}_\phi$$

$$\frac{\partial \underline{e}_a}{\partial x_a} = \frac{\partial}{\partial x_a}\left(\underline{e}_b \times \underline{e}_c\right) = \frac{\partial \underline{e}_b}{\partial x_a} \times \underline{e}_c + \underline{e}_b \times \frac{\partial \underline{e}_c}{\partial x_a}$$

$$= \frac{\underline{e}_a}{s_b}\frac{\partial s_a}{\partial x_b} \times \underline{e}_c + \underline{e}_b \times \frac{\underline{e}_a}{s_c}\frac{\partial s_a}{\partial x_c}$$

$$= -\frac{\underline{e}_b}{s_b}\frac{\partial s_a}{\partial x_b} - \frac{\underline{e}_c}{s_c}\frac{\partial s_a}{\partial x_c}$$

$$
\begin{aligned}
\operatorname{div} \underline{A} &= \left(\frac{\underline{e}_a}{S_a} \frac{\partial}{\partial x_a} + \frac{\underline{e}_b}{S_b} \frac{\partial}{\partial x_b} + \frac{\underline{e}_c}{S_c} \frac{\partial}{\partial x_c} \right) \cdot \left(A_a \underline{e}_a + A_b \underline{e}_b + A_c \underline{e}_c \right) \\
&= \frac{\underline{e}_a}{S_a} \frac{\partial}{\partial x_a} \cdot \left(A_a \underline{e}_a + A_b \underline{e}_b + A_c \underline{e}_c \right) \\
&= \frac{\underline{e}_a}{S_a} \cdot \left(\underline{e}_a \frac{\partial A_a}{\partial x_a} + A_a \frac{\partial \underline{e}_a}{\partial x_a} + \underline{e}_b \frac{\partial A_b}{\partial x_a} + A_b \frac{\partial \underline{e}_b}{\partial x_a} + \underline{e}_c \frac{\partial A_c}{\partial x_a} + A_c \frac{\partial \underline{e}_c}{\partial x_a} \right) \\
&= \frac{1}{S_a} \frac{\partial A_a}{\partial x_a} + \frac{\underline{e}_a}{S_a} \cdot \left(A_b \frac{\underline{e}_a}{S_b} \frac{\partial S_a}{\partial x_b} \right) + \frac{\underline{e}_a}{S_a} \cdot \left(A_c \frac{\underline{e}_a}{S_c} \frac{\partial S_a}{\partial x_c} \right) \\
&= \frac{1}{S_a} \frac{\partial A_a}{\partial x_a} + \frac{A_b}{S_a S_b} \frac{\partial S_a}{\partial x_b} + \frac{A_c}{S_a S_c} \frac{\partial S_a}{\partial x_c} \\
&\quad + \frac{1}{S_b} \frac{\partial A_b}{\partial x_b} + \frac{A_a}{S_a S_b} \frac{\partial S_b}{\partial x_a} + \frac{A_c}{S_b S_c} \frac{\partial S_b}{\partial x_c} \\
&\quad + \frac{1}{S_c} \frac{\partial A_c}{\partial x_c} + \frac{A_a}{S_a S_c} \frac{\partial S_c}{\partial x_a} + \frac{A_b}{S_b S_c} \frac{\partial S_c}{\partial x_b}
\end{aligned}
$$

$$\operatorname{div} A = \frac{1}{S_a S_b S_c}\left[\frac{\partial}{\partial x_a}\left(S_b S_c A_a\right) + \frac{\partial}{\partial x_b}\left(S_c S_a A_b\right) + \frac{\partial}{\partial x_c}\left(S_a S_b A_c\right)\right]$$

$S_r = 1 \qquad S_\theta = r \qquad S_\phi = r\sin\theta$

$$\operatorname{div} A = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(\underline{r^2 \sin\theta}\, A_r\right) + \frac{\partial}{\partial \theta}\left(r\sin\theta\, A_\theta\right) + \frac{\partial}{\partial \phi}\left(r A_\phi\right)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\operatorname{curl} \underline{A} = \frac{1}{S_a S_b S_c}\begin{vmatrix} S_a \underline{e}_a & S_b \underline{e}_b & S_c \underline{e}_c \\ \frac{\partial}{\partial x_a} & \frac{\partial}{\partial x_b} & \frac{\partial}{\partial x_c} \\ S_a A_a & S_b A_b & S_c A_c \end{vmatrix}$$

$$\nabla . \nabla T = \nabla^2 T$$

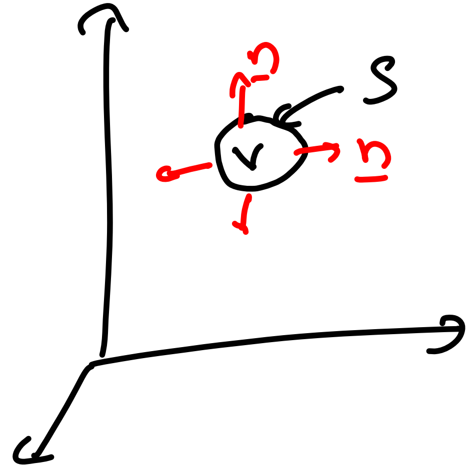
$$= \frac{1}{s_a s_b s_c} \left(\frac{\partial}{\partial x_a}\left(s_b s_c \frac{1}{s_a} \frac{\partial T}{\partial x_a} \right) + \frac{\partial}{\partial x_b}\left(s_a s_c \frac{1}{s_b} \frac{\partial T}{\partial x_b} \right) \right.$$

$$\left. + \frac{\partial}{\partial x_c}\left(\frac{s_a s_b}{s_c} \frac{\partial T}{\partial x_c} \right) \right]$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta} \right) +$$

$$\frac{1}{r^2 \sin^2\theta} \frac{\partial^2 T}{\partial \phi^2}$$

Kinematics:



$\underline{q}$ = Energy/(area × time)

$$\int ds\, \underline{q}\cdot\underline{n} = \int dV\, \mathrm{div}\, \underline{q}$$

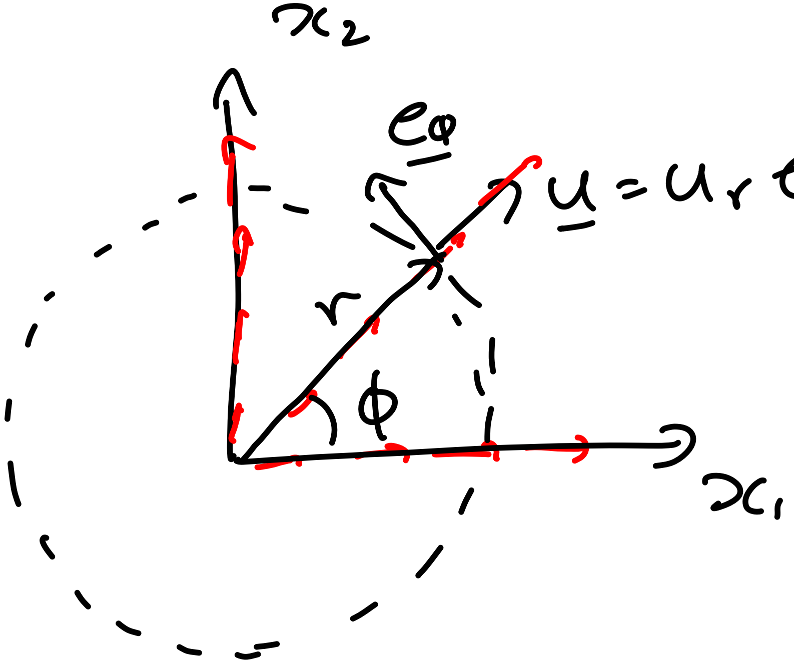
Amount of fluid in volume $\Delta V = \rho\, \Delta V$

$\rho\, \underline{u}$ = Mass/(Area × Time)

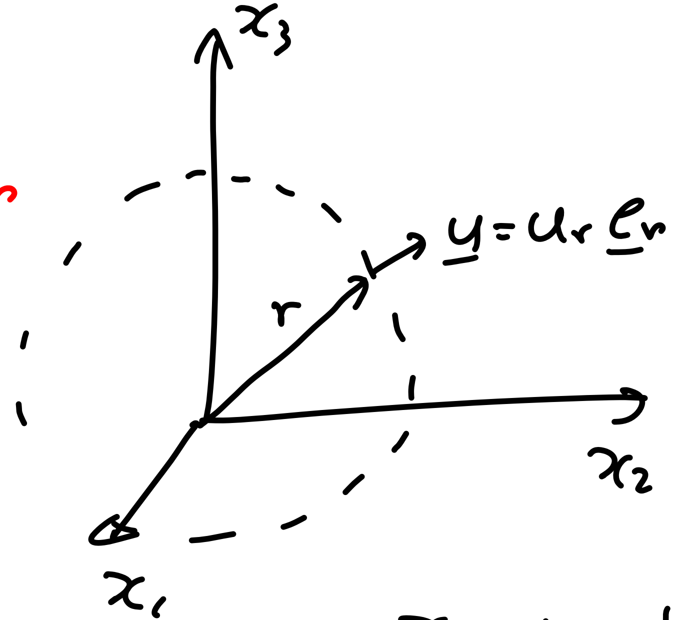
$$\int ds\, (\rho\, \underline{u})\cdot\underline{n} = \int dV\, \mathrm{div}(\rho\, \underline{u})$$

'Incompressible' constant density

$\mathrm{div}\, \underline{u} = 0$ or $\nabla\cdot\underline{u} = 0$



$\underline{u} = u_r \underline{e}_r = C r \underline{e}_r$



$\underline{u} = u_r \underline{e}_r$

$u_1 = \frac{u_r x_1}{r}$ $\qquad u_2 = \frac{u_r x_2}{r}$

$u_1 = C x_1$ $\qquad u_2 = C x_2$

$\mathrm{div}\, \underline{u} = 2C$

$\underline{e}_r = \underline{e}_1 \cos\phi + \underline{e}_2 \sin\phi$

$\underline{e}_\phi = -\underline{e}_1 \sin\phi + \underline{e}_2 \cos\phi$

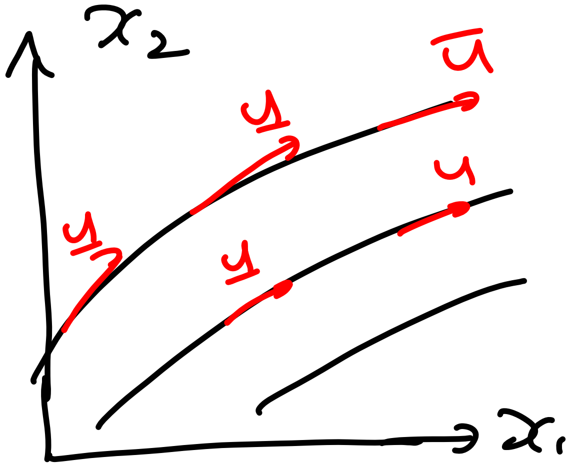
$r = \sqrt{x_1^2 + x_2^2}$; $\underline{x}_1 = \underline{r \cos\phi}$, $\underline{x}_2 = r \underline{\sin}\phi$

$\underline{u} = u_r \underline{e}_r = u_r \left[\underline{e}_1 \cos\phi + \underline{e}_2 \sin\phi\right]$

$\underline{u}_1 = u_r \cos\phi \qquad u_2 = u_r \sin\phi$

Incompressible fluids

$$\operatorname{div} \underline{u} = 0 \implies \underline{u} = \nabla \times \underline{\psi}$$



$$\nabla\psi \cdot \underline{u} = \frac{\partial \psi}{\partial x_1} u_1 + \frac{\partial \psi}{\partial x_2} u_2$$

$$= \frac{\partial \psi}{\partial x_1}\left(\frac{\partial \psi}{\partial x_2}\right) + \frac{\partial \psi}{\partial x_2}\left(\frac{-\partial \psi}{\partial x_1}\right)$$

$$= 0$$

Stream function ψ

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0$$

$$\underline{u} = \underline{e}_1 \frac{\partial \psi}{\partial x_2} - \underline{e}_2 \frac{\partial \psi}{\partial x_1}$$

$$u_1 = \frac{\partial \psi}{\partial x_2} \qquad u_2 = \frac{-\partial \psi}{\partial x_1}$$

$$Q = \int_A^B ds \; \underline{u} \cdot \underline{n}$$

$$= \int_A^B ds \, (u_1 n_1 + u_2 n_2)$$

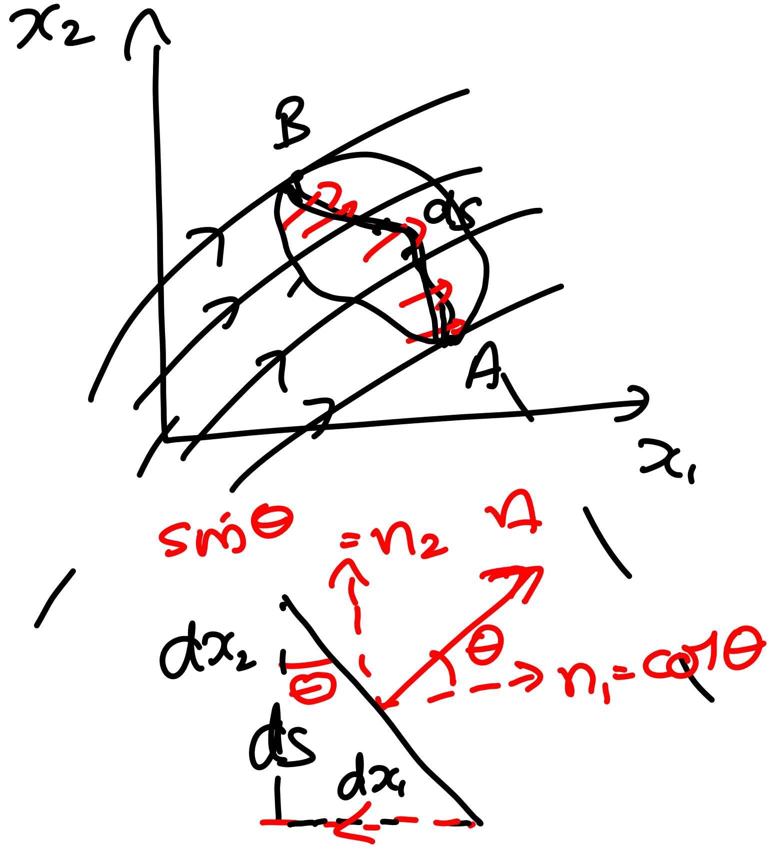
$$= \int_A^B ds \left(\frac{\partial \psi}{\partial x_2} n_1 - \frac{\partial \psi}{\partial x_1} n_2 \right)$$

$$n_1 \, ds = dx_2$$

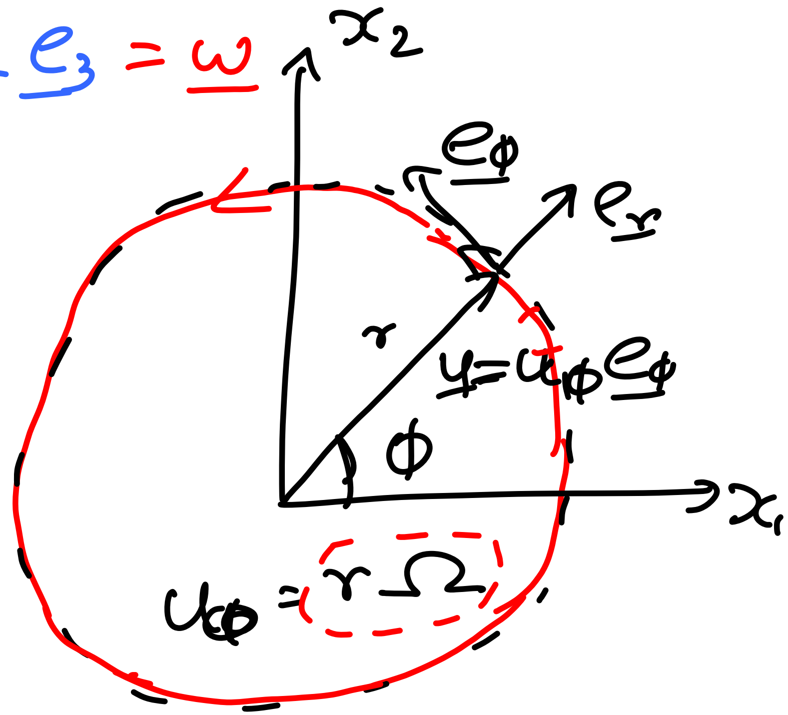
$$n_2 \, ds = -dx_1$$

$$Q = \int_A^B \left[dx_2 \frac{\partial \psi}{\partial x_2} + dx_1 \frac{\partial \psi}{\partial x_1} \right]$$

$$= \int_A^B d\underline{x} \cdot \nabla \psi = \psi(\underline{x}_B) - \psi(\underline{x}_A)$$



$$\nabla \times \underline{u} = \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\underline{e_3} = 2\Omega \underline{e_3} = \underline{\omega}$$



$$\int ds\, \underline{n} \cdot (\nabla \times \underline{u}) = \oint d\underline{x} \cdot \underline{u}$$

$$\int ds\, 2\Omega = 2\Omega \pi r^2$$

$$\oint d\underline{x} \cdot \underline{u} = \int_{2\pi} r\, d\phi\, u_\phi$$

$$= \int_0 r\, d\phi\, \Omega r$$

$$= 2\pi r^2 \Omega$$

If $\nabla \times u = 0$; $\underline{u} = \nabla \phi$

Velocity potential

$$u = \underline{e_1}\frac{\partial \phi}{\partial x_1} + \underline{e_2}\frac{\partial \phi}{\partial x_2} + \underline{e_3}\frac{\partial \phi}{\partial x_3}$$

$$u_1 = \frac{\partial \phi}{\partial x_1} \qquad u_2 = \frac{\partial \phi}{\partial x_2} \qquad u_3 = \frac{\partial \phi}{\partial x_3}$$

$$u_1 = \frac{\partial \psi}{\partial x_2} \qquad u_2 = -\frac{\partial \psi}{\partial x_1}$$

$$(\nabla \phi)\cdot(\nabla \psi) = \left(\frac{\partial \phi}{\partial x_1}\right)\left(\frac{\partial \psi}{\partial x_1}\right) + \left(\frac{\partial \phi}{\partial x_2}\right)\left(\frac{\partial \psi}{\partial x_2}\right)$$

$$= u_1(-u_2) + (u_2)(u_1)$$

$$= 0$$

Gradient of velocity

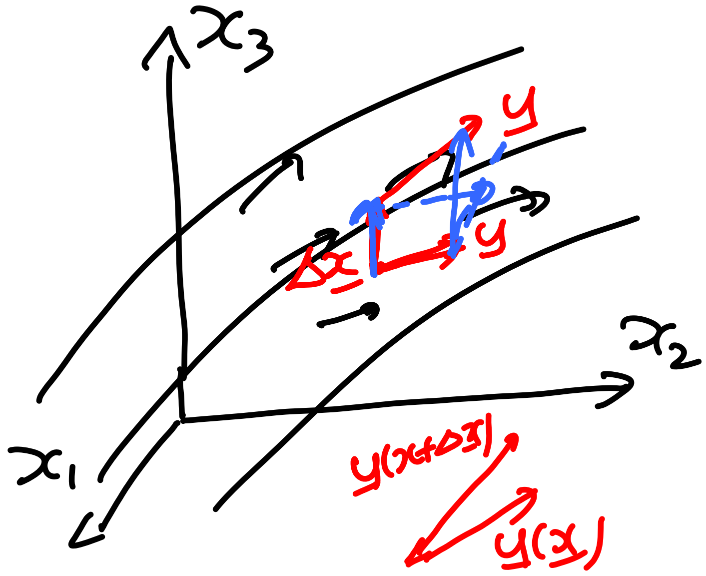
$$\nabla \underline{u} = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial u_i}{\partial x_j} \underline{e}_i \, \underline{e}_j$$

$$= \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

$$\underline{u}(\underline{x} + \Delta \underline{x}) - \underline{u}(\underline{x}) = \Delta \underline{x} \cdot \nabla \underline{u}$$

$$= \Delta x_j \left(\frac{\partial u_i}{\partial x_j} \right)$$

Rate of deformation tensor:



$$\nabla y = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

$$(\nabla u^T) = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} & \frac{\partial u_2}{\partial x_3} & \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

$$\frac{\partial u_i}{\partial x_j} = S_{ij} + A_{ij}$$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$A_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$S_{ij} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) & \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) & 0 & 0 \\ 0 & \frac{1}{3}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) & 0 \\ 0 & 0 & \frac{1}{3}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) \end{pmatrix} + E_{ij}$$

KINEMATICS

$$\underline{A}\cdot\underline{B} = A_1B_1 + A_2B_2 + A_3B_3$$

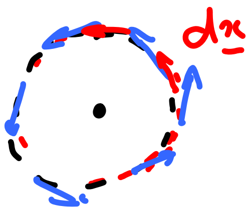
$$\underline{A}\times\underline{B} = \epsilon_{ijk}A_jB_k$$

$$\nabla\underline{u} = \frac{\partial}{\partial x_i}u_j$$

$$\nabla\cdot\underline{u} = \mathrm{div}\,\underline{u} = \frac{\partial u_i}{\partial x_i}$$

$$\int ds\,\underline{n}\cdot\underline{u} = \int dV(\mathrm{div}\,\underline{u})$$

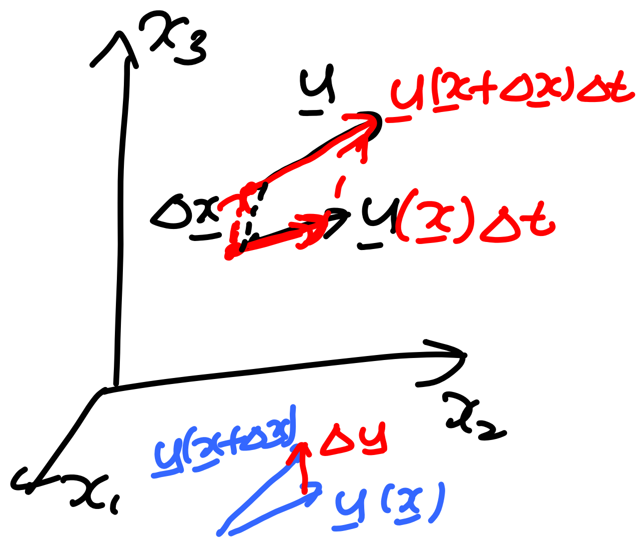
$$\underline{\omega} = \nabla\times\underline{u} = \epsilon_{ijk}\frac{\partial}{\partial x_j}u_k$$



$$\int ds\,\underline{n}\cdot\nabla\times\underline{u} = \oint d\underline{x}\cdot\underline{u}$$

Rate of deformation tensor:

$$\nabla \underline{u} = \frac{\partial u_i}{\partial x_j} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

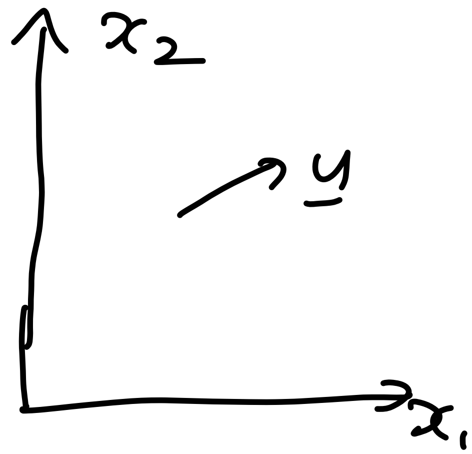


$$\underbrace{\left[u(\underline{x}+\Delta\underline{x}) - u(\underline{x})\right]}_{\Delta u_i} = \Delta\underline{x} \cdot \nabla\underline{u}$$

$$= \Delta x_j \boxed{\left(\frac{\partial u_i}{\partial x_j}\right)}$$

$$\left[u(\underline{x}+\Delta\underline{x}) - u(\underline{x})\right]\Delta t = \left[\Delta x_j \frac{\partial u_i}{\partial x_j} \Delta t\right]$$

$$\nabla \underline{y} = \frac{\partial u_i}{\partial x_j} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{pmatrix}$$



$$= S_{ij} + A_{ij}$$

$$S_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \left(\frac{\partial u_i}{\partial x_j} \right)^T \right] = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right]$$

$$A_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} - \left(\frac{\partial u_i}{\partial x_j} \right)^T \right] = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right]$$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} \end{pmatrix}$$

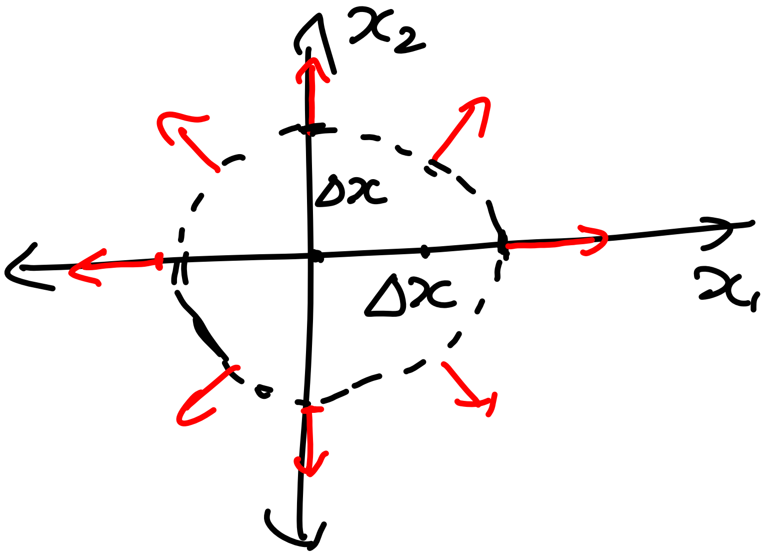
$$= \begin{pmatrix} \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) & 0 \\ 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) \end{pmatrix} + \overbrace{\begin{pmatrix} \frac{\partial u_1}{\partial x_1} - \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} - \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) \end{pmatrix}}^{E_{ij}}$$

$$= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + E_{ij}$$

$$= \frac{1}{2}(\nabla \cdot \underline{u})\,\delta_{ij} + E_{ij}$$

$$\frac{\partial u_i}{\partial x_j} = A_{ij} + E_{ij} + \frac{1}{2}(\nabla \cdot \underline{u})\delta_{ij}$$

Isotropic $\begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix}$



$\Delta u_1 = s\,\Delta x_1$

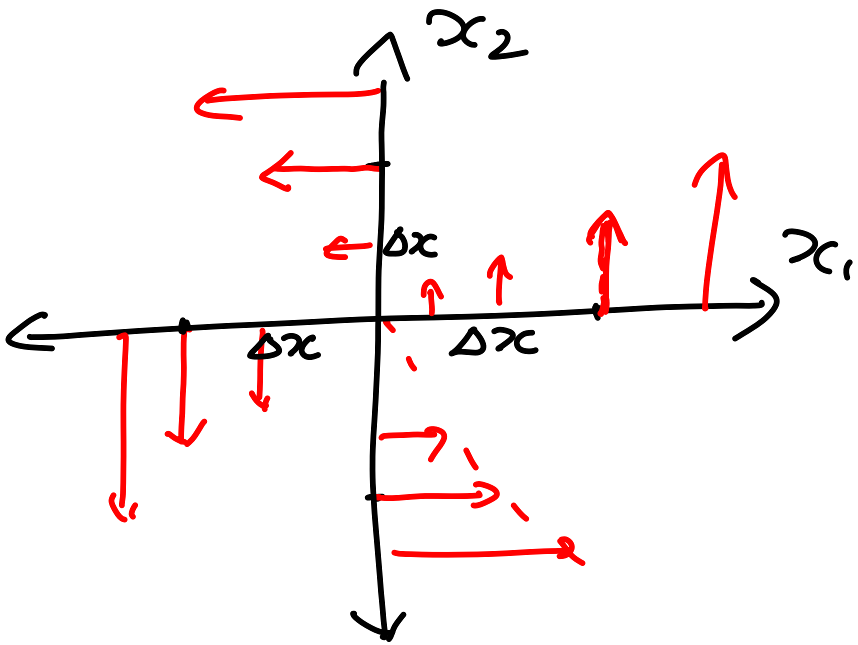
$\Delta u_2 = s\,\Delta x_2$

$u_r = c\,r$

$$\begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix} = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix}$$

$$\Delta u_1 = -a \Delta x_2$$

$$\Delta u_2 = a \Delta x_1$$



$$\omega_i = \epsilon_{ijk} \frac{\partial}{\partial x_j}(u_k) = \nabla \times \underline{u}$$

$$= \epsilon_{ikj} \frac{\partial u_j}{\partial x_k}$$

$$\omega_i = \frac{1}{2}\left(\epsilon_{ijk} \frac{\partial u_k}{\partial x_j} + \epsilon_{ikj} \frac{\partial u_j}{\partial x_k} \right)$$

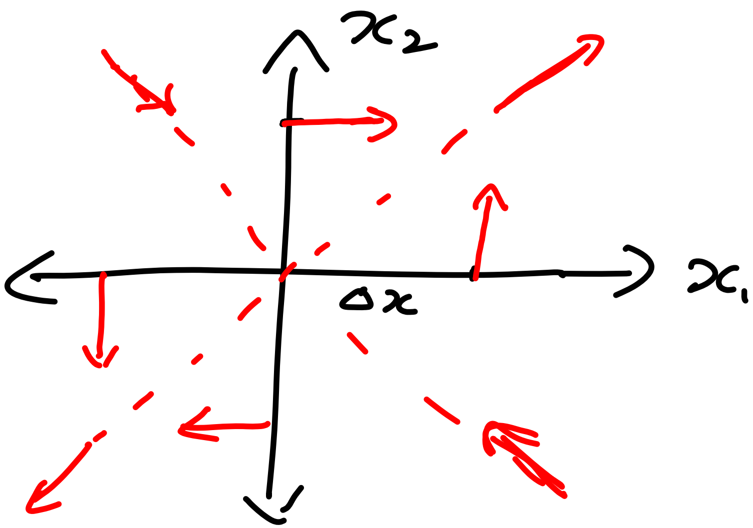
$$= \frac{1}{2} \epsilon_{ijk} \left(\frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k} \right) = \epsilon_{ijk} A_{kj}$$

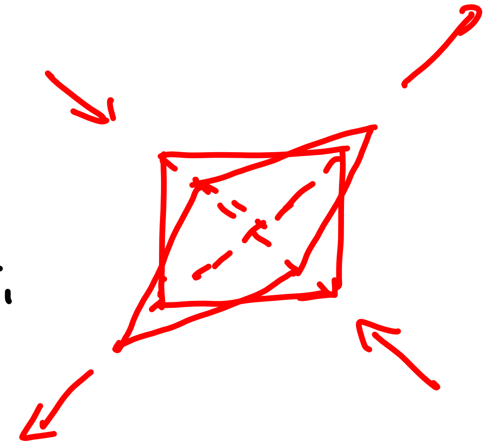
Symmetric traceless

$$\begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix} = \begin{pmatrix} 0 & S \\ S & 0 \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix}$$

$$\Delta u_1 = S \Delta x_2$$

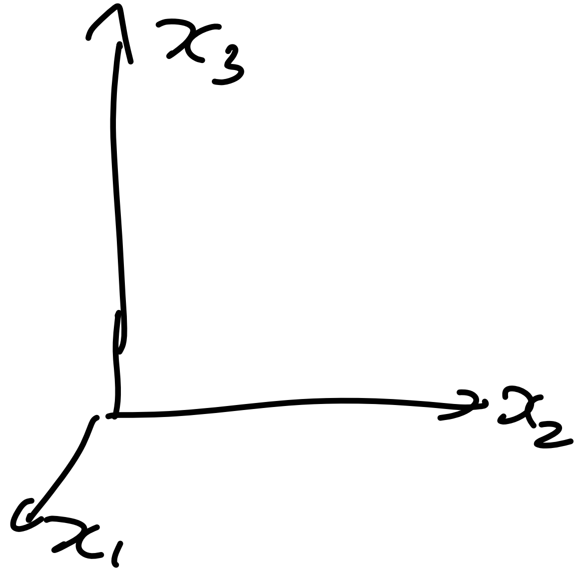
$$\Delta u_2 = S \Delta x_1$$





Pure extensional strain

$$\frac{\partial u_i}{\partial x_j} = A_{ij} + \frac{1}{3}(\nabla\cdot\underline{u})\,\delta_{ij} + E_{ij}$$



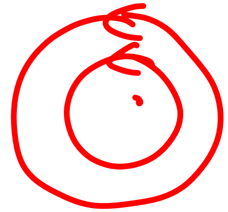
$$\begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

$$\mathrm{Trace}\,(\nabla\underline{u}) = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \nabla\cdot\underline{u}$$

$$\frac{\partial u_i}{\partial x_j} = A_{ij} + E_{ij} + \frac{1}{3}(\nabla\cdot\underline{u})\delta_{ij}$$

$$\frac{1}{3}(\nabla\cdot\underline{u})\delta_{ij}$$

$$\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3} = \frac{\partial u_i}{\partial x_i} = \delta_{ij}\frac{\partial u_i}{\partial x_j}$$



$$A_{ij}$$

$$\omega_i = \frac{1}{2}\epsilon_{ijk}A_{kj}$$

$$\delta_{ij}\frac{\partial u_i}{\partial x_j} = \delta_{ij}A_{ij} + \delta_{ij}E_{ij} + \frac{1}{3}(\nabla\cdot\underline{u})\delta_{ij}\delta_{ij}$$

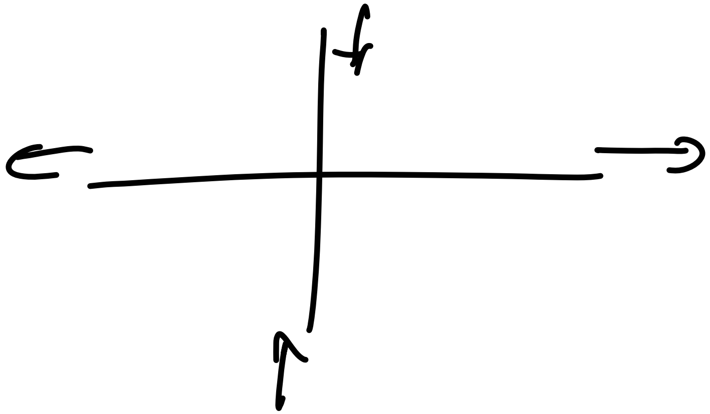
$$= 0 + E_{ii} + \frac{1}{3}(\nabla\cdot\underline{u})\delta_{ii}$$

$$\cancel{\frac{\partial u_i}{\partial x_i}} = 0 + E_{ii} + \cancel{(\nabla\cdot\underline{u})}$$

$$E_{ii} = 0$$

Symmetric traceless: ① One +ve & one -ve & one 0

$$E_{ij} = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{12} & e_{22} & e_{23} \\ e_{13} & e_{23} & e_{33} \end{pmatrix}$$

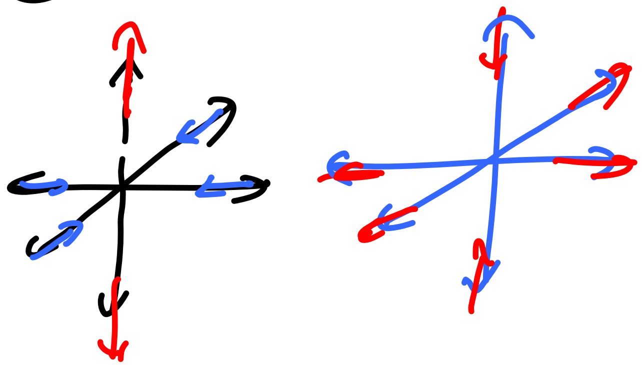


① Three eigenvalues

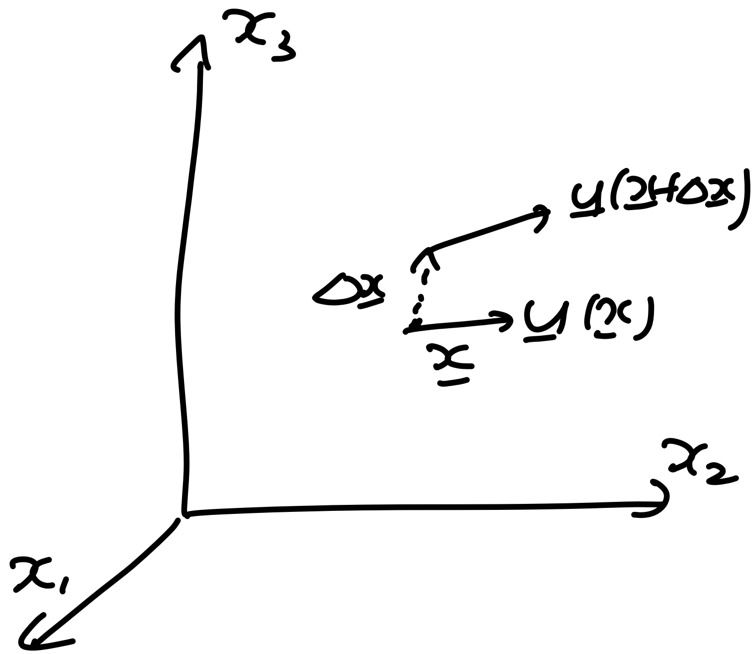
② Three orthogonal eigenvectors.

③ Sum of eigenvalues is zero

② One +ve & two -ve



Rate of deformation tensor:



$$\nabla \underline{u} = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial u_i}{\partial x_j}\, \underline{e}_i\, \underline{e}_j$$

$$\Delta \underline{u} = \nabla_{\underline{x}} \cdot \nabla \underline{u}$$

$$\nabla \underline{u} = S_{ij} + A_{ij}$$

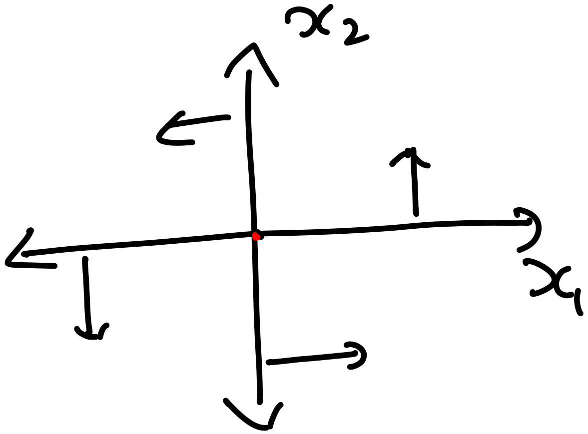
$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

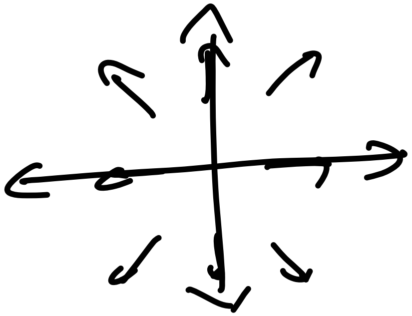
$$A_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

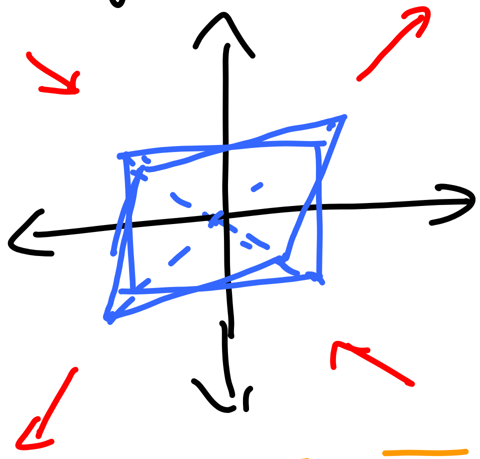
$$S_{ij} = E_{ij} + \tfrac{1}{3}\,\delta_{ij}\left(\frac{\partial u_k}{\partial x_k}\right)$$

Sum of diagonal elements $= E_{ii} = \sum_{i=1}^{3} E_{ii}$

$$\frac{\partial u_i}{\partial x_j} = A_{ij} + \frac{1}{3}\delta_{ij}\left(\frac{\partial u_k}{\partial x_k}\right) + E_{ij}$$

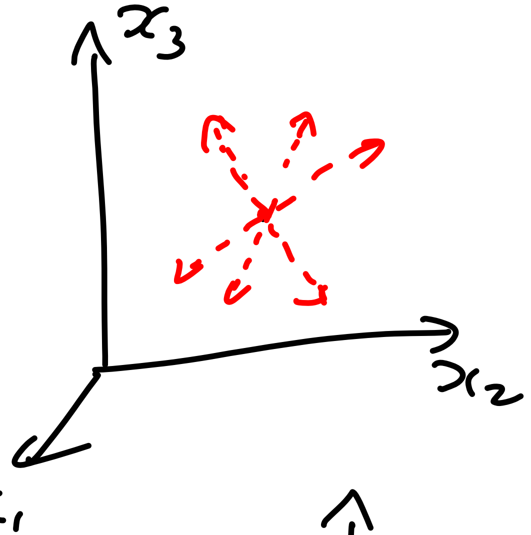


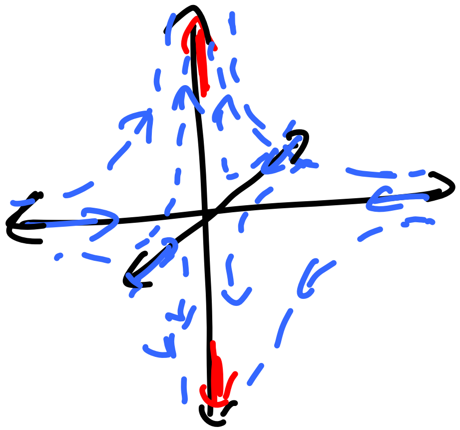




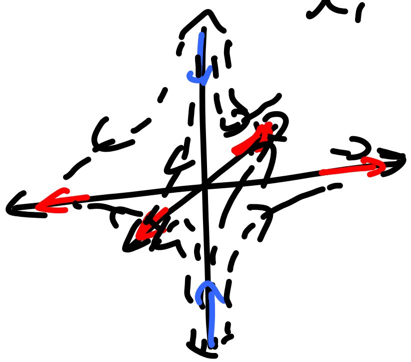
$$\omega_i = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j} = \epsilon_{ijk}A_{kj}$$

$$E_{ij} = \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{pmatrix}$$

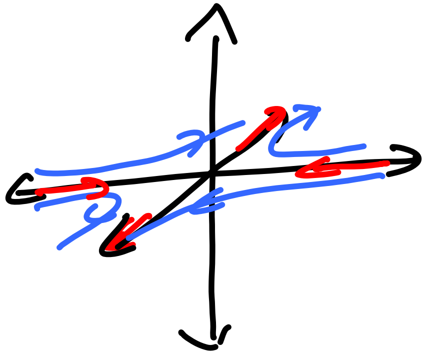




Uniaxial extension



Biaxial extension



Planar extension

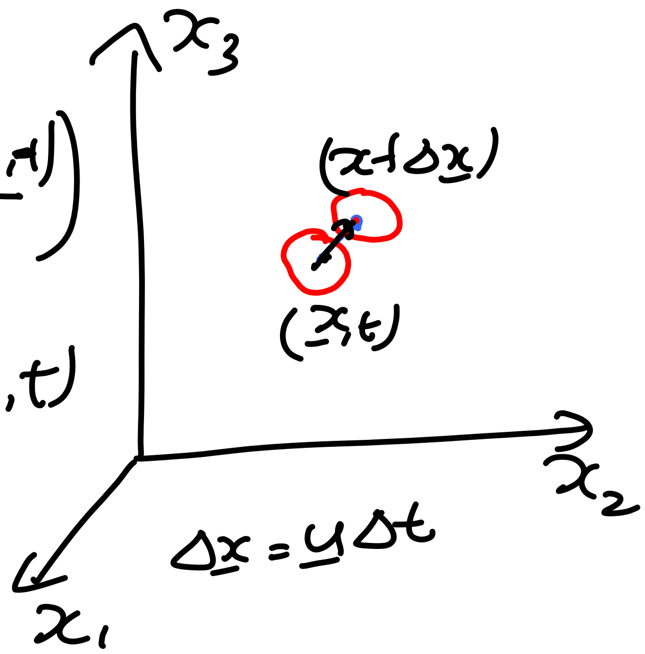
Substantial derivative.

$$\frac{\partial T}{\partial t} = \lim_{\Delta t \to 0} \left(\frac{T(x_1, x_2, x_3, t+\Delta t) - T(x_1, x_2, x_3, t)}{\Delta t} \right)$$

'Eulerian reference frame' $T(\underline{x}, t)$

'Lagrangian reference frame'

$T(X(t), t)$



$$\frac{DT}{Dt} = \lim_{\Delta t \to 0} \left[\frac{T(\underline{x} + \underline{u}\Delta t, t+\Delta t) - T(\underline{x}, t)}{\Delta t} \right]$$

$$= \left[\frac{\partial T}{\partial t} + u_1 \frac{\partial T}{\partial x_1} + u_2 \frac{\partial T}{\partial x_2} + u_3 \frac{\partial T}{\partial x_3} \right] = \frac{\partial T}{\partial t} + \underline{u} \cdot \nabla T$$

Mass in

$= \rho\, ds\, \underline{u}\cdot\underline{n}\, \Delta t$

Rate of mass in

$= \boxed{\rho\, ds\, \underline{u}\cdot\underline{n}}$

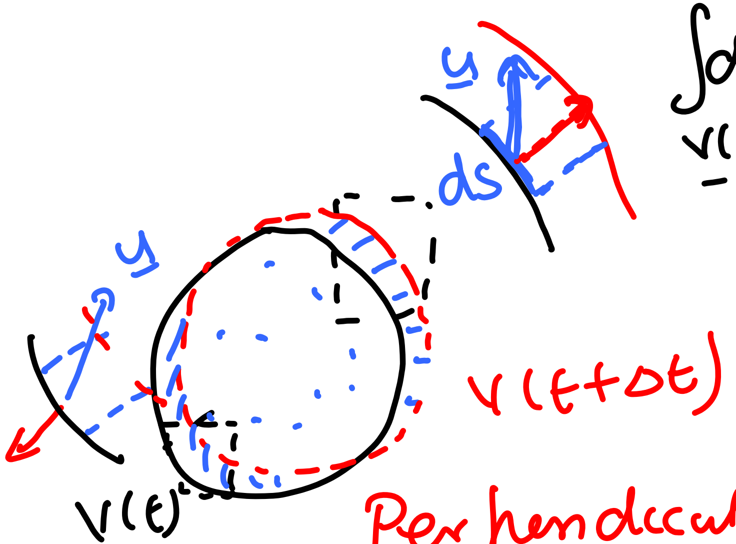
$$\text{Mass} = \int_{V(t)} dV \rho$$

Rate of change of mass $= 0$

$$= \frac{d}{dt}\left[\int_{V(t)} dV \rho\right] = 0$$

$$\int_{V(t)} dV \rho = \int dV \frac{\partial \rho}{\partial t} + \boxed{\int ds\, \rho\, \underline{u}\cdot\underline{n}}$$

Liebniz rule



Perpendicular distance × area

$= \underline{u}\cdot\underline{n}\, \Delta t\, ds$

$$\frac{d}{dt}\int_{V(t)} dV\,\rho = \int dV\,\frac{\partial \rho}{\partial t} + \int dS\,\rho\,\underline{u}\cdot\underline{n} = 0$$

$$\int dV\,\frac{\partial \rho}{\partial t} + \int dV\,\nabla\cdot(\rho\,\underline{u}) = 0$$

Mass conservation equation

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\,\underline{u}) = 0$$

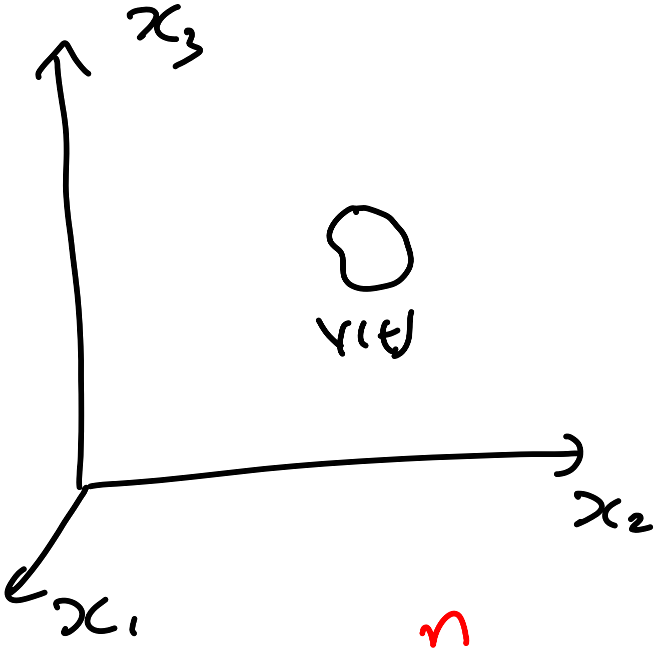
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0$$

$$\frac{\partial \rho}{\partial t} + u_i\frac{\partial \rho}{\partial x_i} + \rho\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{D\rho}{Dt} + \rho\frac{\partial u_i}{\partial x_i} = 0$$

Incompressible

$$\frac{\partial u_i}{\partial x_i} = 0 \qquad \nabla\cdot\underline{u} = 0$$



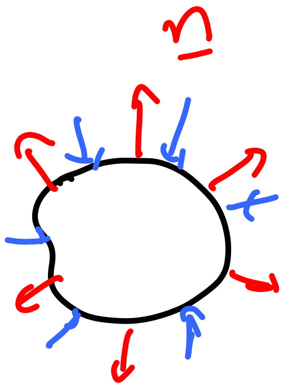
$$\frac{d}{dt}\int dV\, c = -\int ds\, \underline{j}\cdot\underline{n}$$

$$\int dV \left[\frac{\partial c}{\partial t}\right] + \int ds\, c\, \underline{u}\cdot\underline{n} = -\int ds\, \underline{j}\cdot\underline{n}$$

$$\int dV \left(\frac{\partial c}{\partial t}\right) + \int dV \frac{\partial}{\partial x_i}(c u_i) = -\int dV \frac{\partial}{\partial x_i}(j_i)$$

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x_i}(c u_i) = -\frac{\partial j_i}{\partial x_i}$$

$$\frac{\partial c}{\partial t} + \nabla\cdot(\underline{u}\, c) = -\nabla\cdot\underline{j}$$



$$\underline{j} = -D\nabla c$$

$$\frac{\partial c}{\partial t} + \nabla\cdot(\underline{u}\, c) = \nabla\cdot(D\nabla c)$$

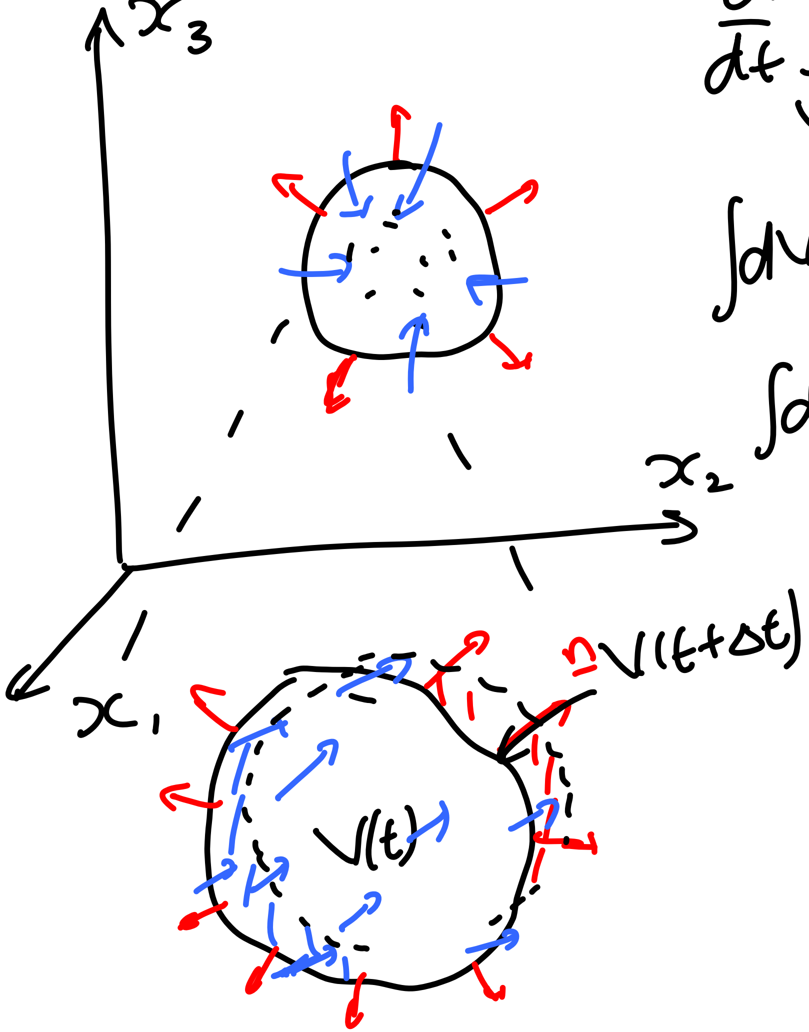
$$= D\nabla^2 c$$

$$\rho c_v \left[\frac{\partial T}{\partial t} + \frac{\partial}{\partial x_i} \left(u_i T \right) \right] = \frac{\partial}{\partial x_i} \left(k \frac{\partial T}{\partial x_i} \right)$$

$$\rho c_v \left[\frac{\partial T}{\partial t} + \nabla \cdot \left(\underline{u} T \right) \right] = \nabla \cdot \left(k \nabla T \right)$$

$$= k \nabla^2 T$$

Mass conservation equation:



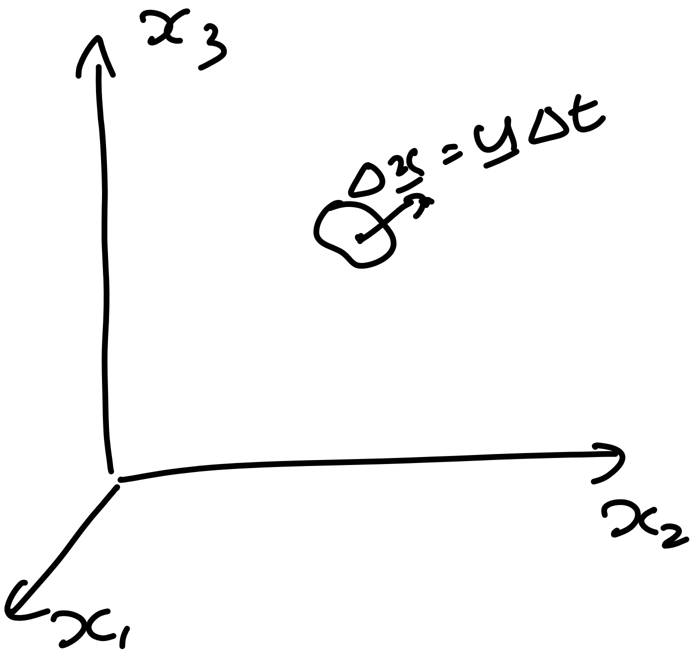
$$\frac{d}{dt}\int_{V(t)} dV\, c = \int dS\,(-\underline{j}\cdot\underline{n}) + \int dV\, S(\underline{x})$$

$$\int dV\left(\frac{\partial c}{\partial t}\right) + \int dS\,(c\,\underline{u}\cdot\underline{n}) = \int dS\,(-\underline{j}\cdot\underline{n}) + \int dV\, S(\underline{x})$$

$$\int dV\left[\frac{\partial c}{\partial t} + \nabla\cdot(c\,\underline{u})\right] = \int dV\left[-\nabla\cdot\underline{j}\right] + \int dV\,(S)$$

$$\frac{\partial c}{\partial t} + \nabla\cdot(c\,\underline{u}) = -\nabla\cdot\underline{j} + S$$

$$\underline{j} = -D\nabla c$$



$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underline{u}) = 0$$

$$\frac{\partial \rho}{\partial t} + \underline{u}\cdot\nabla\rho + \rho\nabla\cdot\underline{u} = 0$$

$$\boxed{\frac{D\rho}{Dt}} + \rho\nabla\cdot\underline{u} = 0$$

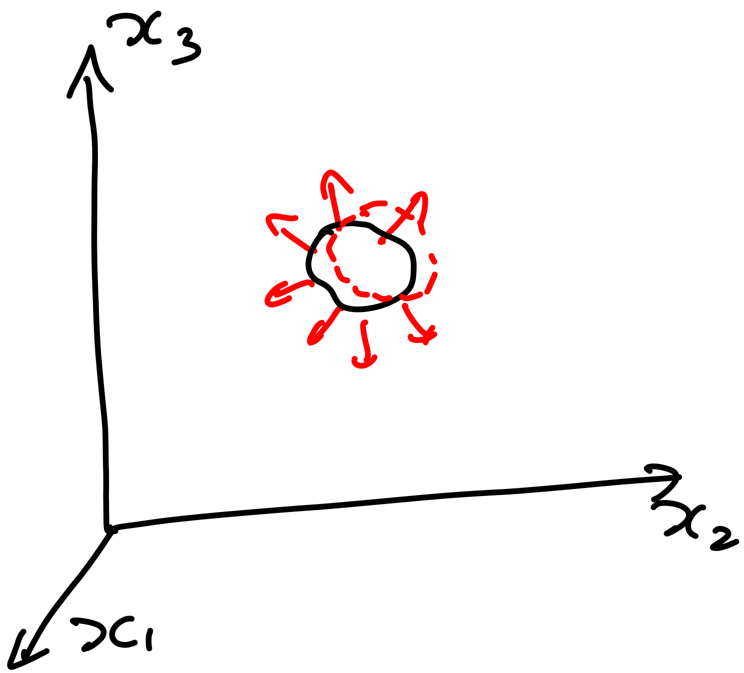
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0$$

Incompressible fluid

$$\nabla\cdot\underline{u} = 0$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

Momentum conservation:

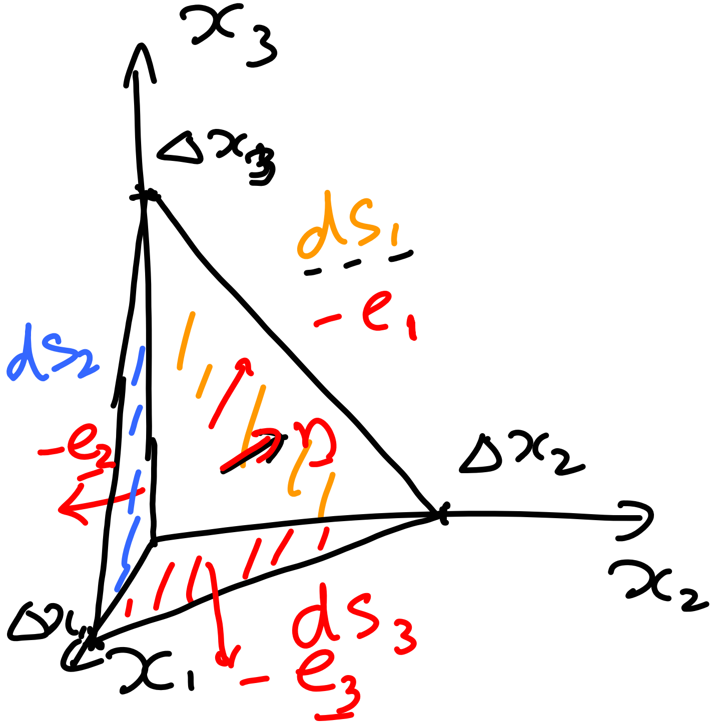


Rate of change of momentum = Sum of applied forces

$$\frac{d}{dt}\int_{V(t)} dV\, \rho u_i = \int_{V(t)} dV\, \rho a_i + \int_S dS\, R_i$$

$$\frac{d}{dt}\int_{V(t)} dV\, \rho u_i = \int_{V(t)} dV\, \frac{\partial}{\partial t}(\rho u_i) + \int dS\, (u_j n_j)(\rho u_i)$$

$$= \int_{V(t)} dV\, \frac{\partial}{\partial t}(\rho u_i) + \int dV\, \frac{\partial}{\partial x_j}(\rho u_i u_j)$$



$$R_i(\underline{n}) = -R_i(-n_i)$$

$$R_i(\underline{n}) = T_{ij}\, n_j$$

Cauchy construction.

Total surface force

$$= ds\, R_i(\underline{n}) + ds_1 R_i(-\underline{e}_1) + ds_2 R_i(-\underline{e}_2) + ds_3 R_i(-\underline{e}_3)$$

Total surface force

$$= ds\, R_i(\underline{n}) - ds_1 R_i(\underline{e}_1) - ds_2 R_i(\underline{e}_2) - ds_3 R(\underline{e}_3) = 0$$

$$\underline{n} = n_1 \underline{e}_1 + n_2 \underline{e}_2 + n_3 \underline{e}_3$$

$$ds_1 = n_1\, ds\,;\quad ds_2 = n_2\, ds\,;\quad ds_3 = n_3\, ds$$

Total surface force

$$= ds\left(R_i(\underline{n}) - n_1 R_i(\underline{e}_1) - n_2 R_i(\underline{e}_2) - n_3 R_i(\underline{e}_3)\right) = 0$$

$$R_i(\underline{n}) = n_1 R_i(\underline{e}_1) + n_2 R_i(\underline{e}_2) + n_3 R_i(\underline{e}_3)$$

$$= T_{i1} n_1 + T_{i2} n_2 + T_{i3} n_3$$

$$= T_{ij} n_j$$

T_{ij} = Force/Area in i direction acting at a surface with outward unit normal in j direction

$$\int ds\, R_i = \int ds\, T_{ij} n_j = \int dv \left(\frac{\partial}{\partial x_j} T_{ij} \right)$$

$$\frac{d}{dt}\int_{V(t)} dV\,(\rho u_i) = \int_{V(t)} dV\,\rho a_i + \int_{V(t)} dS\,R_i$$

$$\int dV \left(\frac{\partial}{\partial t}(\rho u_i)\right) + \int dS\,(n_j\,\rho u_i u_j) = \int dV\,(\rho a_i) + \int dS\,T_{ij}\,n_j$$

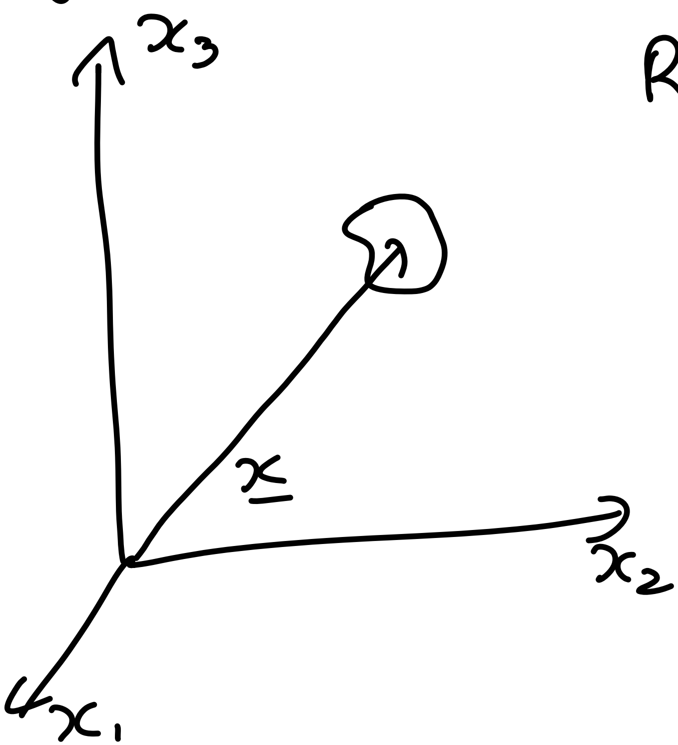
$$\int dV\,\frac{\partial}{\partial t}(\rho u_i) + \int dV\,\frac{\partial}{\partial x_j}(\rho u_i u_j) = \int dV(\rho a_i) + \int dV\,\frac{\partial}{\partial x_j}(T_{ij})$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \rho a_i + \frac{\partial}{\partial x_j}(T_{ij})$$

$$u_i\left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\right) + \rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = \rho a_i + \frac{\partial}{\partial x_j}(T_{ij})$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = \rho a_i + \frac{\partial}{\partial x_j}(T_{ij})$$

Angular momentum $= \underline{x} \times \rho \underline{u} = \epsilon_{ijk} x_j \rho u_k$



Rate of change of angular momentum = Sum of applied torques

$$\frac{d}{dt}\int_{V(t)} dV\left(\epsilon_{ijk}x_j\rho u_k\right) = \int dV\,\epsilon_{ijk}x_j\rho a_k + \int dS\,\epsilon_{ijk}x_j R_k$$

$$= \int dV\,\epsilon_{ijk}x_j\rho a_k + \int dS\,\epsilon_{ijk}x_j T_{kl}n_l$$

$$\int dV\,\frac{\partial}{\partial t}\left(\epsilon_{ijk}x_j\rho u_k\right) + \int dS\,(u_l n_l)(\epsilon_{ijk}x_j\rho u_k)$$

$$= \int dV\,\epsilon_{ijk}x_j\rho a_k + \int dS\,\epsilon_{ijk}x_j T_{kl}n_l$$

$$\int dV\,\epsilon_{ijk}x_j\frac{\partial}{\partial t}(\rho u_k) + \int dV\frac{\partial}{\partial x_l}\left(\epsilon_{ijk}x_j\rho u_k u_l\right)$$

$$= \int dV\left(\epsilon_{ijk}x_j\rho u_k\right) + \int dV\frac{\partial}{\partial x_l}\left(\epsilon_{ijk}x_j T_{kl}n_l\right)$$

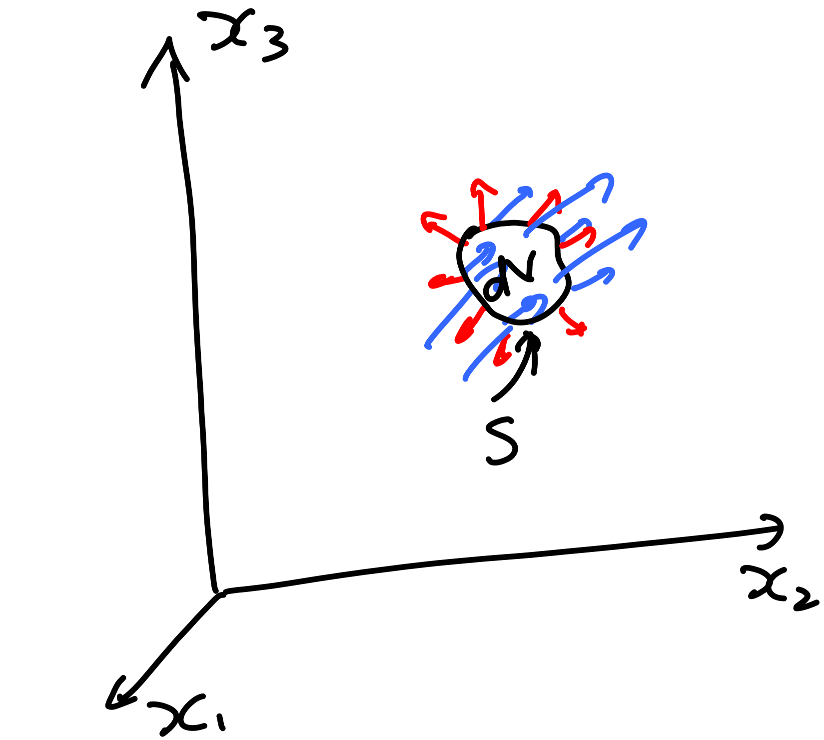
$$\int dV\,\epsilon_{ijk}x_j\frac{\partial}{\partial t}(\rho u_k) + \int dV\,\epsilon_{ijk}\left[x_j\frac{\partial}{\partial x_l}(\rho u_k u_l) + \rho u_k u_l\delta_{jl}\right]$$

$$= \int dV\,\epsilon_{ijk}x_j\rho u_k) + \int dV\,\epsilon_{ijk}\left(x_j\frac{\partial}{\partial x_l}T_{kl} + T_{kl}\delta_{jl}\right)\Big]$$

$$\int dV\, \epsilon_{ijk}\, x_j \left[\frac{\partial}{\partial t}(\rho u_k) + \frac{\partial}{\partial x_l}(\rho u_k u_l)\right] + \int dV\, \epsilon_{ijk}\, x_j\, \rho u_k\, y_j$$

$$= \int dV\, \epsilon_{ijk}\, x_j\, \rho u_r + \int dV \left[\epsilon_{ijk}\, x_j \frac{\partial}{\partial x_l}(T_{kl}) + \epsilon_{ijk} T_{kj}\right]$$

Mass conservation:



$$\frac{d}{dt}\int_{V(t)} dV\, \rho = 0$$

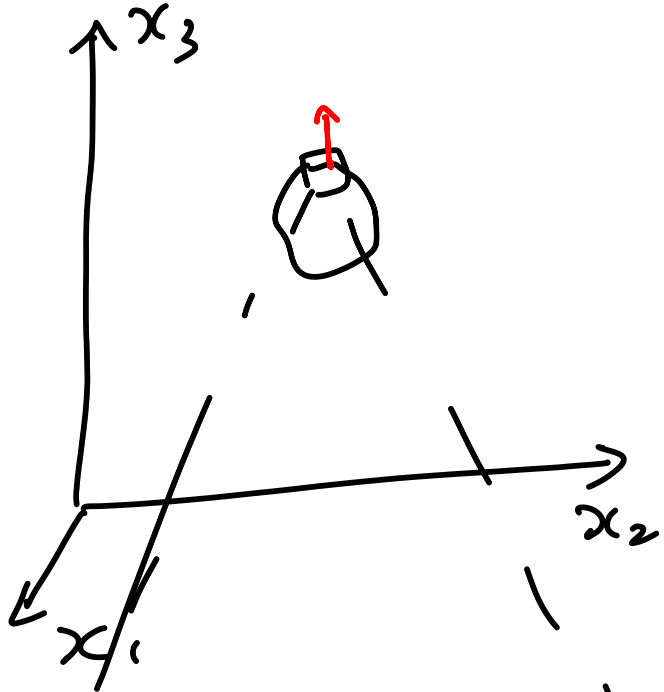
$$\int dV\, \frac{\partial \rho}{\partial t} + \int ds\, \rho \underline{u}.\underline{n} = 0$$

$$\int dV \left[\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \underline{u})\right] = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \underline{u}) = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0$$

Momentum conservation:



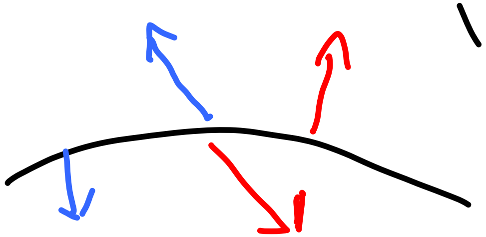
$$\frac{d}{dt}\int_{V(t)} dV(\rho u_i) = \int_V dV \rho a_i + \int_S dS R_i$$

$$\int dV \frac{\partial}{\partial t}(\rho u_i) + \int dS(\rho u_i)(u_j n_j) = \int dV \rho a_i + \int dS R_i$$

$$\int dV \frac{\partial}{\partial t}(\rho u_i) + \int dV \frac{\partial}{\partial x_j}(\rho u_i u_j) = \int dV \rho a_i + \int dS R_i$$

$$\int dV \left(\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j)\right) = \int dV(\rho a_i) + \int dS T_{ij} n_j$$

$$\int dV \frac{\partial}{\partial x_j}(T_{ij})$$



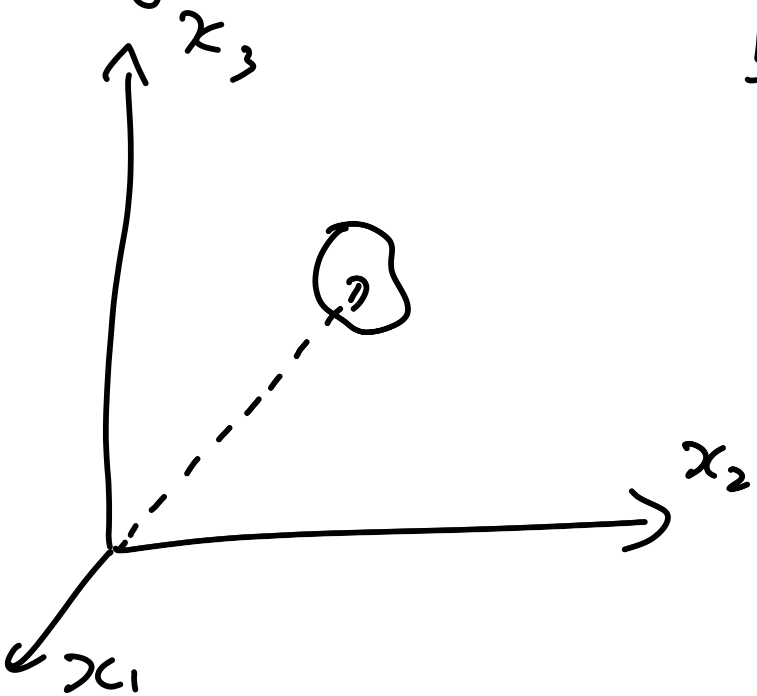
$$R_i(n) = -R_i(-n) = T_{ij} n_j$$

Momentum conservation eqn:

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \rho a_i + \frac{\partial}{\partial x_j} T_{ij}$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = \rho a_i + \frac{\partial}{\partial x_j} T_{ij}$$

Angular momentum conservation:



$$\underline{L} = \int_{V(t)} dV\, \underline{x}\times(\rho \underline{u}) = \int_{V(t)} dV\, \epsilon_{ijk}\, x_j \rho u_k$$

$$L_i = \epsilon_{ijk} \int dV\, x_j \rho u_k$$

$$\frac{d}{dt}\int dV\, \epsilon_{ijk}\, x_j \rho u_k = \int dV\, \epsilon_{ijk}\, x_j \rho a_k + \int dS\, \epsilon_{ijk}\, x_j T_{kl} n_l$$

$$\int dV\, \epsilon_{ijk} \frac{\partial}{\partial t}(x_j \rho u_k) + \int dS\, \epsilon_{ijk}\, x_j \rho u_k u_l n_l = \int dV\, \epsilon_{ijk}\, x_j \rho a_k + \int dV \frac{\partial}{\partial x_l}(\epsilon_{ijk}\, x_j T_{kl})$$

$$\int dV\, \epsilon_{ijk}\, x_j \frac{\partial}{\partial t}(\rho u_k) + \int dV\, \epsilon_{ijk} \frac{\partial}{\partial x_l}(x_j \rho u_k u_l) = \int dV\, \epsilon_{ijk}\, x_j \rho a_k + \int dV\, \epsilon_{ijk} \frac{\partial}{\partial x_l}(x_j T_{kl})$$

$$\int dV\, \epsilon_{ijk} x_j \left[\frac{\partial}{\partial t}(\rho u_k)\right] + \int dV\, \epsilon_{ijk}\left[x_j \frac{\partial}{\partial x_l}(\rho u_k u_l) + \rho u_k u_j\right]$$

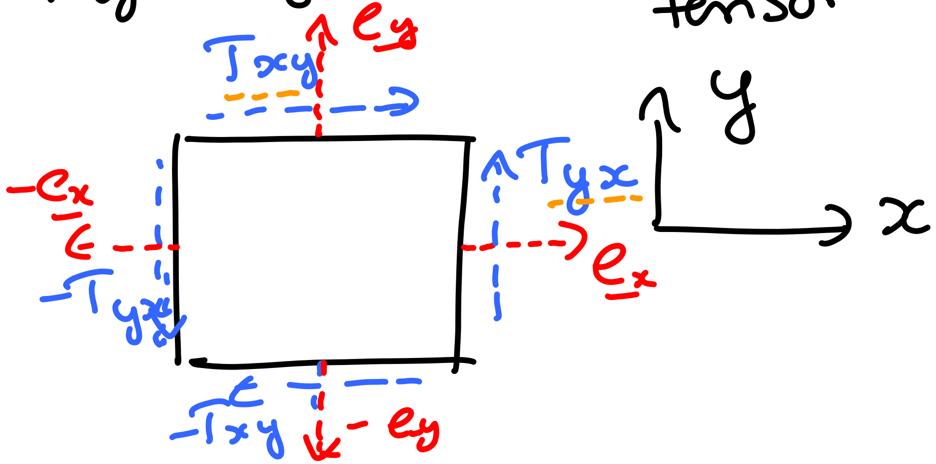
$$= \int dV\, \epsilon_{ijk} x_j (\rho a_k) + \int dV\, \epsilon_{ijk}\left[x_j \frac{\partial}{\partial x_l} T_{kl} + T_{kj}\right]$$

$$\epsilon_{ijk} x_j \left[\frac{\partial}{\partial t}(\rho u_k) + \frac{\partial}{\partial x_l}(\rho u_k u_l)\right] + \cancel{\epsilon_{ijk} \rho u_k u_j}^{\,0}$$

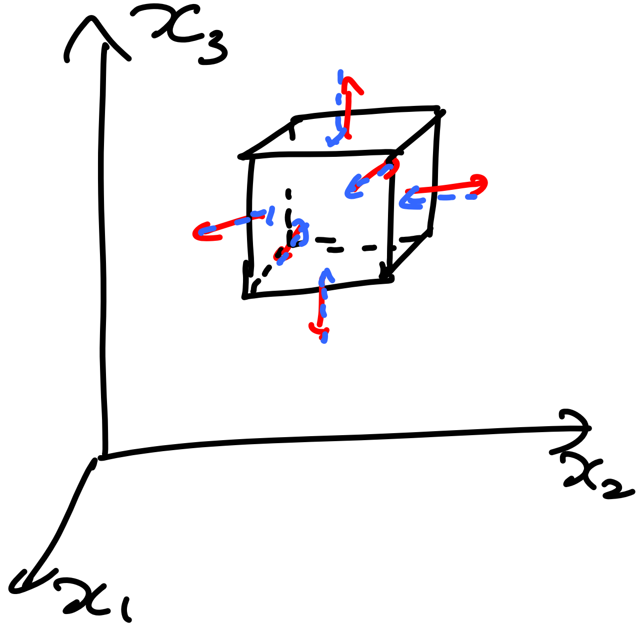
$$= \epsilon_{ijk} x_j\, \rho a_k + \epsilon_{ijk} x_j \frac{\partial}{\partial x_l} T_{kl} + \boxed{\epsilon_{ijk} T_{kj}}$$

$$\epsilon_{ijk} T_{kj} = 0$$

$$T_{ij} = T_{ji} \Rightarrow \text{Symmetric tensor}$$



Stress tensor:

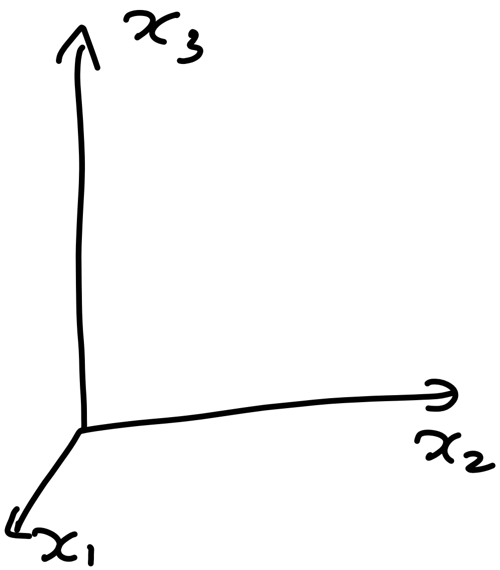


$$\underline{\underline{T}} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{T_{11}+T_{22}+T_{33}}{3} & 0 & 0 \\ 0 & \frac{T_{11}+T_{22}+T_{33}}{3} & 0 \\ 0 & 0 & \frac{T_{11}+T_{22}+T_{33}}{3} \end{pmatrix} + \underline{\underline{\tau}}$$

$T_{ij} = \tau_{ij} + \delta_{ij}\,(-p)$ | $\tau_{ii} = 0$

$$T_{ij} = -p\,\delta_{ij} + \tau_{ij} + \mu_b\,\delta_{ij}\left(\frac{\partial u_k}{\partial x_k}\right)$$



$$\frac{\partial u_i}{\partial x_j} = A_{ij} + E_{ij} + \frac{1}{3}\,\delta_{ij}\left(\frac{\partial u_k}{\partial x_k}\right)$$

Additional postulate:

Stress tensor is linear function of rate of deformation tensor

$$\boxed{\tau_{ij} = 2\mu E_{ij}}$$ Newton's law

μ = Coefficient of viscosity

$$\boxed{\tau_{xy} = \mu \frac{\partial u_x}{\partial y}}$$

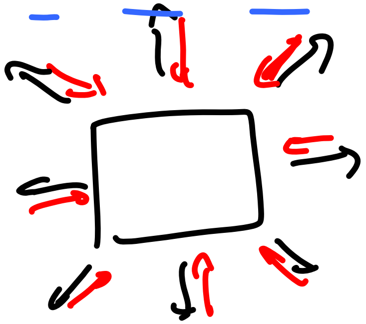
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0$$

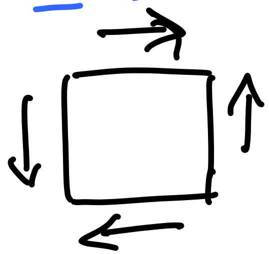
$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = \rho a_i + \frac{\partial}{\partial x_j}\left(-p\delta_{ij} + 2\mu E_{ij} + \mu_b \delta_{ij} \frac{\partial u_k}{\partial x_k}\right)$$

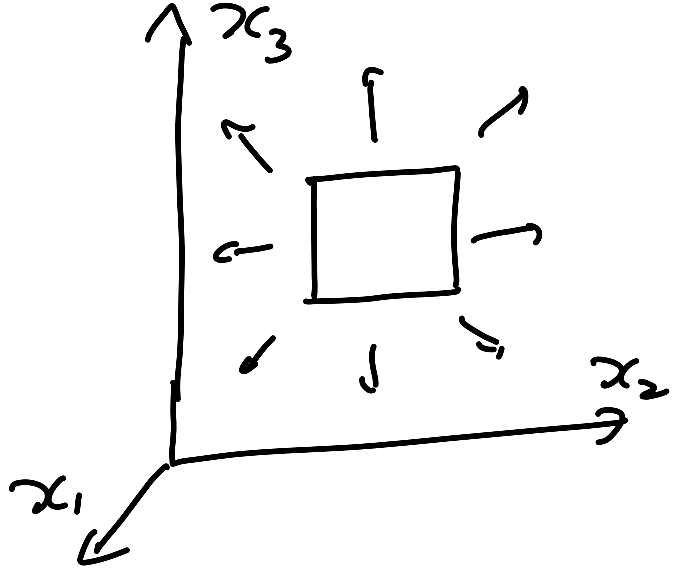
$$\rho \frac{Du_i}{Dt} = \rho a_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}(2\mu E_{ij}) + \frac{\partial}{\partial x_i}\left(\mu_b \frac{\partial u_k}{\partial x_k}\right)$$

Stress Tensor:

$$T_{ij} = \overset{-p\,\delta_{ij}}{\frac{1}{3}\delta_{ij}T_{kk}} + \tau_{ij}$$







$$T_{ij} = -p\,\delta_{ij} + \tau_{ij}$$

$$\frac{\partial u_i}{\partial x_j} = A_{ij} + E_{ij} + \frac{1}{3}\delta_{ij}\left(\frac{\partial u_k}{\partial x_k}\right)$$

$$A_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right); \quad E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}$$

$$\tau_{ij} = 2\mu E_{ij}$$

$$T_{ij} = -p\,\delta_{ij} + 2\mu E_{ij} + \mu_b \left(\frac{\partial u_k}{\partial x_k}\right)\delta_{ij}$$

Newtonian fluid

$$= -p\,\delta_{ij} + 2\mu\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right] + \mu_b\left(\frac{\partial u_k}{\partial x_k}\right)\delta_{ij}$$

$$= -p\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \left(\mu_b - \frac{2}{3}\mu\right)\delta_{ij}\left(\frac{\partial u_k}{\partial x_k}\right)$$

$$T_{ij} = \mu^{(g)}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - p\,\delta_{ij}$$

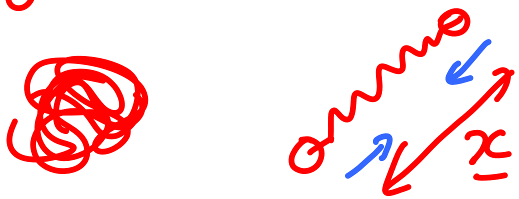
$\mu^{(g)}$ = fn.(rate of deformation tensor)

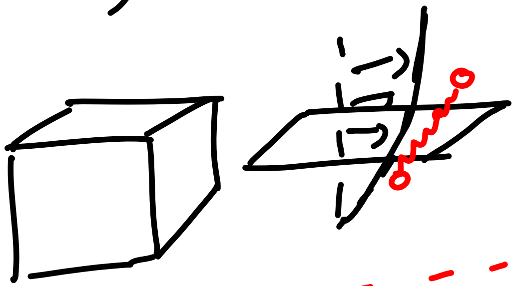
$$\underline{\underline{E}} = \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix}$$

$I_1 = \mathrm{Trace}(\underline{\underline{E}})$; $I_2 = \underline{\underline{E}} : \underline{\underline{E}}$; $I_3 = \mathrm{Det}(\underline{\underline{E}})$

$$u^{(\theta)} = u^{(\theta)}(I_1, I_2, I_3)$$

Polymer solution:





$\underline{\underline{q}} = \underline{x}\,\underline{x}$

$q_{ij} = x_i x_j$

$$\underline{\underline{I}}^{p} = \frac{\eta_0 k_n}{\tau k T}\left(\underline{\underline{q}} - \underline{\underline{q}}^{eq}\right)$$

Mass conservation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = \rho a_i + \frac{\partial}{\partial x_j}(T_{ij})$$

$$\frac{\partial}{\partial x_j}(T_{ij}) = \frac{\partial}{\partial x_j}\left(-p\delta_{ij} + 2\mu E_{ij} + \mu_b \delta_{ij}\frac{\partial u_k}{\partial x_k}\right)$$

$$= -\frac{\partial p}{\partial x_i} + 2\mu \frac{\partial}{\partial x_j}\left(\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)$$

$$+ \mu_b \frac{\partial}{\partial x_j}\left(\frac{\partial u_k}{\partial x_k}\right)\delta_{ij}$$

$$= -\frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) + \mu \frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_i}\right)$$

$$- \frac{2}{3}\mu \frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right) + \mu_b \frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right)$$

$$\frac{\partial T_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} + \left(\mu_b + \frac{1}{3}\mu\right)\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right)$$

$$\rho\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)=-\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 u_i}{\partial x_j^2}+\left(\mu_b+\frac{1}{3}\mu\right)\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right)$$

$$\rho\left(\frac{\partial \underline{u}}{\partial t}+\underline{u}\cdot\nabla\,\underline{u}\right)=-\nabla p+\mu\nabla^2\underline{u}+\left(\mu_b+\frac{1}{3}\mu\right)\nabla(\nabla\cdot\underline{u})$$

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_i}(\rho u_i)=0 \qquad \frac{\partial \rho}{\partial t}+u_i\frac{\partial \rho}{\partial x_i}+\rho\frac{\partial u_i}{\partial x_i}=0$$

$$\frac{D\rho}{Dt}+\rho\frac{\partial u_i}{\partial x_i}=0$$

Equation of state

$$p=nkT=\frac{\rho}{m}kT$$

Incompressible: $\rho=$ constant

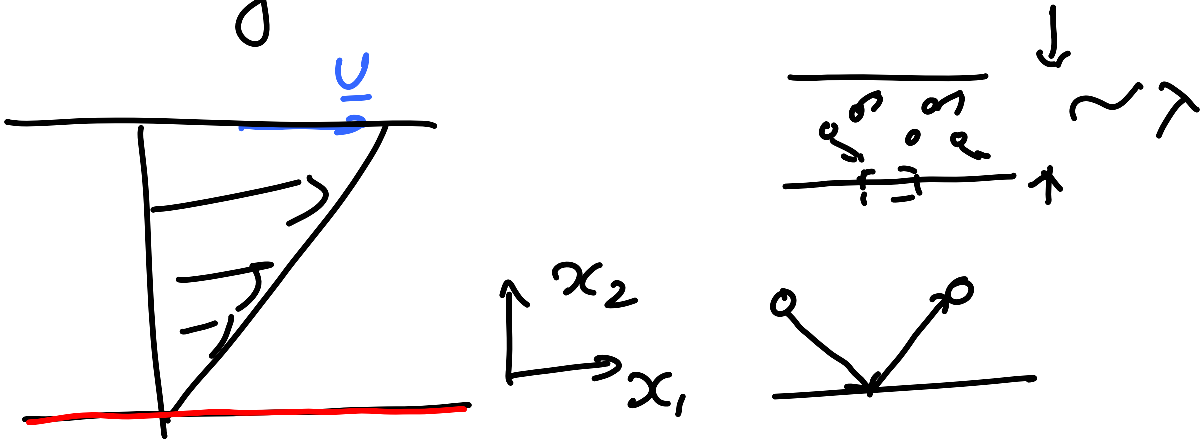
$\nabla\cdot\underline{u}=0$ Mass conservation

$$\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\frac{\mu}{\rho}\frac{\partial^2 u_i}{\partial x_j^2}$$

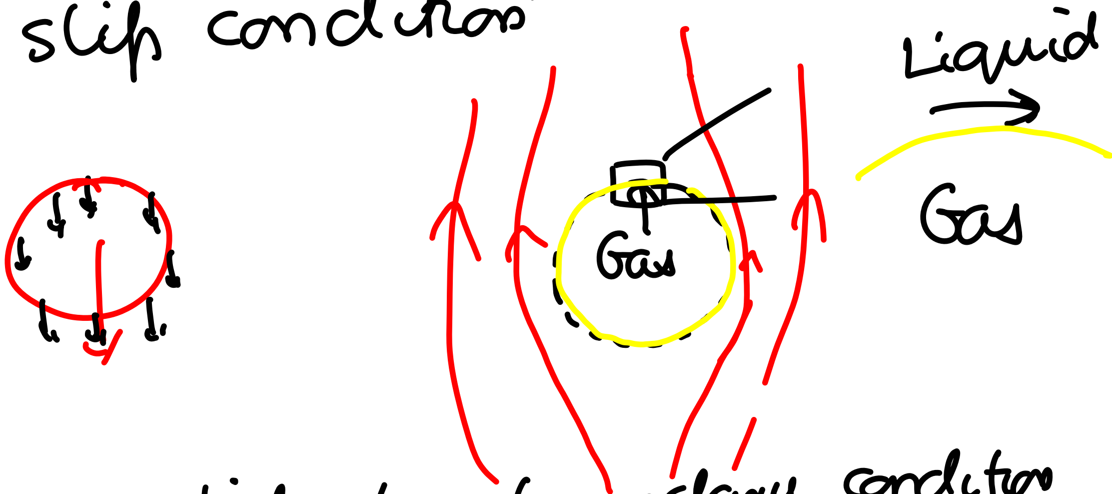
Navier Stokes eqns.

$$\frac{\partial c}{\partial t}+\nabla\cdot(\underline{u}c)=D\nabla^2 c$$

Boundary conditions:



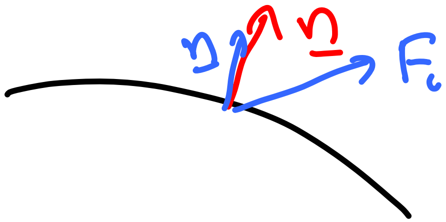
'No-slip condition'



Zero tangential stress boundary condition

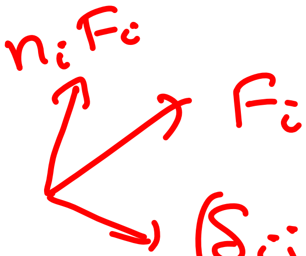
$$\left(\text{Normal stress outside}\right) - \left(\begin{matrix}\text{Normal stress} \\ \text{inside}\end{matrix}\right) = \gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$F_i = T_{ij} n_j$$



$$n_i F_i = n_i T_{ij} n_j$$

$$n_i T_{ij} n_j \Big|_{\text{liquid}} - n_i T_{ij} n_j \Big|_{\text{gas}} = \gamma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$$

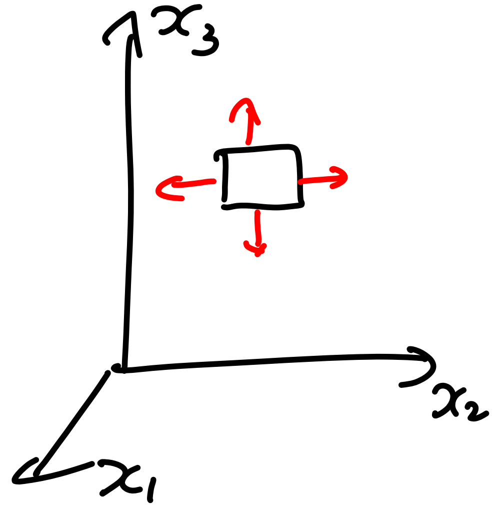


$$(\delta_{ij} - n_i n_j) F_j = (\delta_{ij} - n_i n_j) T_{jk} n_k = 0$$

$$(\delta_{ij} - n_i n_j) F_j$$

$$n_i (\delta_{ij} - n_i n_j) = n_i - n_i n_j^2$$

Mass conservation eqn:



$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0$$

$$\frac{D\rho}{Dt} + \rho \frac{\partial u_i}{\partial x_i} = 0 \qquad \frac{\partial u_i}{\partial x_i} = \nabla \cdot \underline{u} = 0$$

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \frac{\partial T_{ij}}{\partial x_j} + \rho a_i$$

$$T_{ij} = T_{ji} = -p\,\delta_{ij} + \tau_{ij} + \mu_b\,\delta_{ij}\frac{\partial u_k}{\partial x_k}$$

$$\tau_{ij} = 2\mu E_{ij} = 2\mu\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right]$$

$$\frac{\partial u_i}{\partial x_i} = 0 \qquad \nabla \cdot \underline{u} = 0$$

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho u_i u_j\right) = \frac{\partial}{\partial x_j}\left(-p\delta_{ij} + 2\mu E_{ij}\right) + \rho a_i$$

$$\rho\frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} + u_i\left(\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)}\right)$$

$$= -\frac{\partial p}{\partial x_i} + \cancel{2}\mu \frac{\partial}{\partial x_j}\left(\frac{1}{\cancel{2}}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right) + \rho a_i$$

$$= -\frac{\partial p}{\partial x_i} + \mu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) + \boxed{\mu\frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_i}\right)} + \rho a_i$$

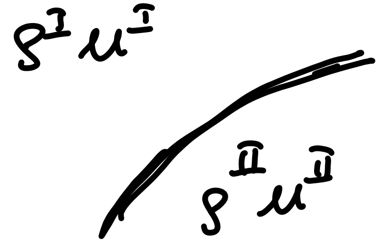
$$= -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2}{\partial x_j^2}(u_i)$$

Navier-Stokes mass & momentum eqns for an incompressible fluid:

$$\nabla \cdot \underline{u} = 0 \qquad \frac{\partial u_i}{\partial x_i} = 0$$

$$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2}$$

$$\rho \left(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} \right) = -\nabla p + \mu \nabla^2 \underline{u}$$

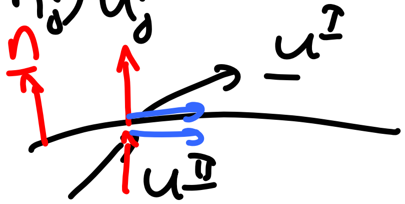


Boundary conditions:

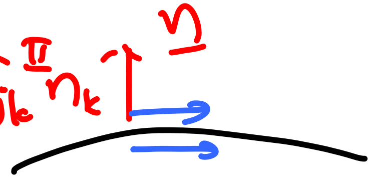
Continuity of velocity & stress

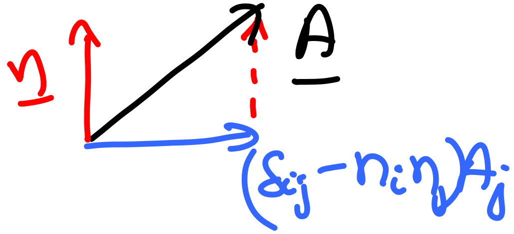
$$u_i^{I} n_i = u_i^{II} n_i \qquad (\delta_{ij} - n_i n_j) u_j^{I} = (\delta_{ij} - n_i n_j) u_j^{II}$$

$$\underline{u}^{I} \cdot \underline{n} = \underline{u}^{II} \cdot \underline{n} \qquad (\underline{\underline{I}} - \underline{n}\,\underline{n}) \cdot \underline{u}^{I} = (\underline{\underline{I}} - \underline{n}\,\underline{n}) \cdot \underline{u}^{II}$$



$$(\delta_{ij} - n_i n_j) T_{jk}^{I} n_k = (\delta_{ij} - n_i n_j) T_{jk}^{II} n_k$$





$$(\delta_{ij} - n_i n_j) \tau_{jk}^{I} n_k = (\delta_{ij} - n_i n_j) \tau_{jk}^{II} n_k$$

$$n_j T_{jk}^{I} n_k = n_j T_{jk}^{II} n_k + \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

Solid - Fluid interface $\qquad \underline{u} = \underline{U} + \underline{R} \times \underline{\Omega}$

Gas - liquid interface: $(\delta_{ij} - n_i n_j) \tau_{jk} n_k = 0$

$$n_j T_{jk}^{I} n_k - n_j T_{jk}^{II} n_k = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

$$n_j T_{jk}^{I} n_k - \left(-p^{II} \right) = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

$$u_i \times \rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\tau_{ij} + \rho a_i$$

$$\rho\left(\frac{\partial}{\partial t}\left(\tfrac{1}{2}u_i^2\right) + u_j \frac{\partial}{\partial x_j}\left(\tfrac{1}{2}u_i^2\right)\right) = -u_i \frac{\partial p}{\partial x_i} + u_i \frac{\partial}{\partial x_j}(\tau_{ij}) + \rho a_i u_i$$

$$\frac{1}{2}u_i^2\left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\right) = 0$$

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u_i^2\right) + \frac{\partial}{\partial x_j}\left(\tfrac{1}{2}\rho u_i^2 u_j\right) = -\frac{\partial}{\partial x_i}(p u_i) + p\frac{\partial u_i}{\partial x_i} + \frac{\partial}{\partial x_j}(\tau_{ij} u_i) - \tau_{ij}\frac{\partial u_i}{\partial x_j} + \rho a_i u_i$$

$$\frac{\partial}{\partial t}(e_k) + \frac{\partial}{\partial x_j}(u_j e_k) = -\frac{\partial}{\partial x_i}(p u_i) + \frac{\partial}{\partial x_j}(\tau_{ij} u_i) + p\frac{\partial u_i}{\partial x_i} - \tau_{ij}\frac{\partial u_i}{\partial x_j} + \rho a_i u_i$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot u = 0 \qquad -\frac{p}{\rho}\frac{D\rho}{Dt}$$

$$D = \underline{\tau_{ij} \frac{\partial u_i}{\partial x_j}} = 2\mu E_{ij} \left(\frac{\partial u_i}{\partial x_j} \right)$$

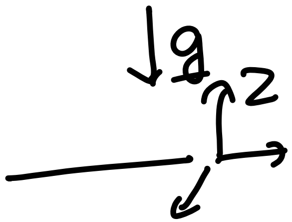
$$= 2\mu E_{ij} \left(S_{ij} + A_{ij} \right)$$

$$= 2\mu E_{ij} S_{ij}$$

$$= 2\mu \left[S_{ij} - \frac{1}{3} \cancelto{0}{\delta_{ij} S_{kk}} \right] S_{ij}$$

$$\geqslant 0$$

Navier-Stokes equations.



$$\nabla \cdot \underline{u} = 0$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} + \rho a_i$$

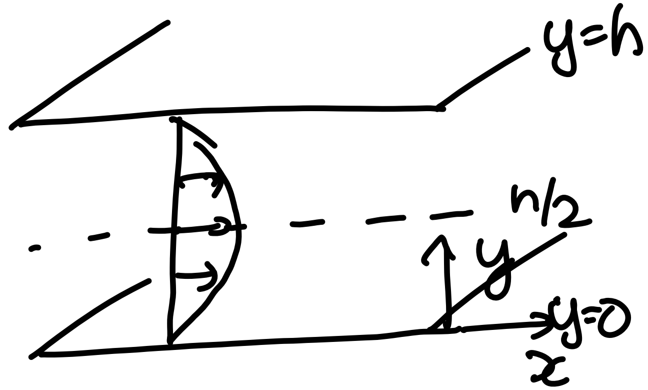
Hydrostatics:

$$-\frac{\partial p}{\partial x_i} + \rho a_i = 0 \Rightarrow \frac{\partial p}{\partial x_i} = \rho a_i$$

$$p = p_0 + \rho a_j x_j \Rightarrow \frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_i}(\rho a_j x_j) = \rho a_j \delta_{ij}$$

$$= p_0 - \rho g z$$

Unidirectional flows:



$$\nabla \cdot \underline{u} = 0 \Rightarrow \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2}$$

Independent of y

$$\rho\left(\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right)$$

$$\rho\left(\frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\right)$$

$$\frac{\partial u_x}{\partial x} = 0 \qquad\qquad 0 = -\frac{\partial p}{\partial y}$$

'Fully developed'

$$\rho \frac{\partial u_x}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u_x}{\partial y^2}$$

Fully developed steady $-\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u_x}{\partial y^2} = 0$

$$u_x = -\frac{h^2}{2\mu}\frac{\partial p}{\partial x}\left(\frac{y}{h}\right)\left(1-\frac{y}{h}\right)$$

$$u_x^{max} = -\frac{\partial p}{\partial x}\frac{h^2}{8\mu}$$

$$u_x = 4\,u_x^{max}\left(\frac{y}{h}\right)\left(1-\frac{y}{h}\right)$$

$$u_x^{av} = \frac{2}{3}u_x^{max}$$

$$f = \left(-\frac{\partial p}{\partial x}\right)\Big/\left(\frac{8u^2_{av}}{2h}\right) = \frac{24}{Re}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)+\frac{\partial u_x}{\partial x}=0 \qquad \frac{\partial u_x}{\partial x}=0$$

Unidirectional $\Rightarrow$ Fully developed

$$\rho\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)=-\frac{\partial p}{\partial x_i}+\mu\nabla^2 u_i$$

$$\rho\left(\frac{\partial u_x}{\partial t}+\boxed{u_x\frac{\partial u_x}{\partial x}}+\boxed{u_r\frac{\partial u_x}{\partial r}}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_x}{\partial r}\right)+\boxed{\frac{\partial^2 u_x}{\partial x^2}}\right)$$

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+u_x\frac{\partial u_r}{\partial x}\right)=-\frac{\partial p}{\partial r}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)+\frac{\partial^2 u_r}{\partial x^2}\right)$$

$$-\frac{\partial p}{\partial r}=0$$

$$\boxed{\rho\left(\frac{\partial u_x}{\partial t}\right)}=-\boxed{\frac{\partial p}{\partial x}}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_x}{\partial r}\right)\right]$$

$$0=-\frac{\partial p}{\partial x}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_x}{\partial r}\right)\right)$$

$$u_x = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right)\left(1 - \frac{r^2}{R^2}\right)$$

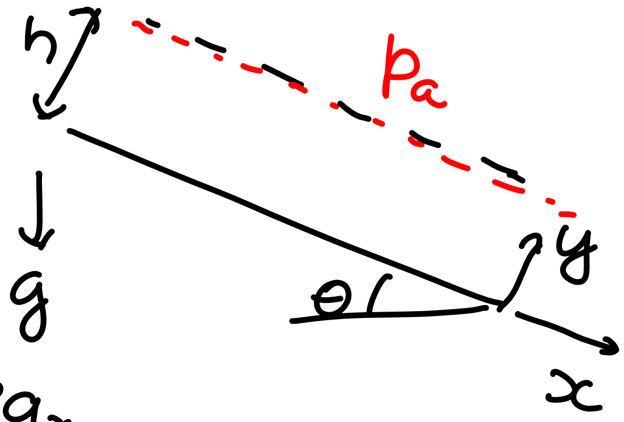
$$u_x^{max} = -\frac{R^2}{4\mu}\frac{\partial p}{\partial x}$$

$$u_x^{av} = \frac{u_x^{max}}{2}$$

$$f = \left(-\frac{\partial p}{\partial x}\right) \Big/ \left(\rho u_{av}^2 / 2D\right) = \frac{64}{Re}$$

$$= \left(-\frac{\partial p}{\partial x}\right) \Big/ \left(2\rho u_{av}^2 / D\right) = \frac{16}{Re}$$

Flow down inclined plane



$$\frac{\partial u_x}{\partial x} + \cancel{\frac{\partial u_y}{\partial y}} = 0$$

$$\rho\left(\frac{\partial u_x}{\partial t} + \boxed{u_x \frac{\partial u_x}{\partial x}} + \boxed{u_y \frac{\partial u_x}{\partial y}}\right) = \boxed{-\frac{\partial p}{\partial x}} + \mu\left(\boxed{\frac{\partial^2 u_x}{\partial x^2}} + \frac{\partial^2 u_x}{\partial y^2}\right) + \rho g_x$$

$$\rho\left(\frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\right) + \rho g_y$$

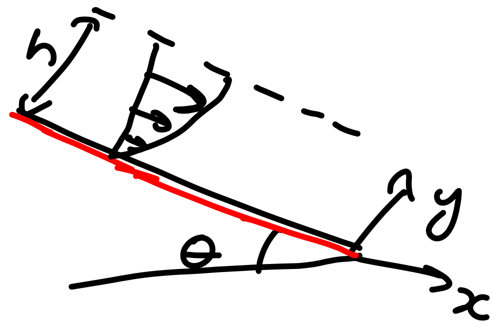
$$-\frac{\partial p}{\partial y} + \rho g_y = 0 \Rightarrow p = p_0(x) + \rho g_y y$$

$$p = p_0 + \rho g_y y$$

Normal stress : At $y = h$, $p = p_a$

$$p_0(x) = p_a - \rho g_y h \Bigg] \Rightarrow \frac{\partial p_0}{\partial x} = 0$$

$$\rho \frac{\partial u_x}{\partial t} = \mu \frac{\partial^2 u_x}{\partial y^2} + \rho g_x$$



Steady $\mu \frac{\partial^2 u_x}{\partial y^2} + \rho g_x = 0$

$$u_x = -\frac{\rho g_x}{\mu}\frac{y^2}{2} + C_0 + C_1 y$$

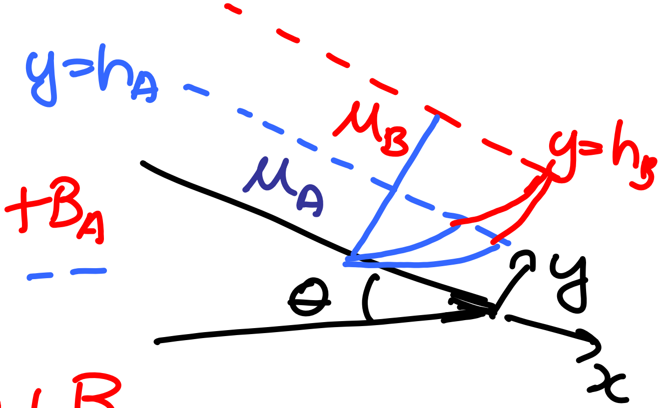
At $y=0$, $u_x = 0$

At $y=h$, $\mu \frac{\partial u_x}{\partial y} = 0$

$$u_x = \left(\frac{\rho g_x h^2}{\mu}\right)\left(\frac{y}{h} - \frac{y^2}{2h^2}\right)$$

$$u_x^{max} = \frac{\rho g_x h^2}{2\mu}$$

Flow down inclined plane:



$$\mu_A \frac{\partial^2 u_x^A}{\partial y^2} = -\rho g_x \qquad u_x^A = -\frac{\rho g_x y^2}{2\mu_A} + A_A y + B_A$$

$$\mu_B \frac{\partial^2 u_x^B}{\partial y^2} = -\rho g_x \qquad u_x^B = -\frac{\rho g_x y^2}{2\mu_B} + A_B y + B_B$$

At $y=0$, $u_x^A = 0 \implies B_A = 0$

At $y=h_B$, $\mu_B \frac{\partial u_x^B}{\partial y} = 0 \implies -\frac{\rho g_x h}{\mu_B} + A_B = 0$

At $y=h_A$ $\quad \mu_A \left(\frac{\partial u_x^A}{\partial y}\right) = \mu_B \left(\frac{\partial u_x^B}{\partial y}\right)$

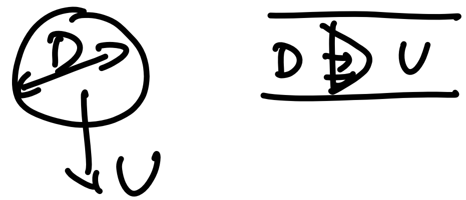
$$u_x^A = u_x^B$$

$$A_A = \rho g_x \left(h_A \left(\frac{1}{\mu_A} - \frac{1}{\mu_B}\right) + \frac{h_B}{\mu_B}\right)$$

$$B_B = \frac{\rho g_x h_A^2}{2} \left(\frac{1}{\mu_A} - \frac{1}{\mu_B}\right)$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2}$$



$$Re = \frac{\text{Inertia}}{\text{Viscosity}} = \frac{\rho U L}{\mu} = \frac{UL}{\nu}$$

Characteristic Length L

Characteristic velocity U

$x_i^* = x_i/L$

$u_i^* = u_i/U$

$t^* = (tU/L)$

$p^* = (p/(\mu U/L))$

$p^{**} = (p/(\rho U^2))$

$$\frac{U}{L}\frac{\partial u_i^*}{\partial x_i^*} = 0 \qquad \frac{\partial u_i^*}{\partial x_i^*} = 0$$

$$\frac{\rho U^2}{L}\left(\frac{\partial u_i^*}{\partial t^*} + u_j^* \frac{\partial u_i^*}{\partial x_j^*}\right) = -\frac{1}{L}\frac{\partial p}{\partial x_i^*} + \frac{\mu U}{L^2}\frac{\partial^2 u_i}{\partial x_j^2}$$

$$Re\left(\frac{\rho U L}{\mu}\right)\left(\frac{\partial u_i^*}{\partial t^*} + u_j^* \frac{\partial u_i^*}{\partial x_j^*}\right) = -\frac{\partial p^*}{\partial x_i^*} + \frac{\partial^2 u_i^*}{\partial x_j^{*2}}$$

$$\frac{\partial u_i^*}{\partial t^*} + u_j^* \frac{\partial u_i^*}{\partial x_j^*} = -\frac{\partial p^{**}}{\partial x_i^*} + \frac{1}{Re}\frac{\partial^2 u_i^*}{\partial x_j^{*2}}$$

① Low Reynolds number:

Neglect inertia

$$0 = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2}$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

② High Reynolds number Potential flow

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_i}\right) = -\frac{\partial p}{\partial x_i}$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

③ Boundary layer theory.

Navier-Stokes equations:

$$\nabla \cdot \underline{u} = 0 \qquad \frac{\partial u_i}{\partial x_i} = 0$$

$$\rho \left(\frac{\partial u_i}{\partial t} + \boxed{u_j \frac{\partial u_i}{\partial x_j}} \right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2}$$

$$u_i^* = (u_i/U); \quad x_i^* = (x_i/L); \quad t^* = t/(L/U)$$

$$\boxed{Re \left(\frac{\partial u_i^*}{\partial t^*} + u_j^* \frac{\partial u_i^*}{\partial x_j^*} \right)} = -\frac{\partial p^*}{\partial x_i^*} + \frac{\partial^2 u_i^*}{\partial x_j^{*2}}$$

$$p^* = p/(\mu U/L); \quad Re = \frac{\rho U L}{\mu} = \frac{UL}{\nu}$$

Small Reynolds number $Re \ll 1$

Stokes equations:

$$\nabla \cdot \underline{u} = 0 \qquad \frac{\partial u_i}{\partial x_i} = 0$$

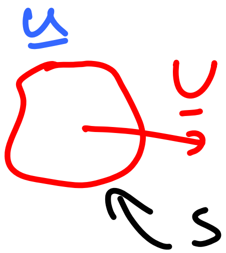
$$-\nabla p + \mu \nabla^2 \underline{u} = 0 \qquad -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} = 0$$

$$T_{ij} = -p\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Linear Quasi-steady.

$$D\nabla^2 C = 0$$

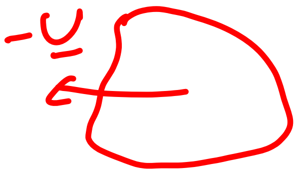
$$\alpha \nabla^2 T = 0$$



$$\left[\nabla\cdot\underline{u} = 0\right]$$

$$\left[-\nabla p + \mu\nabla^2\underline{u} = 0\right]$$

$\underline{u} = \underline{U}$ on the surface S



$$\nabla\cdot\underline{u}' = 0$$

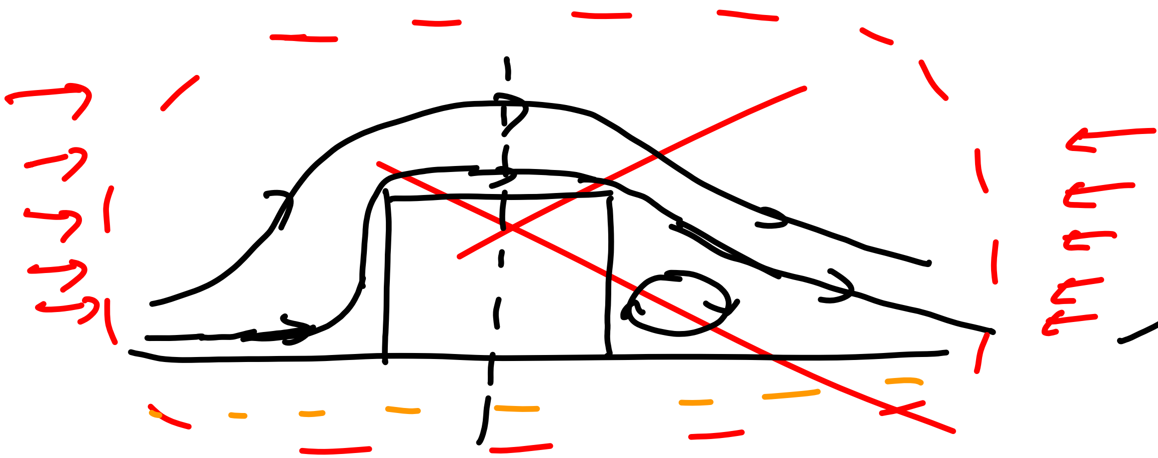
$$-\nabla p' + \mu\nabla^2\underline{u}' = 0$$

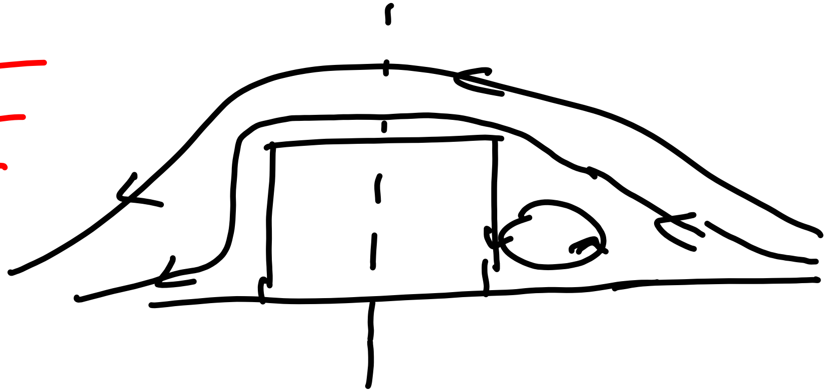
$\underline{u}' = -\underline{U}$ on the surface S

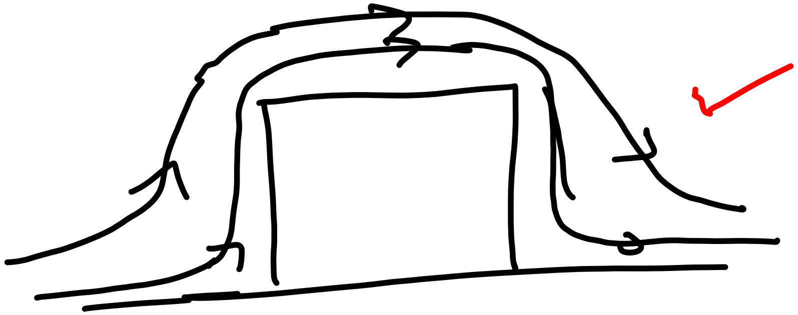
$\underline{u}' = -\underline{u}$

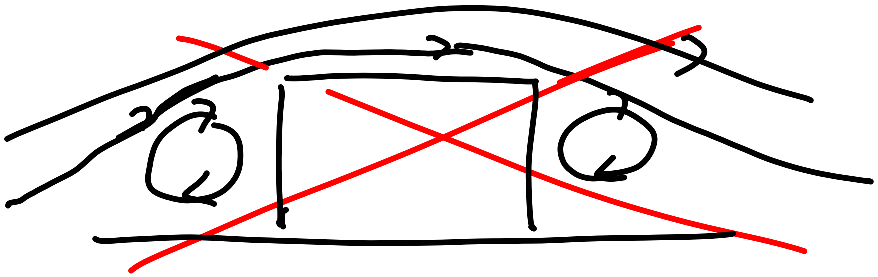
$p' = -p$

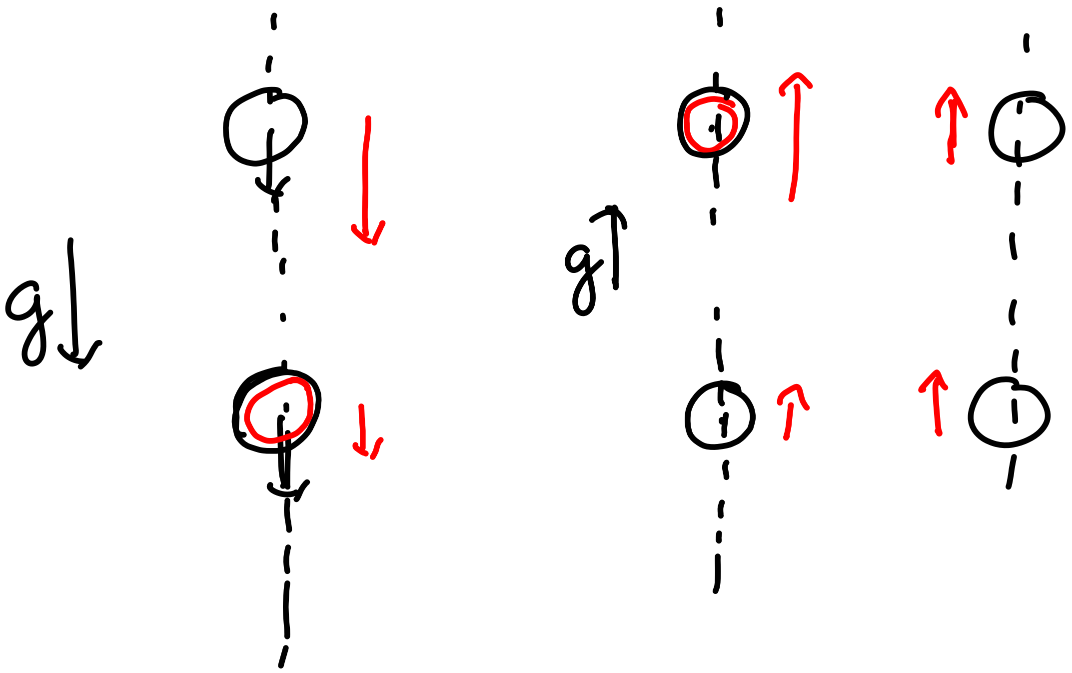
$\alpha = -1$

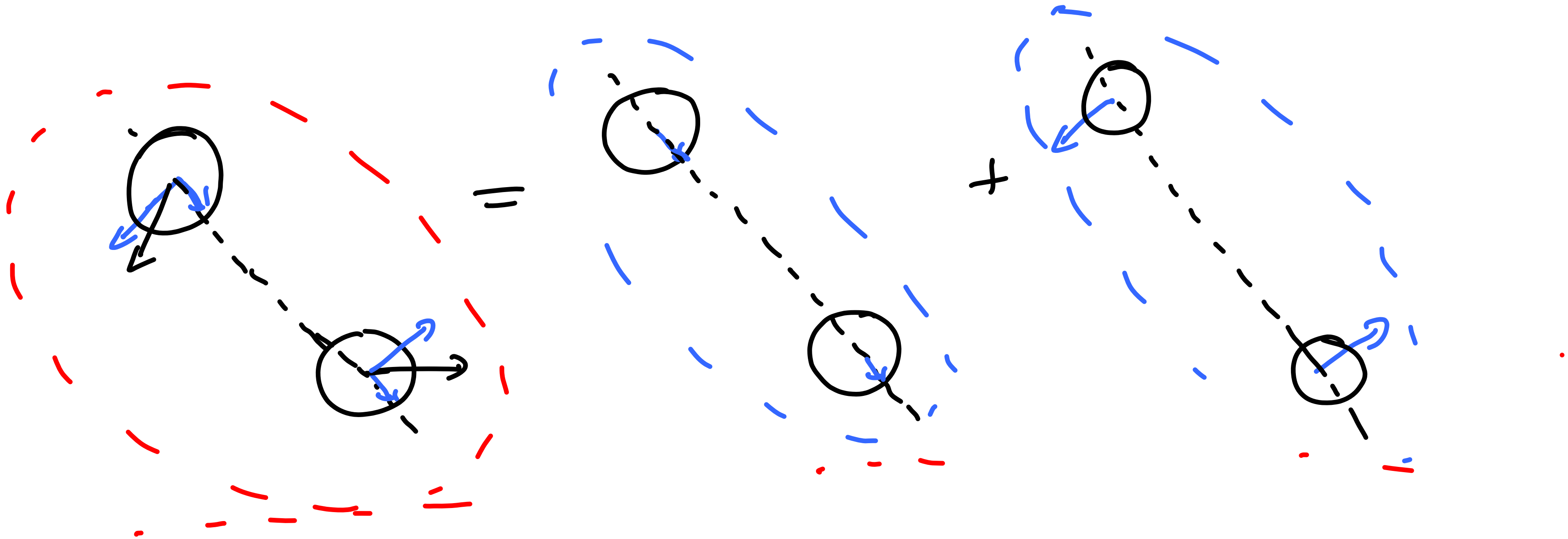












Linearity $\Longrightarrow$ Superposition

$$\nabla \cdot \underline{u} = 0 \qquad \frac{\partial u_i}{\partial x_i} = 0$$

$$-\nabla p + \mu \nabla^2 \underline{u} = 0 \qquad \frac{\partial}{\partial x_i}\left[-\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} = 0\right]$$

$$\nabla^2 \tau = 0$$

$$-\frac{\partial^2 p}{\partial x_i^2} + \mu \frac{\partial}{\partial x_i}\left(\frac{\partial^2 u_i}{\partial x_j^2}\right) = 0$$

$$-\frac{\partial p}{\partial x_i^2} + \mu \frac{\partial^2}{\partial x_j^2}\left(\frac{\partial u_i}{\partial x_i}\right) = 0$$

$$\frac{\partial^2 p}{\partial x_i^2} = 0 \Rightarrow \nabla^2 p = 0$$

$$-\nabla p + \mu \nabla^2 \underline{u} = 0$$

$$\nabla^2 \underline{u}^g = 0$$

$$-\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i^p}{\partial x_j^2} = 0$$

$$u_i^p = c\, p\, x_i$$

$$\frac{\partial u_i^p}{\partial x_j} = c\left[x_i \frac{\partial p}{\partial x_j} + p\,\delta_{ij}\right]$$

$$\frac{\partial}{\partial x_j}\left[\frac{\partial u_i^p}{\partial x_j}\right] = c\left[x_i \frac{\partial^2 p}{\partial x_j^2} + \frac{2\,\partial p\,\delta_{ij}}{\partial x_j}\right]$$

$$\frac{\partial}{\partial x_j}\left[\frac{\partial u_i^{p}}{\partial x_j}\right] = 2c\,\frac{\partial p}{\partial x_i} = \frac{1}{\mu}\frac{\partial p}{\partial x_i}$$

$$u_i^{p} = \frac{1}{2\mu}\,p\,x_i \qquad u_i = u_i^{g} + \frac{1}{2\mu}\,p\,x_i$$

Low Reynolds number viscous flows:

$$\nabla \cdot \underline{u} = 0 \qquad \frac{\partial u_i}{\partial x_i} = 0$$

$$-\nabla p + \mu \nabla^2 \underline{u} = 0 \qquad -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} = 0$$

$$\underline{\underline{T}} = -\nabla p + \mu\left(\nabla \underline{u} + \nabla \underline{u}^T\right) \qquad T_{ij} = -p\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

'Quasi-steady' 'Linear'

$$\frac{\partial}{\partial x_i}\left[-\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2}\right] = 0$$

$$\frac{\partial^2 p}{\partial x_i^2} = 0 \quad \text{or} \quad \boxed{\nabla^2 p = 0}$$

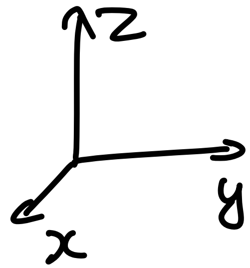
$$-\nabla p + \mu \nabla^2 \underline{u} = 0$$

$$\boxed{\mu \nabla^2 \underline{u}^{(q)} = 0}$$

$$-\nabla p + \mu \nabla^2 \underline{u}^p = 0$$

$$u_i^p = \frac{1}{2\mu} p x_i$$

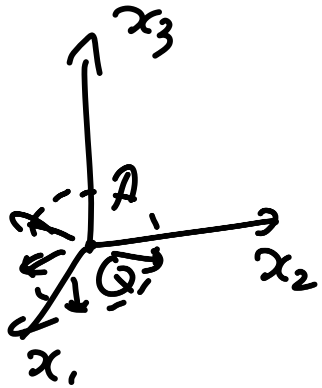
$$u_i = u_i^{(g)} + \frac{1}{2\mu} p x_i$$



$$\nabla^2 T = 0$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

$$T = X(x)\,Y(y)\,Z(z)$$



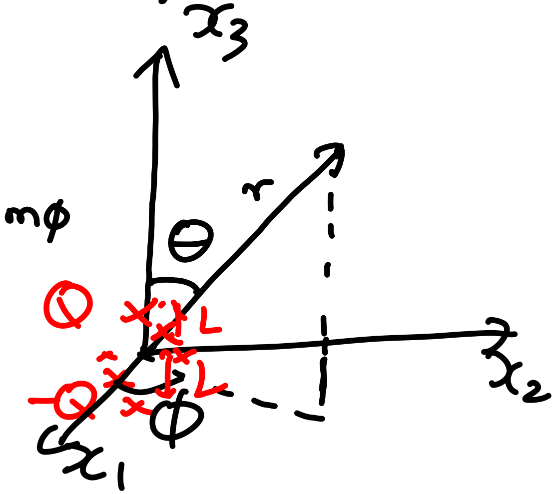
Point source:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) = 0$$

$$T = \frac{c_1}{r} + c_2 = \frac{Q}{4\pi k r} + T_\infty$$

$$Q = \int ds\, q_r = \int ds \left(-k\frac{\partial T}{\partial r}\right)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}=0$$



$$T = R(r)\,F(\theta)\,H(\phi)$$

$$=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[\frac{A_{nm}}{r^{n+1}}+B_{nm}r^n\right]P_n^m(\cos\theta)e^{im\phi}$$

$P_n^m(\cos\theta) \Rightarrow$ Legendre polynomial

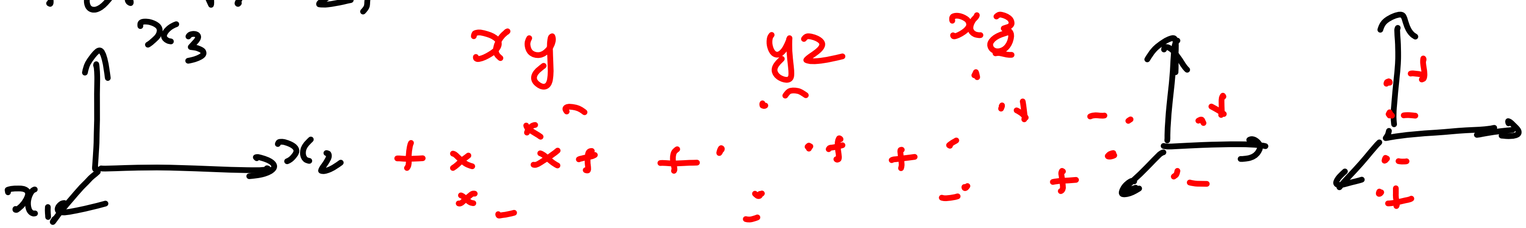
For $n=0 \Longrightarrow T = \frac{A}{r}+B$

For $n=1$ $m=0 \Rightarrow T = \frac{A_{10}}{r^2}\cos\theta + B_{10}r\cos\theta$

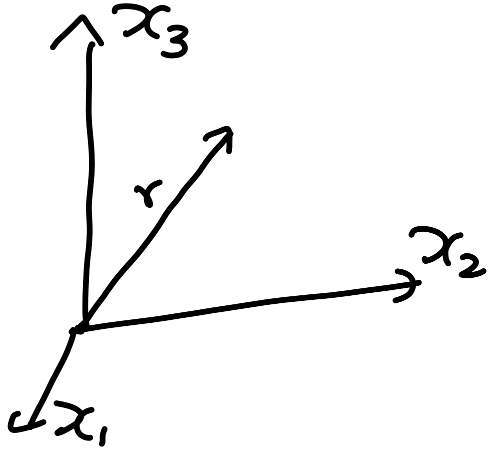
$m=1$
$m=-1$

In the Limit $L\to 0$ & $QL \to$ Finite

For $n=2$, $m=-2,-1,0,+1,+2$



$\nabla^2 \underline{\Phi}^{(0)} = 0$ $\qquad$ $\underline{\Phi}^{(0)} = \dfrac{c}{r}$ $\quad$ Fundamental solution



$$\nabla(\nabla^2 \underline{\Phi}^{(0)}) = 0$$

$$\nabla^2(\nabla \underline{\Phi}^{(0)}) = 0$$

$\nabla^2(\underline{\Phi}^{(1)}) = 0$ $\qquad$ $\underline{\Phi}_i^{(1)}$ = Vector solution of Laplace equation

$$\nabla^2(\underline{\Phi}_i^{(1)}) = 0$$

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$\underline{\Phi}^{(0)} = \frac{C}{r} = \frac{C}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$\nabla(\underline{\Phi}^{(0)}) = \left(\underline{e}_1 \frac{\partial}{\partial x_1} + \underline{e}_2 \frac{\partial}{\partial x_2} + \underline{e}_3 \frac{\partial}{\partial x_3}\right)\left(\frac{C}{r}\right)$$

$$\nabla r = \left(e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3} \right) \left(\sqrt{x_1^2 + x_2^2 + x_3^2} \right)$$

$$= \frac{e_1 x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} + \frac{e_2 x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}} + \frac{e_3 x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$\frac{\partial r}{\partial x_i} = \sum_{i=1}^{3} \frac{e_i x_i}{r} = \frac{x_i}{r}$$

$$\frac{\partial}{\partial x_i} \left(\Phi^{(0)} \right) = \frac{\partial}{\partial x_i} \left(\frac{c}{r} \right) = -\frac{c}{r^2} \frac{\partial r}{\partial x_i} = -\frac{c x_i}{r^3}$$

$$\Phi_i^{(1)} = -\frac{c x_i}{r^3}$$

$$\nabla^2 \Phi_i^{(1)} = 0 \Longrightarrow \frac{\partial}{\partial x_j} \left(\nabla^2 \Phi_i^{(1)} \right) = 0$$

$$\nabla^2 \left(\frac{\partial}{\partial x_j} \Phi_i^{(1)} \right) = 0 \Longrightarrow \nabla^2 \left(\Phi_{ij}^{(2)} \right) = 0$$

$$\Phi^{(2)}_{ij} = \frac{\partial}{\partial x_j}\left(\frac{-C x_i}{r^3}\right) = \frac{-C}{r^3}\frac{\partial x_i}{\partial x_j} - C x_i \frac{\partial}{\partial x_j}\left(\frac{1}{r^3}\right)$$

$$= -\frac{C\,\delta_{ij}}{r^3} - C x_i \left[\frac{-3}{r^4}\right]\left[\frac{\partial r}{\partial x_j}\right]$$

$$= -\frac{C\,\delta_{ij}}{r^3} + \frac{3C\,x_i x_j}{r^5}$$

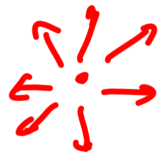
$$= C\left[-\frac{\delta_{ij}}{r^3} + \frac{3x_i x_j}{r^5}\right]$$

$$\Phi^{(3)}_{ijk} = \frac{\partial}{\partial x_k}\left(\Phi^{(2)}_{ijk}\right) = C\left[-\frac{3\delta_{ij}x_k}{r^5} - \frac{3\delta_{ik}x_j}{r^5} - \frac{3\delta_{jk}x_i}{r^5}\right.$$

$$\left. + \frac{15x_i x_j x_k}{r^7}\right]$$

Physical interpretation:

$$\underline{\Phi}^{(0)} = \frac{1}{r}$$



$$T = \sum \frac{A_{nm}}{r^{n+1}} P_n^m(\cos\theta)\, e^{im\phi}$$

$n=0;\ m=0$

$$\underline{\Phi}^{(1)} = -\frac{x_i}{r^3}$$

$$= -\frac{x_1}{r^3}\underline{e}_1 - \frac{x_2}{r^3}\underline{e}_2 - \frac{x_3}{r^3}\underline{e}_3$$

$$= -\frac{r\sin\theta\cos\phi}{r^3}\underline{e}_1 - \frac{r\sin\theta\sin\phi}{r^3}\underline{e}_2 - \frac{r\cos\theta}{r^3}\underline{e}_3$$

+ −

$n=1$, $m=0$ $\qquad \frac{A_{10}\cos\theta}{r^2}$

$n=1$, $m=1$ $\qquad \frac{A_{11}}{r^2}\sin\theta\cos\phi$

$\frac{A_{1-1}}{r^2}\sin\theta\sin\phi$

$$\Phi^{(2)}_{ij} = -\frac{\delta_{ij}}{r^3} + \frac{3x_i x_j}{r^5}$$

$$= \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_i}(\Phi^{(0)})\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}\Phi^{(0)}\right) = \Phi^{(2)}_{ji}$$

$$\begin{pmatrix} \phi^{(2)}_{11} & \phi^{(2)}_{12} & \phi^{(2)}_{13} \\ \phi^{(2)}_{21} & \phi^{(2)}_{22} & \phi^{(2)}_{23} \\ \phi^{(2)}_{31} & \phi^{(2)}_{32} & \phi^{(2)}_{33} \end{pmatrix}$$

$$\delta_{ij}\,\Phi^{(2)}_{ij} = \Phi^{(2)}_{ii} = \delta_{ij}\left[-\frac{\delta_{ij}}{r^3} + \frac{3x_i x_j}{r^5}\right]$$

$$= \left[-\frac{\delta_{ii}}{r^3} + \frac{3x_i^2}{r^5}\right]$$

$$= \left[\frac{-3}{r^3} + \frac{3}{r^3}\right] = 0$$

$$\delta_{ij}\,\Phi^{(2)}_{ij} = \delta_{ij}\left[\frac{\partial}{\partial x_j}\left(\frac{\partial \Phi^{(0)}}{\partial x_i}\right)\right] = \frac{\partial}{\partial x_i}\left(\frac{\partial \Phi^{(0)}}{\partial x_i}\right) = 0$$

$$\frac{A_{nm}}{r^{n+1}} P_n^m(\cos\theta)\, e^{im\phi}$$

$$n=2 \Rightarrow \frac{A_{2m}}{r^3} P_2^m(\cos\theta)\, e^{im\phi}$$

$$T = \sum \left[\frac{A_{nm}}{r^{n+1}} + \boxed{B_{nm}\, r^{n}} \right] P_n^m(\cos\theta)\, e^{im\phi}$$

$\times r^{2n+1}$

$$\Phi^{(0)} = \frac{1}{r} \qquad n=0 \qquad 1 \qquad\qquad T = \frac{Q}{4\pi r} + T_\infty$$

$$\Phi^{(1)}_i = \frac{x_i}{r^3} \qquad n=1 \qquad x_i$$

$$\Phi^{(2)}_{ij} = \left[\frac{-\delta_{ij}}{r^3} + \frac{3x_i x_j}{r^5} \right] \qquad n=2 \qquad \left[-r^2\delta_{ij} + 3x_i x_j \right]$$

Viscous flows:

$$\nabla \cdot \underline{u} = 0 \qquad \frac{\partial u_i}{\partial x_i} = 0$$

$$-\nabla p + \mu \nabla^2 \underline{u} = 0 \qquad -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} = 0$$

$$T_{ij} = -p\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\nabla^2 p = 0 \qquad \nabla^2 \underline{u}^{(g)} = 0$$

$$u_i^{(p)} = \frac{1}{2\mu}\, p\, x_i$$

$$u_i = u_i^{(g)} + u_i^{(p)}$$

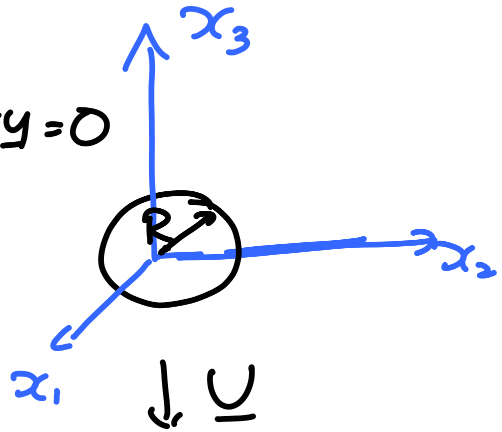
$\nabla^2 p = 0 \qquad \nabla^2 u_i^{(g)} = 0$

$\nabla \cdot \underline{u} = 0$

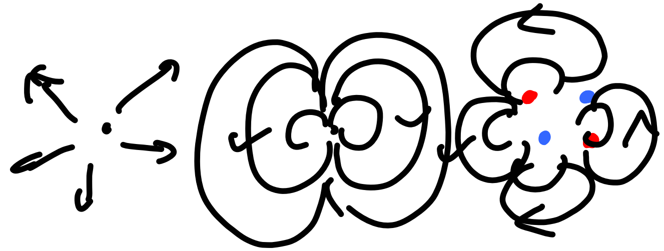
$-\nabla p + \mu \nabla^2 \underline{u} = 0$

$$T = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left(\frac{A_{nm}}{r^{n+1}} + B_{nm} r^n \right) P_n^m(\cos\theta)\, e^{im\phi}$$

At $r = R$, $u_i = U_i$



$$T = \frac{Q}{4\pi k r} + T_\infty$$



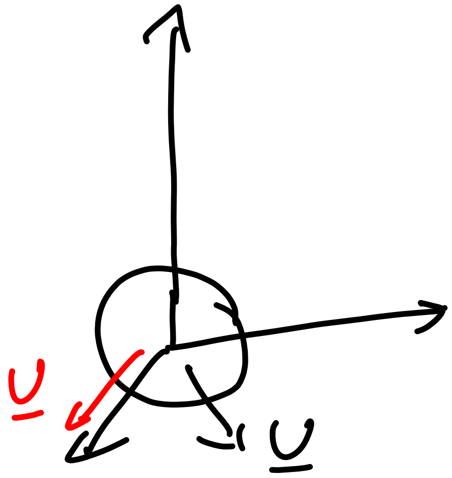
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \bar{\Phi}^{(0)}}{\partial r} \right) = 0$$

$$\bar{\Phi}^{(0)} = \frac{1}{r} \qquad \frac{\partial}{\partial x_i}\left(\nabla^2 \bar{\Phi}^{(0)} \right) = 0 \qquad n=0 \qquad \bar{\Phi}^{(0)} = 1$$

$$\bar{\Phi}_i^{(1)} = \frac{x_i}{r^3} \qquad \nabla^2 \left(\frac{\partial \bar{\Phi}^{(0)}}{\partial x_i} \right) = 0 \qquad n=1 \qquad \bar{\Phi}_i^{(1)} = x_i$$

$$\bar{\Phi}_{ij}^{(2)} = \frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5} \qquad n=2 \qquad \bar{\Phi}_{ij}^{(2)} = \delta_{ij} r^2 - 3x_i x_j$$

$$\nabla^2 p = 0 \qquad \nabla^2 u_i^{(g)} = 0$$



$$p = A_3 U_j \Phi_j^{(1)} = A_3 \frac{U_j x_j}{r^3}$$

$$u_i^{(g)} = A_1 U_i \Phi^{(0)} + A_2 U_j \Phi_{ij}^{(2)}$$

$$= A_1 \frac{U_i}{r} + A_2 U_j \left(\frac{\delta_{ij}}{r^3} - \frac{3 x_i x_j}{r^5} \right)$$

$$u_i = u_i^{(g)} + \frac{1}{2\mu} p x_i$$

$$= A_1 \frac{U_i}{r} + A_2 U_j \left(\frac{\delta_{ij}}{r^3} - \frac{3 x_i x_j}{r^5} \right) + \frac{A_3}{2\mu} \frac{x_i U_j x_j}{r^3}$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$u_i = A_1 U_i \Phi^{(0)} + A_2 U_j \Phi_{ij}^{(2)} + \frac{A_3}{2\mu} \underline{x_i} U_j \Phi_j^{(1)}$$

$$\frac{\partial u_i}{\partial x_i} = A_1 U_i \frac{\partial \Phi^{(0)}}{\partial x_i} + A_2 U_j \frac{\partial}{\partial x_i}\left(\Phi^{(2)}_{ij}\right) + \frac{A_3}{2\mu}\left[\delta_{ii} U_j \Phi^{(1)}_j + x_i U_j \frac{\partial \Phi^{(1)}_j}{\partial x_i}\right]$$

$$= A_1 U_i\left(-\frac{x_i}{r^3}\right) + \frac{A_3}{2\mu}\left[3U_j \Phi^{(1)}_j + x_i U_j\left(\frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}\right)\right]$$

$$= \frac{-A_1 U_i x_i}{r^3} + \frac{A_3}{2\mu}\left[\frac{3U_j x_j}{r^3} + \frac{U_j x_j}{r^3} - \frac{3x_i^2 x_j}{r^5}\right] = 0$$

$$\Longrightarrow \quad A_1 = \frac{A_3}{2\mu}$$

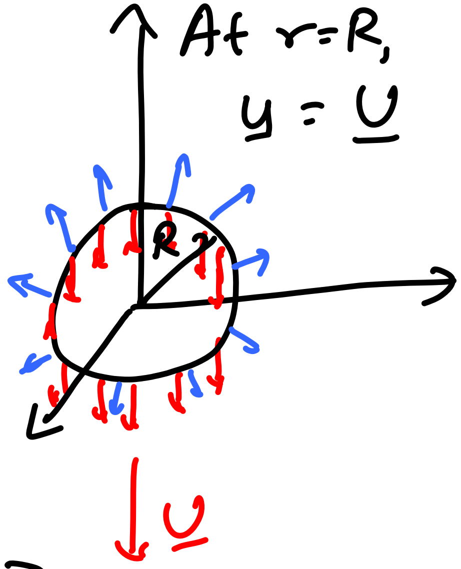
$$u_i = A_1\left[\frac{U_i}{r} + \frac{x_i x_j U_j}{r^3}\right] + A_2 U_j\left[\frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}\right]$$

$$= U_i\left[\frac{A_1}{r} + \frac{A_2}{r^3}\right] + U_j x_i x_j\left[\frac{A_1}{r^3} - \frac{3A_2}{r^5}\right]$$

$$\underline{u} = \underline{U}\left[\frac{A_1}{r} + \frac{A_2}{r^3}\right] + \underline{x}\,(\underline{U}\cdot\underline{x})\left[\frac{A_1}{r^3} - \frac{3A_2}{r^5}\right]$$

At $r = R$, $\dfrac{A_1}{R} + \dfrac{A_2}{R^3} = 1$

$$\frac{A_1}{R^3} - \frac{3A_2}{R^5} = 0$$



$$A_1 = \frac{3R}{4} \; ; \; A_2 = \frac{R^3}{4}$$

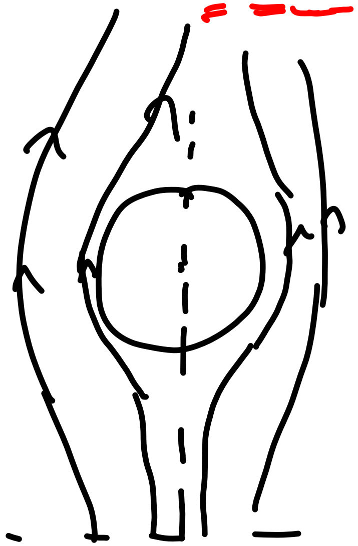
$$\underline{u} = \frac{3R}{4}\left[\frac{U_i}{r} + \frac{U_j x_i x_j}{r^3}\right] + \frac{R^3}{4} U_j \left[\frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}\right]$$

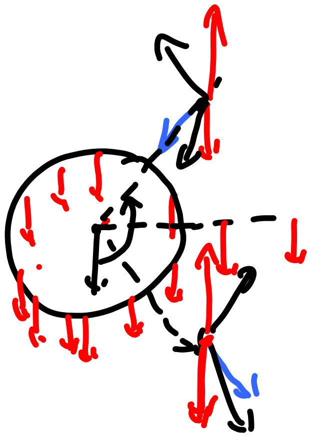
$$p = A_3 \frac{U_j x_j}{r^3} = \frac{3}{2} \frac{\mu R U_j x_j}{r^3}$$

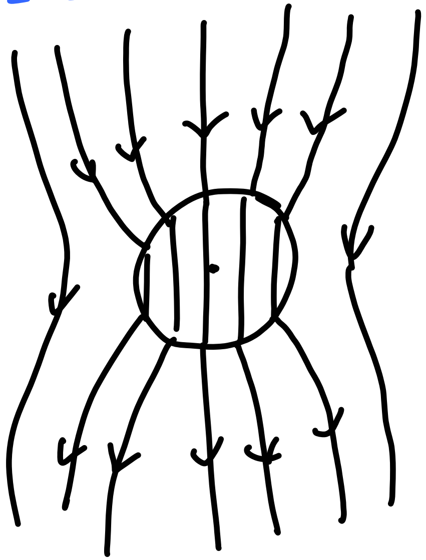
$$u_i = \frac{3R}{4}\left[\frac{U_i}{r} + \frac{U_j x_i x_j}{r^3}\right] + \frac{R^3}{4} U_j\left[\frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}\right]$$

$$= U_i\left[\frac{3R}{4r} + \frac{R^3}{4r^3}\right] + U_j x_i x_j\left[\frac{3R}{4r^3} - \frac{3R^3}{4r^5}\right]$$

$$\underline{u} = \underline{U}\left[\frac{3R}{4r} + \frac{R^3}{4r^3}\right] + \underline{x}(\underline{U}\cdot\underline{x})\left[\frac{3R}{4r^3} - \frac{3R^3}{4r^5}\right]$$







Stokes equations:

$$\nabla \cdot \underline{u} = 0 \qquad \frac{\partial u_i}{\partial x_i} = 0$$

$$-\nabla p + \mu \nabla^2 \underline{u} = 0 \qquad -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} = 0$$

$$\frac{\partial}{\partial x_i}\left(-\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2}\right) = 0$$

$$\nabla^2 p = 0 \qquad \nabla^2 u_i^{(g)} = 0$$

$$u_i^{(b)} = \frac{1}{2\mu} p x_i$$

$$u_i = u_i^{(g)} + \frac{1}{2\mu} p x_i$$

$$\nabla \cdot \underline{u} = 0$$

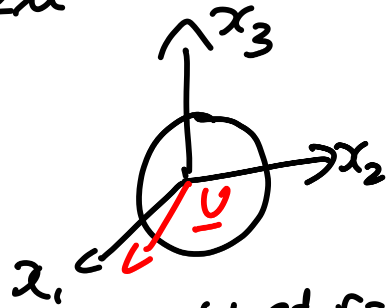
$$-\nabla p + \mu \nabla^2 \underline{u} = 0$$

$$\nabla^2 p = 0$$

$$\nabla^2 \underline{u}^{(g)} = 0$$

$$u_i^{(p)} = \frac{1}{2\mu} p x_i$$

$$u_i = u_i^{(g)} + \frac{1}{2\mu} p x_i$$



$u_i = U_i$ at $r = R$

$$\Phi^{(0)} = \frac{1}{r}$$

$$\Phi_i^{(1)} = \frac{x_i}{r^3}$$

$$\Phi_{ij}^{(2)} = \left[\frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}\right]$$

$$\Phi^{(0)} = 1$$

$$\Phi^{(1)} = x_i$$

$$\Phi_{ij}^{(2)} = r^2 \delta_{ij} - 3x_i x_j$$

$$p = A_3 \frac{U_j x_j}{r^3}$$

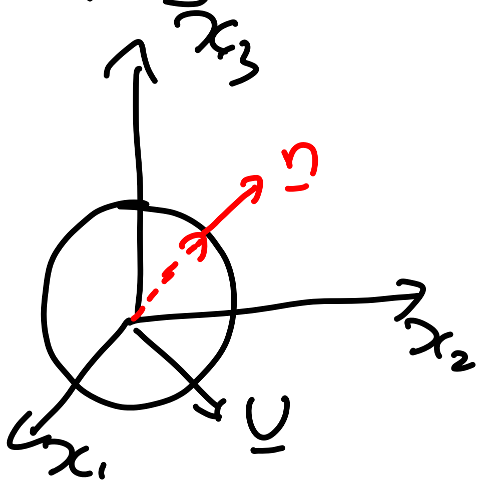
$$u_i = \frac{A_1 U_i}{r} + A_2 U_j \left(\frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}\right) + \frac{A_3}{2\mu} x_i \frac{U_j x_j}{r^3}$$

$$u_i = \frac{3U_jR}{4}\left[\frac{\delta_{ij}}{r} + \frac{x_ix_j}{r^3}\right] + \frac{U_jR^3}{4}\left[\frac{\delta_{ij}}{r^3} - \frac{3x_ix_j}{r^5}\right]$$

$$p = \frac{3\mu U_j R}{2}\left(\frac{x_j}{r^3}\right)$$

$$F_i = \int ds\, T_{ik} n_k = \int ds\, T_{ik}\left(\frac{x_k}{r}\right)$$

$$T_{ik} = -p\,\delta_{ik} + \mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$



$$n_k = \frac{x_k}{r}$$

$$u_i = \frac{3}{4} U_j R \left(\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3} \right) + \frac{U_j R^3}{4} \underline{\Phi}^{(2)}_{ij}$$

$$\mu \frac{\partial u_i}{\partial x_k} = \frac{3\mu}{4} U_j R \left[-\frac{\delta_{ij} x_k}{r^3} + \frac{\delta_{ik} x_j}{r^3} + \frac{\delta_{jk} x_i}{r^3} - \frac{3 x_i x_j x_k}{r^5} \right] + \frac{\mu U_j R^3}{4} \underline{\Phi}^{(3)}_{ijk}$$

$$\mu \frac{\partial u_k}{\partial x_i} = \frac{3\mu}{4} U_j R \left[-\frac{\delta_{jk} x_i}{r^3} + \frac{\delta_{ik} x_j}{r^3} + \frac{\delta_{ij} x_k}{r^3} - \frac{3 x_i x_j x_k}{r^5} \right] + \frac{\mu U_j R^3}{4} \underline{\Phi}^{(3)}_{ijk}$$

$$-p\,\delta_{ik} = -\frac{3}{2} \frac{\mu U_j x_j}{r^3} \delta_{ik}$$

$$T_{ik} = \frac{3}{4} \mu U_j R \left[-\frac{6 x_i x_j x_k}{r^5} \right] + \frac{\mu U_j R^3}{4} \left[2 \underline{\Phi}^{(3)}_{ijk} \right]$$

$$= -\frac{9}{2} \frac{\mu U_j R x_i x_j x_k}{r^5} + \frac{\mu U_j R^3}{2} \underline{\Phi}^{(3)}_{ijk}$$

$$T_{ik} = -\frac{9}{2}\,\frac{\mu U_j R x_i x_j x_k}{r^5} + \frac{\mu U_j R^3}{2}\,\Phi^{(3)}_{ijk}$$

$$\Phi^{(3)}_{ijk} = \frac{\partial}{\partial x_k}\left[\frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}\right]$$

$$= \left[\frac{-3\delta_{ij} x_k}{r^5} - \frac{3\delta_{ik} x_j}{r^5} - \frac{3\delta_{jk} x_i}{r^5} + \frac{15 x_i x_j x_k}{r^7}\right]$$

$$T_{ik}\frac{x_k}{r} = -\frac{9}{2}\,\frac{\mu U_j R x_i x_j x_k^2}{r^6} + \frac{\mu U_j R^3}{2}\left[-3\frac{\delta_{ij} x_k^2}{r^6} - 3\frac{x_i x_j}{r^6} - 3\frac{x_i x_j}{r^6} + 15\frac{x_i x_j x_k^2}{r^8}\right]$$

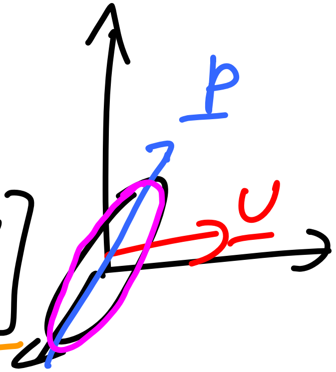
$$= -\frac{9}{2}\,\frac{\mu U_j R x_i x_j}{r^4} + \frac{\mu U_j R^3}{2}\left[\frac{-3\delta_{ij}}{r^4} - \frac{6x_i x_j}{r^6} + \frac{15 x_i x_j}{r^6}\right]$$

$$= -\frac{9}{2}\,\frac{\mu U_j R x_i x_j}{r^4} - \frac{3}{2}\,\frac{\mu U_j R^3 \delta_{ij}}{r^4} + \frac{9}{2}\,\frac{\mu U_j R^3 x_i x_j}{r^6}$$

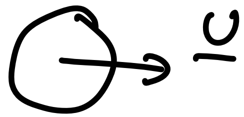
$$T_{ik}\frac{x_k}{r}\Big|_{r=R} = -\frac{3}{2}\,\frac{\mu U_j \delta_{ij}}{R} = -\frac{3}{2}\,\frac{\mu U_i}{R}$$

$$F_i = \int dS\, T_{ik} n_k = \left(-\frac{3}{2}\,\frac{\mu U_i}{R}\right)\left(4\pi R^2\right) = -6\pi\mu R U_i$$

$$\underline{F} = -6\pi\mu R \underline{U}$$ Stokes law



$$u_i = \frac{3U_jR}{4}\left[\frac{\delta_{ij}}{r} + \frac{x_ix_j}{r^3}\right] + \frac{U_jR^3}{4}\left[\frac{\delta_{ij}}{r^3} - \frac{3x_ix_j}{r^5}\right]$$



$$p = \frac{3}{2}\frac{\mu U_j x_j R}{r^3}$$

$$F_i = 6\pi\mu R U_i \Longrightarrow U_i = \frac{F_i}{6\pi\mu R}$$

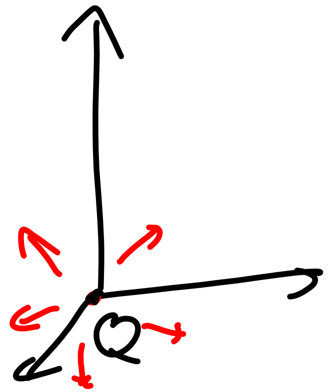
$$u_i = \frac{F_j}{8\pi\mu}\left[\frac{\delta_{ij}}{r} + \frac{x_ix_j}{r^3}\right] \qquad p = \frac{F_j x_j}{4\pi\mu r^3}$$

$$= J_{ij}F_j \qquad\qquad = K_i F_i$$

Oseen tensor $J_{ij} = \frac{1}{8\pi\mu}\left(\frac{\delta_{ij}}{r} + \frac{x_ix_j}{r^3}\right)$ $\quad K_i = \frac{x_i}{4\pi\mu r^3}$

$$\boxed{k\nabla^2 T + Q\,\delta(\underline{x}) = 0}$$

$$T = \frac{Q}{4\pi k r}$$

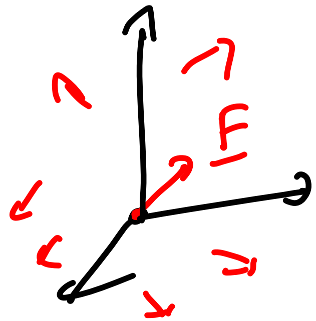


$$\boxed{u_i = J_{ij} F_j} \qquad p = K_i F_i$$

$$\nabla \cdot \underline{u} = 0$$

$$\boxed{-\nabla p + \mu \nabla^2 \underline{u} + \underline{F}\,\delta(\underline{x}) = 0}$$

$$J_{ij} = \frac{1}{8\pi\mu}\left[\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3}\right]$$



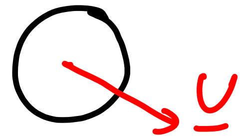
Low Reynolds number Stokes flow:

$$\nabla \cdot \underline{u} = 0$$

$$-\nabla p + \mu \nabla^2 \underline{u} = 0$$

$$\nabla^2 p = 0 \qquad \underline{u} = \underline{u}^{(g)} + \frac{1}{2\mu} p \underline{x}$$

$$\nabla^2 \underline{u}^{(g)} = 0$$



$\underline{u} = \underline{U}$ at $r = R$

$$\Phi^{(0)} = 1$$

$$\Phi^{(1)} = x_i$$

$$\Phi^{(2)} = (r^2 \delta_{ij} - 3 x_i x_j)$$

$$\underline{\Phi}^{(0)} = \frac{1}{r}$$

$$\underline{\Phi}^{(1)} = \frac{x_i}{r^3}$$

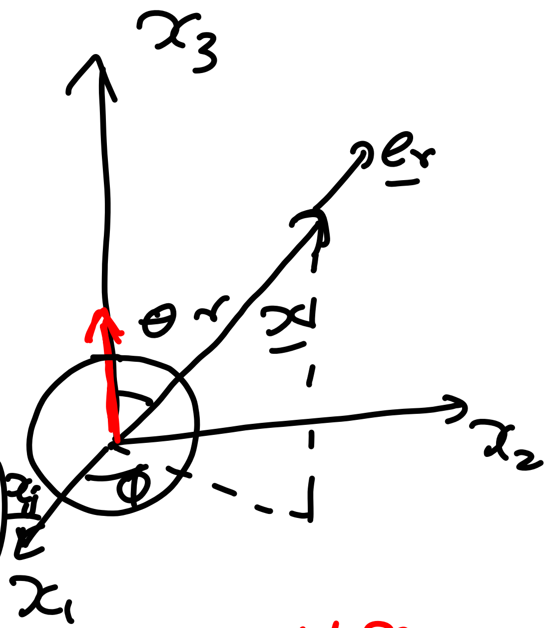
$$\underline{\Phi}^{(2)} = \left(\frac{\delta_{ij}}{r^3} - \frac{3 x_i x_j}{r^5} \right)$$

$$u_i = \frac{3R}{4} U_j \left(\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3} \right) + \frac{R^3 U_j}{4} \left(\frac{\delta_{ij}}{r^3} - \frac{3 x_i x_j}{r^5} \right)$$

$$p = \frac{3R}{2} \mu \frac{U_j x_j}{r^3}$$

$$\boxed{F_i = 6 \pi \mu R U_i}$$

$$u_i = \frac{3U_jR}{4}\left(\frac{\delta_{ij}}{r} + \frac{x_ix_j}{r^3}\right) + \frac{R^3U_j}{4}\left(\frac{\delta_{ij}}{r^3} - \frac{3x_ix_j}{r^5}\right)$$



$$u_r = \underline{u}\cdot\underline{e}_r = u_i\left(\frac{x_i}{r}\right)$$

$$= \frac{3U_jR}{4}\left(\frac{\delta_{ij}}{r} + \frac{x_ix_j}{r^3}\right)\frac{x_i}{r} + \frac{R^3}{4}U_j\left(\frac{\delta_{ij}}{r^3} - \frac{3x_ix_j}{r^5}\right)\frac{x_i}{r}$$

$$U_jx_j = \underline{U}\cdot\underline{x}$$
$$= Ur\cos\theta$$

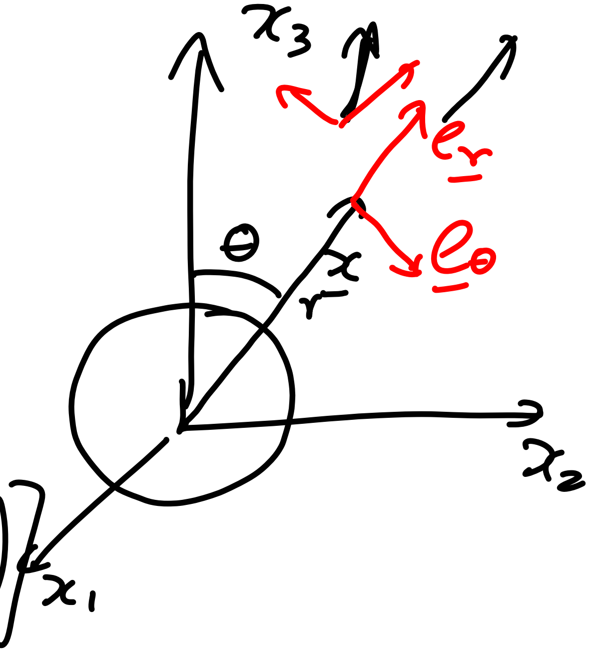
$$= \frac{3U_jR}{4}\left(\frac{x_j}{r^2} + \frac{x_i^2x_j}{r^4}\right) + \frac{R^3U_j}{4}\left[\frac{x_j}{r^4} - 3\frac{x_i^2x_j}{r^6}\right]$$

$$= \frac{3U_jR}{4}\left(\frac{2x_j}{r^2}\right) + \frac{R^3U_j}{4}\left[-\frac{2x_j}{r^4}\right]$$

$$= \frac{3}{2}\,\frac{RU_jx_j}{r^2} - \frac{R^3U_jx_j}{r^4}$$

$$= \frac{3}{2}\,\frac{RU\cos\theta}{r} - \frac{R^3U\cos\theta}{r^3}$$

$$u_{ti} = \left(\delta_{ij} - \frac{x_i x_j}{r^2}\right) u_j$$



$$= \left[\delta_{ij} - \frac{x_i x_j}{r^2}\right]\left[\frac{3RU_k}{4}\left(\frac{\delta_{jk}}{r} + \frac{x_j x_k}{r^3}\right) + \frac{R^3 U_k}{4}\left(\frac{\delta_{jk}}{r^3} - \frac{3x_j x_k}{r^5}\right)\right]$$

$$= \left[\delta_{ik} - \frac{x_i x_k}{r^2}\right] U_k \left[\frac{3R}{4r} + \frac{R^3}{4r^3}\right]$$

$$(u_{ti} u_{ti}) = U^2 \sin^2\theta \left[\frac{3R}{4r} + \frac{R^3}{4r^3}\right]^2$$

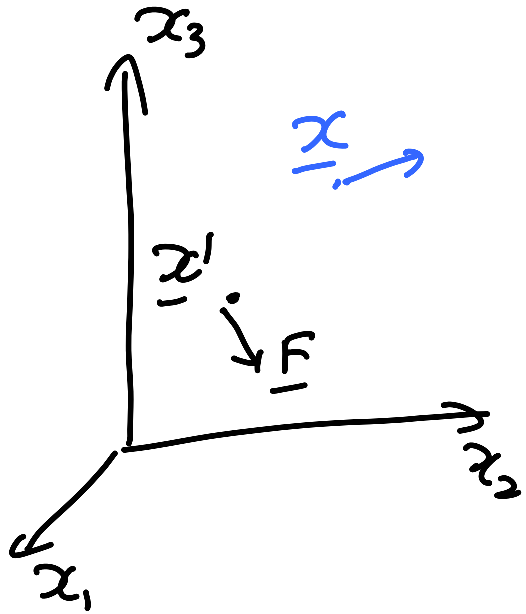
$$u_\theta = -U^2 \sin\theta \left[\frac{3R}{4r} + \frac{R^3}{4r^3}\right]$$

$$u_i(\underline{x}) = J_{ij}(\underline{x} - \underline{x}')\, F_j(\underline{x}')$$

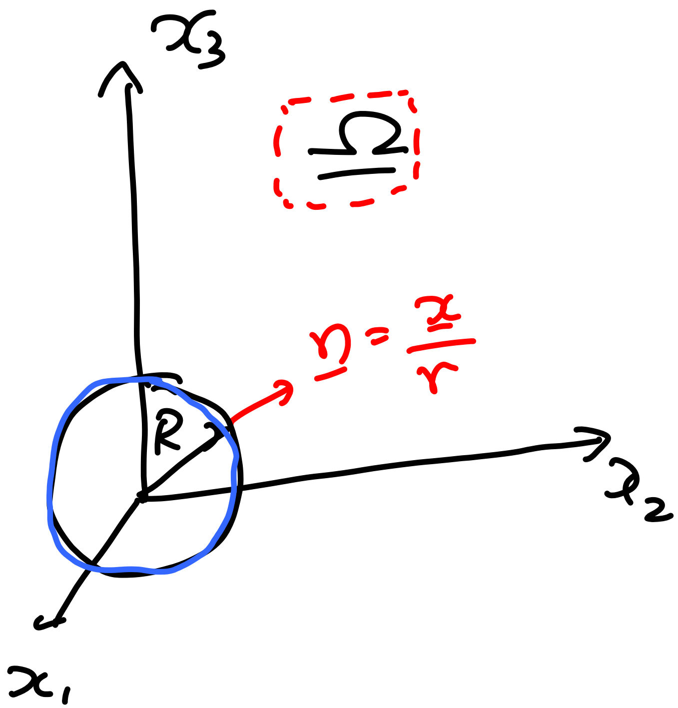
$$p(\underline{x}) = K_i(\underline{x} - \underline{x}')\, F_i(\underline{x}')$$

$$J_{ij} = \frac{1}{8\pi\mu}\left[\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3}\right]$$

$$K_i = \frac{1}{4\pi}\frac{x_i}{r^3}$$



Particle rotating in Stokes flow



$$\underline{u} = \underline{\Omega} \times \underline{x} \quad \text{at} \quad r = R$$

$$u_i = \epsilon_{ijk}\Omega_j x_k \quad \text{at} \quad r = R$$

$$u_i^{(g)} = A_1 \epsilon_{ijk}\Omega_j \underline{\Phi}^{(1)}_k + A_2 \underline{\Phi}^{(3)}_{ijk}\epsilon_{jkl}\Omega_l$$

$$p = A_3 \epsilon_{ijk}\Omega_k \underline{\Phi}^{(2)}_{ij} = 0$$

$$u_i^{(g)} = R^3 \epsilon_{ijk}\Omega_j \left(\frac{x_k}{r^3}\right) = u_i$$

$$u_i = \frac{\epsilon_{ijk}\Omega_j x_k R^3}{r^3} \qquad p = 0$$

$$\underline{L} = \int ds\,(\underline{x} \times \underline{F}) \qquad L_i = \int ds\, \epsilon_{ijk}\, x_j\, F_k$$

$$= \int ds\, \epsilon_{ijk}\, x_j\, T_{kl}\, n_l$$

$$T_{kl} = \mu \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right)$$

$$u_k = \epsilon_{kmn} \frac{\Omega_m x_n}{r^3} R^3$$

$$\frac{\partial u_k}{\partial x_l} = R^3 \epsilon_{kmn} \Omega_m \left[\frac{\delta_{nl}}{r^3} - \frac{3 x_n x_l}{r^5} \right]$$

$$\frac{\partial u_l}{\partial x_k} = R^3 \epsilon_{lmn} \Omega_m \left[\frac{\delta_{nk}}{r^3} - \frac{3 x_n x_k}{r^5} \right]$$

$$T_{kl} n_l = \mu R^3 \epsilon_{kmn} \Omega_m \left[\frac{\delta_{nl}}{r^3} - \frac{3 x_n x_l}{r^5} \right] \frac{x_l}{r}$$

$$+ \mu R^3 \epsilon_{lmn} \Omega_m \left[\frac{\delta_{nk}}{r^3} - \frac{3 x_n x_k}{r^5} \right] \frac{x_l}{r}$$

$$T_{kl} n_l = \mu R^3 \left[\epsilon_{kmn} \Omega_m \frac{x_n}{r^4} - \frac{3 \epsilon_{kmn} \Omega_m x_n x_l^2}{r^6} \right]$$

$$+ \mu R^3 \epsilon_{lmk} \Omega_m \frac{x_l}{r^4}$$

$$= \mu R^3 \left[\cancel{\epsilon_{kml} \Omega_m \frac{x_l}{r^4}} + \cancel{\frac{\epsilon_{lmk} \Omega_m x_l}{r^4}} \right]$$

$$- 3 \mu R^3 \frac{\epsilon_{kmn} \Omega_m x_n}{r^4}$$

$$= \boxed{\frac{-3 \mu R^3 \epsilon_{kmn} \Omega_m x_n}{r^4}}$$

$$L_i = \int dS \, \epsilon_{ijk} x_j \left[\frac{-3 \mu R^3 \epsilon_{kmn} \Omega_m x_n}{r^4} \right]$$

$$\epsilon_{ijk} \epsilon_{kmn} = \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}$$

$$L_i = \frac{-3 \mu R^3}{r^4} \int dS \left(\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm} \right) \left[\Omega_m x_j x_n \right]$$

$$= \frac{-3 \mu R^3}{r^4} \int dS \left(\Omega_i x_j^2 - x_i x_j \Omega_j \right)$$

$$= -\frac{3\mu R^3}{r^4}\left[\Omega_i \int ds\, x_j^2 - \Omega_j \int ds\, x_i x_j\right]$$

$$= -\frac{3\mu R^3}{r^4}\left[\Omega_i \int ds\, r^2 - \Omega_j \int ds\, x_i x_j\right]$$

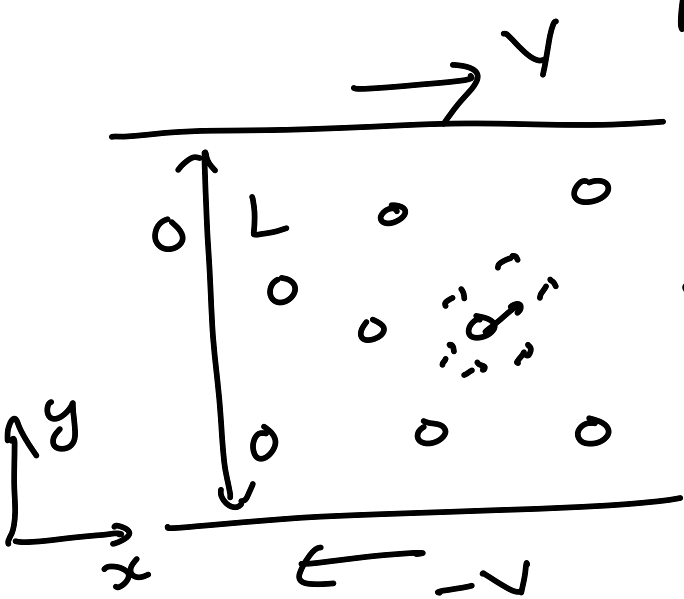
$$L_i\Big|_{r=R} = -\frac{3\mu R^3}{R^4}\left[\Omega_i R^2 (4\pi R^2) - \Omega_j \frac{4}{3}\pi R^4 \delta_{ij}\right]$$

$$= -\frac{3\mu R^3}{R^4}\left[\frac{8}{3}\pi R^4 \Omega_i\right]$$

$$= -8\pi\mu R^3 \Omega_i$$

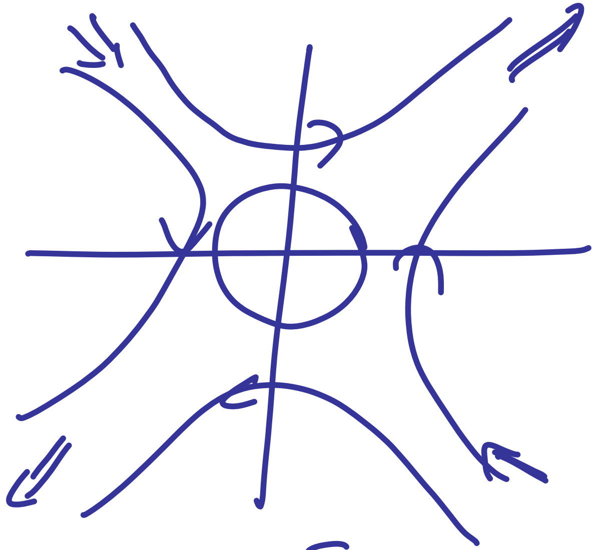
$$u_i = \frac{\epsilon_{ijk}\Omega_j x_k R^3}{r^3} = \frac{\epsilon_{ijk} L_j x_k}{8\pi\mu r^3}$$

Effective viscosity of a suspension:



$$\tau_{xy} = \mu_{eff}\left(\frac{2V}{L}\right)$$

Fluid viscosity μ



$$u_i = E_{ij} x_j$$

'Dilute limit'

$$\langle T_{ij} \rangle = \frac{1}{V}\int dV\, T_{ij} = \frac{1}{V}\left[\int_{\text{Fluid}} dV\, T_{ij} + \int_{\text{Particle}} dV\, T_{ij}\right]$$

$$T_{ij} = \left[T_{ij} + p\,\delta_{ij} - 2\mu E_{ij}\right] - p\,\delta_{ij} + 2\mu E_{ij}$$

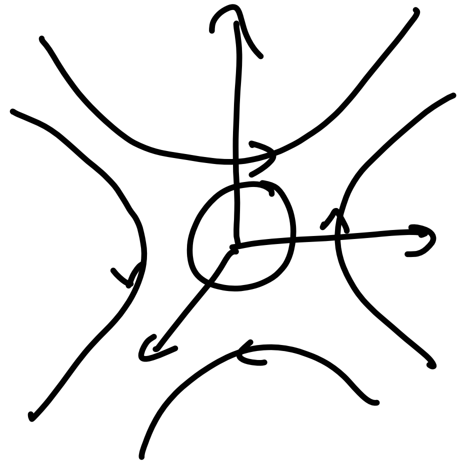
$$\langle T_{ij}\rangle = \frac{1}{V}\left[\int dV\,(T_{ij}+p\delta_{ij}-2\mu E_{ij}) + \int_V dV\,(-p\delta_{ij}+2\mu E_{ij})\right]$$

$$T_{ij} = -p\delta_{ij} + 2\mu E_{ij}$$

$$\langle T_{ij}\rangle = \frac{1}{V}\int_{\text{particles}} dV\,\left[T_{ij}+p\delta_{ij}-2\mu E_{ij}\right] - \langle p\rangle\delta_{ij} + 2\mu\langle E_{ij}\rangle$$

$$= \frac{1}{V}\int_{\text{particles}} dV\, T_{ij} + \langle p_p\rangle\delta_{ij} - \langle p\rangle\delta_{ij} + 2\mu\langle E_{ij}\rangle$$

$$\frac{1}{V}\int_{\text{particles}} dV\, T_{ij} = \frac{N}{V}\int_{\text{1 particle}} dV\, T_{ij}$$



$$\frac{\partial}{\partial x_l}\left(T_{il}x_j\right) = \left[\frac{\partial}{\partial x_l}\left(T_{il}\right)\right]x_j + T_{il}\frac{\partial x_j}{\partial x_l}$$

$$= \cancel{x_j\frac{\partial}{\partial x_l}\left(T_{il}\right)} + T_{il}\,\delta_{jl}$$

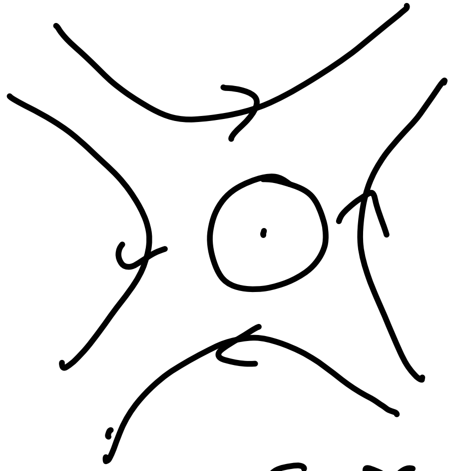
$$= T_{ij}$$

$$\frac{1}{V}\int_{\text{1 particle}} dV\, T_{ij} = \frac{1}{V}\int_{\text{1 particle}} dV\,\frac{\partial}{\partial x_l}\left(T_{il}x_j\right)$$

$$= \frac{1}{V}\int_{\text{1 particle}} dV\, T_{il}\, n_l\, x_j$$

Particle in extensional flow

$$u_i^{(g)} = E_{ij} x_j + A_1 E_{ij} \Phi_j^{(1)} + A_2 E_{jk} \Phi_{ijk}^{(3)}$$



$$u_i = E_{ij} x_j \text{ as } r \to \infty$$

$$u_i = 0 \text{ at } r = R$$

$$p = A_3 E_{jk} \Phi_{jk}^{(2)}$$

$$E_{ii} = E_{ij} \delta_{ij} = 0$$

$$u_i = E_{ij} x_j \left(1 - \frac{R^5}{r^5}\right) + \frac{5}{2} E_{jk} x_i x_j x_k \left(\frac{R^5}{r^7} - \frac{R^3}{r^5}\right)$$

$$p = -\frac{5 \mu R^3 x_j x_k E_{jk}}{r^5}$$

$$\int dS\, T_{il} n_l x_j = \frac{20 \pi R^3 \mu E_{ij}}{3}$$

$$\langle T_{ij} \rangle = \frac{N}{V} \int_{\text{1 particle}} ds\, T_{il} n_l x_j - \langle p^b \rangle \delta_{ij} + \langle p \rangle \delta_{ij} + 2\mu \langle E_{ij} \rangle$$

$$= \frac{N}{V} \left(\frac{20 \pi R^3 \mu E_{ij}}{3} \right) - \langle p^b \rangle \delta_{ij} + \langle p \rangle \delta_{ij} + 2\mu E_{ij}$$

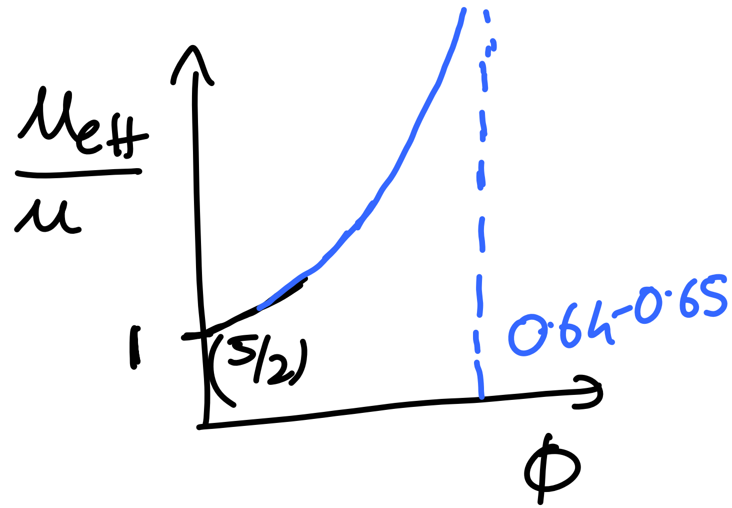
$$= 5 \phi \mu E_{ij} + 2\mu E_{ij} + \delta_{ij} \left(\langle p \rangle - \langle p^b \rangle \right)$$

$$= 2\mu \left(1 + \frac{5\phi}{2} \right) E_{ij} + \delta_{ij} \left(\langle p \rangle - \langle p^b \rangle \right)$$

$$= 2\mu_{eff} E_{ij} + \delta_{ij} \left(\langle p \rangle - \langle p^b \rangle \right)$$

$$\mu_{eff} = \mu \left(1 + \frac{5\phi}{2} \right)$$ 'Einstein viscosity'

$$\mu_{eff} = \mu\left(1 + \frac{5}{2}\phi + 6.5\phi^2\right)$$



Low Reynolds number.

$$\nabla \cdot \underline{u} = 0$$

$$-\nabla p + \mu \nabla^2 \underline{u} = 0$$

Stokes equations:

$$\nabla^2 p = 0$$

$$\nabla^2 \underline{u}^{(g)} = 0$$

$$\underline{u} = \underline{u}^{(g)} + \frac{1}{2\mu} p \underline{x}$$

$$\Phi^{(0)} = \frac{1}{r}$$

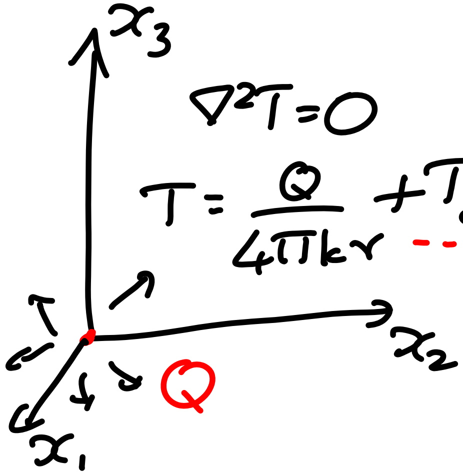
$$\Phi^{(1)}_i = \frac{x_i}{r^3}$$

$$\Phi^{(2)}_{ij} = \left[\frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}\right]$$

$$\bar{\Phi}^{(0)} = 1$$

$$\bar{\Phi}^{(1)}_i = x_i$$

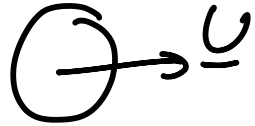
$$\bar{\Phi}^{(2)}_{ij} = (\delta_{ij} r^2 - 3x_i x_j)$$



$$\nabla^2 T = 0$$

$$T = \frac{Q}{4\pi k r} + T_\infty$$

Flow past a sphere:



$$u_i = \frac{3R}{4} U_j \left[\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3} \right] + \frac{R^3 U_j}{4} \left[\frac{\delta_{ij}}{r^3} - \frac{3 x_i x_j}{r^5} \right] \qquad u_i = U_i \text{ at } r = R$$

$$p = \frac{3}{2} \mu \frac{R U_j x_j}{r^3} \qquad F_i = 6\pi \mu R U_i$$

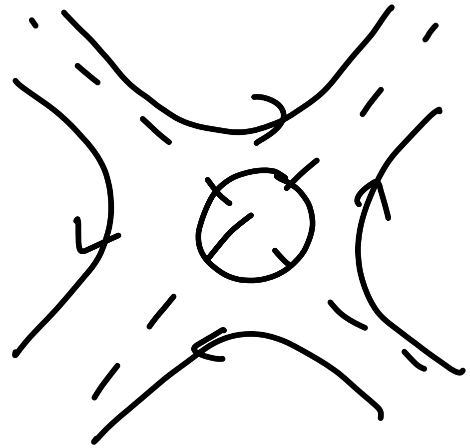
$$u_i = F_j \left[\frac{1}{8\pi\mu} \left(\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3} \right) + \underbrace{\frac{R^2}{24\pi\mu} \left(\frac{\delta_{ij}}{r^3} - \frac{3 x_i x_j}{r^5} \right)}_{\to 0} \right]$$

$$= J_{ij} F_j \quad \text{where } J_{ij} = \frac{1}{8\pi\mu} \left(\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3} \right)$$

$$p = k_i F_i \quad \text{where } k_i = \frac{1}{4\pi} \left(\frac{x_i}{r^3} \right)$$

Particle in shear flow:

$$u_i = E_{ij} x_j \left[1 - \frac{R^5}{r^5}\right] + \frac{5}{2} E_{jk} x_i x_j x_k \left[\frac{R^5}{r^7} - \frac{R^3}{r^5}\right]$$



As $r \to \infty$; $u_i = E_{ij} x_j$

$r = R$; $u_i = 0$

$$F^S_{ij} = \frac{1}{2} \int_S ds \left(f_i x_j + f_j x_i\right)$$

$$= \frac{1}{2} \int_S ds \left[\left(\tau_{il} n_l x_j + \tau_{jl} n_l x_i\right)\right]$$

$$\int_S ds \left[\tau_{il} n_l x_j\right] = \frac{20}{3} \pi R^3 \mu E_{ij} = F^S_{ij}$$

$$u_i = E_{ij} x_j - \frac{3 F^S_{ij} x_j R^2}{20 \pi \mu r^5} + \frac{3 x_i x_j x_k F^S_{jk}}{8 \pi \mu R^3} \left[\frac{R^5}{r^7} - \frac{R^3}{r^5}\right]$$

$$= E_{ij} x_j - \frac{3 F^S_{jk} x_i x_j x_k}{8 \pi \mu r^5}$$

$$u_i = \frac{\epsilon_{ijk}\,\Omega_j\, x_k\, R^3}{r^3}$$

$\underline{\Omega}$

$u_i = \epsilon_{ijk}\Omega_j x_k$ at $r = R$

$$L_i = 8\pi\mu R^3 \Omega_i$$

$$= \epsilon_{ijk}\int_S ds\, x_j f_k$$

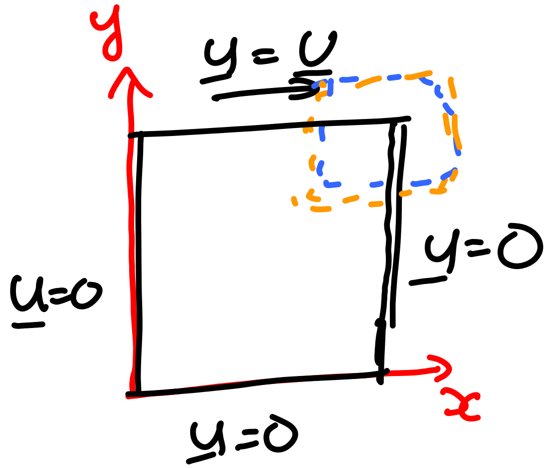
$$= \epsilon_{ijk}\left[\frac{1}{2}\int_S ds\,(x_j f_k - x_k f_j)\right]$$

$$= \epsilon_{ijk}\, F^A_{jk}$$

$$u_i = \frac{\epsilon_{ijk}\, x_k R^3}{r^3}\,\frac{L_j}{8\pi\mu R^3} = \frac{\epsilon_{ijk} L_j x_k}{8\pi\mu r^3}$$

Internal flows:

Two-dimensional internal flows:



$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

$$u_x = \frac{\partial \psi}{\partial y} \qquad u_y = -\frac{\partial \psi}{\partial x}$$

$$u_x = \frac{\partial \psi}{\partial y}$$

$$u_y = -\frac{\partial \psi}{\partial x}$$

$$\begin{array}{l} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} \end{array} \left[\begin{array}{l} -\frac{\partial p}{\partial x} + \mu \nabla^2 u_x = 0 \\ -\frac{\partial p}{\partial y} + \mu \nabla^2 u_y = 0 \end{array} \right| \begin{array}{l} -\frac{\partial p}{\partial x} + \mu \nabla^2 \left(\frac{\partial \psi}{\partial y}\right) = 0 \\ -\frac{\partial p}{\partial y} + \mu \nabla^2 \left(-\frac{\partial \psi}{\partial x}\right) = 0 \end{array} \right]$$

$$\mu \frac{\partial}{\partial y}\left[\nabla^2 \frac{\partial \psi}{\partial y}\right] + \mu \frac{\partial}{\partial x}\left[\nabla^2 \frac{\partial \psi}{\partial x}\right] = 0$$

$$\nabla^2 \left[\nabla^2 \psi\right] = 0 \qquad \text{'Biharmonic equation'}$$

$$\nabla^4 \psi = 0$$

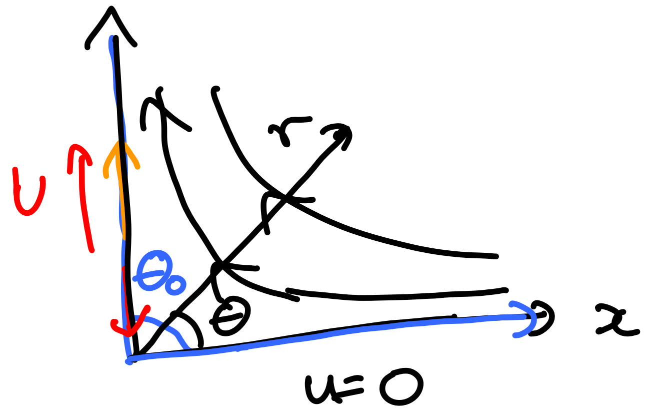
$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0$$

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad u_\theta = -\frac{\partial \psi}{\partial r}$$

$$\nabla^2(\nabla^2 \psi) = 0$$

$$\nabla^2 = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]$$

$$\psi = r f(\theta)$$



$0 \le \theta \le \theta_0$

At $\theta = 0$; $\frac{\partial \psi}{\partial \theta} = \frac{\partial \psi}{\partial r} = 0$

At $\theta = \theta_0$ $\frac{\partial \psi}{\partial r} = 0$; $\frac{1}{r}\frac{\partial \psi}{\partial \theta} = U$

Boundary conditions for f:

At $\theta = 0$ $\frac{df}{d\theta} = 0$; $f(\theta) = 0$

At $\theta = \theta_0$ $\frac{df}{d\theta} = U$ $f(\theta) = 0$

$$\nabla^2 \psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = \frac{1}{r}\left[\frac{d^2 f}{d\theta^2} + f\right]$$

$$\nabla^2(\nabla^2 \psi) = \frac{1}{r^3}\left[\frac{d^4 f}{d\theta^4} + 2\frac{d^2 f}{d\theta^2} + f\right] = 0$$

$$\frac{d^4 f}{d\theta^4} + \frac{2 d^2 f}{d\theta^2} + f = 0$$

$$f = A\cos\theta + B\sin\theta + C\theta\cos\theta + D\theta\sin\theta$$

$$f = \frac{U\left[(\theta_0-\theta)\left(\cos(\theta_0-\theta)-\cos(\theta_0+\theta)\right)-2\theta\theta_0\sin(\theta_0-\theta)\right]}{2(\theta_0^2-\sin^2\theta_0)}$$

$$\psi = r f$$

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{df}{d\theta} \; ; \; u_\theta = \frac{-\partial\psi}{\partial r} = -f(\theta)$$

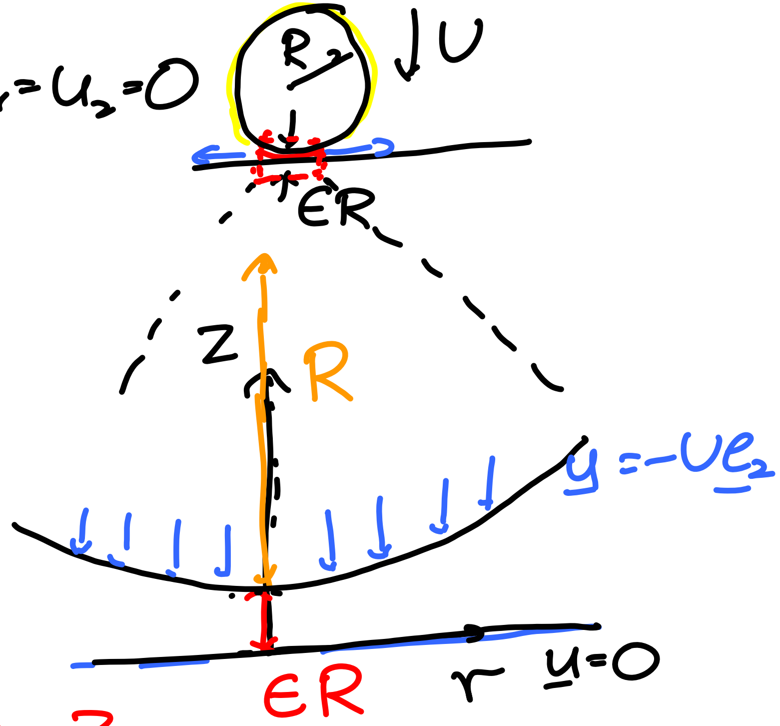
$$\tau_{r\theta} = \mu\left[\frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{2r}\frac{\partial u_r}{\partial\theta}\right]$$

$$= \frac{\mu}{2r}\left[\frac{d^2 f}{d\theta^2} + f(\theta)\right]$$

$$\tau_{r\theta}\Big|_{\theta=\theta_0} = \frac{\mu}{2r}\left[\frac{2\theta_0 - \sin(2\theta_0)}{\theta_0^2-\sin^2\theta_0}\right]$$

Lubrication flows

BC: $z=0,\ u_r=u_z=0$



$$(r_c, z_c) = (0, R(1+\epsilon))$$

$$(r-r_c)^2 + (z-z_c)^2 = R^2$$

$$(z - R(1+\epsilon))^2 = R^2 - r^2$$

$$R(1+\epsilon) - z = \sqrt{R^2-r^2}$$

$$z = R(1+\epsilon) - \sqrt{R^2 - r^2}$$

$$= R(1+\epsilon) - R\left(1 - r^2/R^2\right)^{1/2}$$

$$= R(1+\epsilon) - R\left(1 - \frac{1}{2}\frac{r^2}{R^2}\right) = R\epsilon + \frac{1}{2}\frac{r^2}{R}$$

$$z^* = \frac{z}{R\epsilon}$$

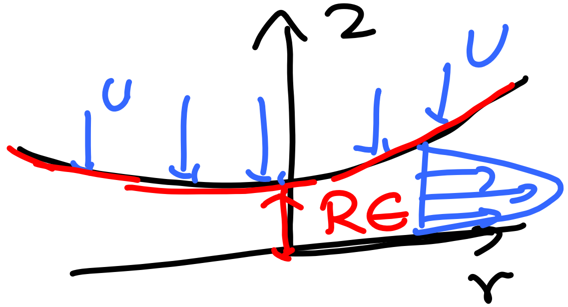
$$r^* = \frac{r}{R\epsilon^{1/2}}$$

$$z^* = 1 + \frac{1}{2}\frac{r^2}{R^2\epsilon} = 1 + \frac{1}{2} r^{*2} + \frac{1}{2!} \times \left(\frac{1}{2}\right)^2 \frac{r^4}{R^4\epsilon}$$

Lubrication flow

$$z^* = \frac{z}{R\epsilon} \qquad r^* = \frac{r}{R\epsilon^{1/2}}$$

$$z^* = h(r^*) = 1 + \frac{1}{2} r^{*2}$$



$$u_z^* = (u_z/U) \qquad u_r^* = \frac{u_r}{(U/\epsilon^{1/2})}$$

At $z^* = 0$, $u_r = 0$, $u_z^* = 0$

At $z^* = h$ $u_r = 0$, $u_z^* = -1$

$$p^* = \frac{p}{(\mu U / R\epsilon^2)}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{\partial u_z}{\partial z} = 0$$

$$\frac{1}{R\epsilon^{1/2}}\left(\frac{1}{r^*}\frac{\partial}{\partial r^*}(r^* u_r)\right) + \frac{U}{R\epsilon}\frac{\partial u_z^*}{\partial z^*} = 0$$

$$\left(\frac{\epsilon^{1/2}}{U}\right)\left(\frac{1}{r^*}\frac{\partial}{\partial r^*}(r^* u_r)\right) + \frac{\partial u_z^*}{\partial z^*} = 0$$

$$\frac{1}{r^*}\frac{\partial}{\partial r^*}(r^* u_r^*) + \frac{\partial u_z^*}{\partial z^*} = 0$$

Scalings $z^* = \frac{z}{R\epsilon}$ $r^* = \frac{r}{R\epsilon^{1/2}}$ $u_z^* = \frac{u_z}{U}$ $u_r^* = \frac{u_r}{(U/\epsilon^{1/2})}$

$t^* = t/(R\epsilon/U)$

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left(\frac{\partial^2 u_r}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)\right)$$

$$\rho\left(\frac{U}{\epsilon^{1/2}}\frac{\partial u_r^*}{\partial t} + \left(\frac{U^2}{\epsilon}\right)\left(\frac{1}{R\epsilon^{1/2}}\right)u_r^*\frac{\partial u_r^*}{\partial r^*} + \left(\frac{U^2}{\epsilon^{1/2}}\right)\left(\frac{1}{R\epsilon}\right)u_z^*\frac{\partial u_r^*}{\partial z^*}\right)$$

$$= -\frac{1}{R\epsilon^{1/2}}\frac{\partial p}{\partial r^*} + \mu\left[\frac{U}{\epsilon^{1/2}}\frac{1}{(R\epsilon)^2}\frac{\partial^2 u_r^*}{\partial z^{*2}} + \left(\frac{U}{\epsilon^{1/2}}\right)\left(\frac{1}{R^2\epsilon}\right)\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial u_r^*}{\partial r^*}\right)\right]$$

$$\frac{\rho U^2}{R\epsilon^{3/2}}\left[\frac{\partial u_r^*}{\partial t^*} + u_r^*\frac{\partial u_r^*}{\partial r^*} + u_z^*\frac{\partial u_r^*}{\partial z^*}\right] = -\frac{1}{R\epsilon^{1/2}}\frac{\partial p}{\partial r^*}$$

$$+ \frac{\mu U}{R^2\epsilon^{5/2}}\left[\frac{\partial^2 u_r^*}{\partial z^{*2}} + \epsilon\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial u_r^*}{\partial r^*}\right)\right]$$

$$\boxed{\left(\frac{\rho U R\epsilon}{\mu}\right)}\left[\frac{\partial u_r^*}{\partial t^*} + u_r^*\frac{\partial u_r^*}{\partial r^*} + u_z^*\frac{\partial u_r^*}{\partial z^*}\right] = -\boxed{\left(\frac{R\epsilon^2}{\mu U}\right)\frac{\partial p}{\partial r^*}}$$

$$+\mu\left[\frac{\partial^2 u_r^*}{\partial z^{*2}} + \frac{\epsilon}{r^*}\frac{\partial}{\partial r}\left(r^*\frac{\partial u_r^*}{\partial r}\right)\right]$$

$$Re_\epsilon\left[\frac{\partial u_r^*}{\partial t^*} + u_r^*\frac{\partial u_r^*}{\partial r^*} + u_z^*\frac{\partial u_r^*}{\partial z^*}\right] = -\frac{\partial p^*}{\partial r^*} + \mu\left[\frac{\partial^2 u_r^*}{\partial z^{*2}} + \frac{\epsilon}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial u_r^*}{\partial r^*}\right)\right]$$

$$Re_\epsilon = \left(\frac{\rho U R \epsilon}{\mu}\right)$$

$$-\frac{\partial p^*}{\partial r^*} + \frac{\partial^2 u_r^*}{\partial z^{*2}} = 0$$

$$\rho\left[\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + u_z\frac{\partial u_z}{\partial z}\right] = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 u_z}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)\right]$$

$$u_z^* = (u_z/U); \quad u_r^* = u_r/(U/\epsilon^{1/2}); \quad z^* = \frac{z}{R\epsilon}; \quad r^* = \frac{r}{R\epsilon^{1/2}} \quad p^* = \frac{p}{(\mu U/R\epsilon^2)}$$

$$\rho\left[\frac{U^2}{R\epsilon}\frac{\partial u_z^*}{\partial t^*} + \left(\frac{U^2}{\epsilon^{1/2}}\right)\left(\frac{1}{R\epsilon^{1/2}}\right)u_r^*\frac{\partial u_z^*}{\partial r^*} + \frac{U^2}{R\epsilon}u_z^*\frac{\partial u_z^*}{\partial z^*}\right]$$

$$= \left(\frac{\mu U}{R^2 \epsilon^3}\right) \frac{\partial p^*}{\partial z^*} + \mu \left[\frac{U}{R^2 \epsilon^2} \frac{\partial^2 u_z^*}{\partial z^{*2}} + \frac{U}{R^2 \epsilon} \frac{1}{r^*} \frac{\partial}{\partial r^*}\left(r^* \frac{\partial u_z^*}{\partial r^*}\right)\right]$$

$$\frac{\rho U^2}{R \epsilon}\left[\frac{\partial u_z^*}{\partial t^*} + u_r^* \frac{\partial u_z^*}{\partial r^*} + u_z^* \frac{\partial u_z^*}{\partial z^*}\right] = \left(\frac{\mu U}{R^2 \epsilon^3}\right) \frac{\partial p^*}{\partial z^*}$$

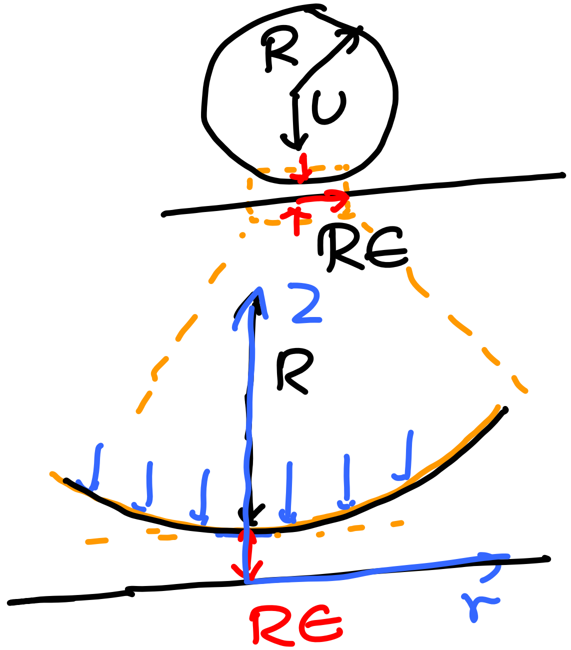
$$+ \frac{\mu U}{R^2 \epsilon^2}\left[\frac{\partial^2 u_z^*}{\partial z^{*2}} + \epsilon \frac{1}{r^*} \frac{\partial}{\partial r^*}\left(r^* \frac{\partial u_z^*}{\partial r^*}\right)\right]$$

$$\left(\frac{\rho U R \epsilon^2}{\mu}\right)\left[\frac{\partial u_z^*}{\partial t^*} + u_r^* \frac{\partial u_z^*}{\partial r^*} + u_z^* \frac{\partial u_z^*}{\partial z^*}\right] = \boxed{-\frac{\partial p^*}{\partial z^*}}$$

$$+ \epsilon \left[\frac{\partial u_z^*}{\partial z^{*2}} + \epsilon \frac{1}{r^*} \frac{\partial}{\partial r^*}\left(r^* \frac{\partial u_z^*}{\partial r^*}\right)\right]$$

$$\frac{\partial p^*}{\partial z^*} = 0 \; ; \; -\frac{\partial p^*}{\partial r^*} + \frac{\partial^2 u_z^*}{\partial z^{*2}} = 0$$

Lubrication flows at low Reynolds number



At $z=0$, $u_r=0$, $u_z=0$

At $z=h(r)$ $u_r=0$ $u_z=-U$

$z^* = (z/R\epsilon)$ $\quad r^* = (r/R\epsilon^{1/2})$

$$(z-z_c)^2 + r^2 = R^2$$

$$(z-R(1+\epsilon))^2 = R^2 - r^2$$

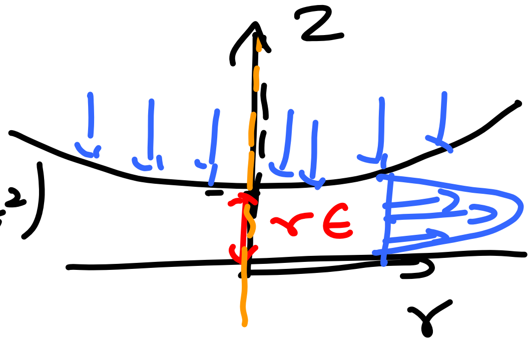
$$R(1+\epsilon) - z = \sqrt{R^2 - r^2}$$

$$\cancel{R} + R\epsilon - z = R\left[\cancel{1} - \frac{r^2}{2R^2}\right]$$

$$z = R\epsilon + \frac{1}{2}\frac{r^2}{R} \qquad z^* = 1 + \frac{1}{2} r^{*2}$$

$$z^* = \frac{z}{R\epsilon} \; ; \quad r^* = \frac{r}{R\epsilon^{1/2}} \; ; \quad h(r^*) = 1 + \frac{1}{2} r^{*2}$$

$$u_z^* = (u_z/U) \qquad u_r^* = u_r/(U/\epsilon^{1/2}) \qquad p^* = \frac{p}{(\mu U/R\epsilon^2)}$$



$$\rho \left[\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} \right] = -\frac{\partial p}{\partial z} + \mu \left[\frac{\partial^2 u_z}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u_z}{\partial r} \right) \right]$$

$$\frac{\rho U R \epsilon^2}{\mu} \left[\frac{\partial u_z^*}{\partial t^*} + u_r^* \frac{\partial u_z^*}{\partial r^*} + u_z^* \frac{\partial u_z^*}{\partial z^*} \right] = -\frac{\partial p^*}{\partial z^*}$$

$$+ \epsilon \frac{\partial^2 u_z^*}{\partial z^{*2}} + \epsilon^2 \frac{1}{r^*} \frac{\partial}{\partial r^*}\left(r^* \frac{\partial u_z^*}{\partial r^*} \right)$$

$$\frac{\rho U R \epsilon}{\mu} \ll 1$$

$$-\frac{\partial p^*}{\partial r^*} + \frac{\partial^2 u_r^*}{\partial z^{*2}} = 0 \qquad \frac{1}{r^*}\frac{\partial}{\partial r^*}(r^* u_r^*) + \frac{\partial u_z^*}{\partial z^*} = 0$$

B.C. At $z^*=0$ $u_r^*=0$, $u_z^*=0$

$z^*=h(r^*)$ $u_r^*=0$, $u_z^*=-1$

$$\frac{\partial p^*}{\partial z^*} = 0$$

$$\frac{\partial^2 u_r^*}{\partial z^{*2}} = \frac{\partial p^*}{\partial r^*}$$

$$u_r^* = \frac{\partial p^*}{\partial r^*}\frac{z^{*2}}{2} + C_1(r^*)z^* + C_2(r^*)$$

$$u_r^* = \frac{\partial p^*}{\partial r^*}\left(\frac{z^{*2}}{2} - \frac{z^* h}{2}\right)$$

$$\int_0^{h(r^*)} dz^* \frac{1}{r^*}\frac{\partial}{\partial r^*}(r^* u_r^*) + \int_0^{h(r^*)} dz \frac{\partial u_z^*}{\partial z^*} = 0$$

$$\int_0^{h(r^*)} dz^* \frac{1}{r^*}\frac{\partial}{\partial r^*}(r^* u_r^*) + \left[u_z^*\Big|_{z=h(r)} - u_z^*\Big|_{z=0} \right] = 0$$

$$\int_0^{h(r^*)} dz^* \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* u_r^*\right) + \left[-1-0\right] = 0$$

$$\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \int_0^{h(r^*)} dz^*\, u_r^*\right) = 1$$

$$\frac{1}{r^*}\frac{\partial}{\partial r^*}\left[r^* \int_0^{h(r^*)} dz^* \left[\left(\frac{\partial p^*}{\partial r^*}\right)\left(\frac{z^{*2}}{2} - \frac{z^* h}{2}\right)\right]\right] = 1$$

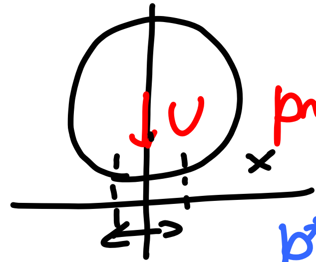
$$\frac{1}{r^*}\frac{\partial}{\partial r^*}\left[r^*\left(-\frac{h^3}{12}\frac{\partial p^*}{\partial r^*}\right)\right] = 1$$

$$\frac{\partial p^*}{\partial r^*} = -\frac{6r^*}{h(r^*)^3} - \frac{C_1}{r^* h(r^*)^3} \qquad C_1 \to 0$$

$$h(r^*) = 1 + \tfrac{1}{2} r^{*2}$$

$$p^* = \frac{3}{\left(1+\tfrac{1}{2}r^{*2}\right)^2} + C_2$$

$p^* \to 0$ as $r^* \to \infty \Rightarrow C_2 = 0$

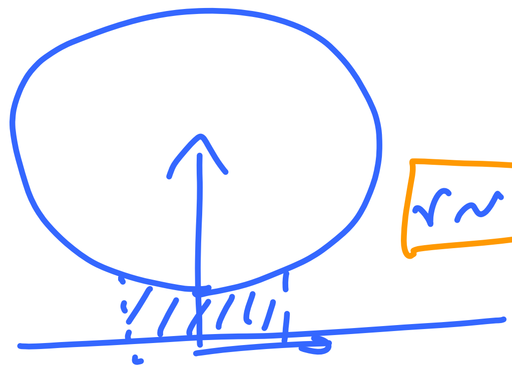


$p \sim \frac{\mu U}{R}$

$p^* \sim \epsilon^2 \cong 0$

$$p^* = \frac{p}{(\mu U / R\epsilon^2)}$$

$$p^* = \frac{3}{\left(1+\frac{1}{2}r^{*2}\right)^2}$$



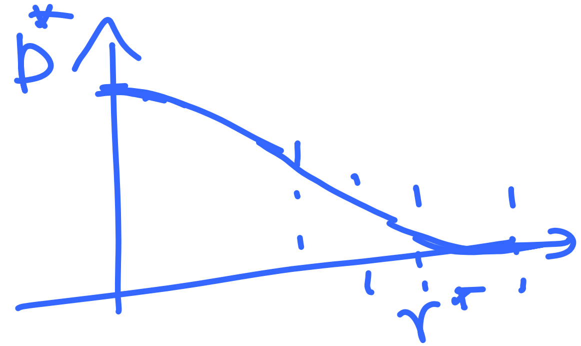
$r \sim R$

$r \sim R\epsilon^{1/2}$

$r^* \sim 1$

$$p \sim \frac{\mu U}{R\epsilon^2} \qquad \text{Area} \sim R^2\epsilon$$

$$\text{Force} \sim \frac{\mu U}{R\epsilon^2} \times R^2\epsilon \sim \boxed{\frac{\mu U R}{\epsilon}}$$



$$p \sim \frac{\mu U}{R} \qquad \text{Area} \sim R^2$$

$$\text{Force} \sim \boxed{\mu U R}$$

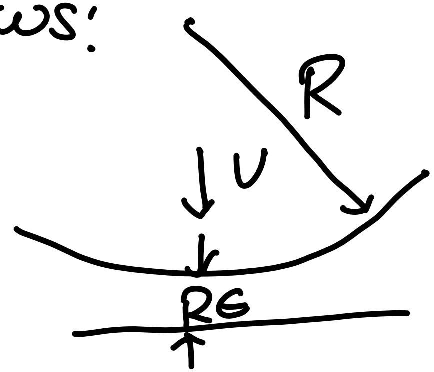
$0 < r < R$

$0 < r^* < \frac{R}{R\epsilon^{1/2}}$

$$F_z = 2\pi \int r\,dr\,p = 2\pi\left(R^2\epsilon\right)\left(\frac{\mu U}{R\epsilon^2}\right)\int_0^\infty r^* dr^*\, p^*$$

$$= 2\pi \frac{\mu R U}{\epsilon}\int_0^\infty r^* dr^* \frac{3}{\left(1+\frac{1}{2}r^{*2}\right)^2} = \boxed{\frac{6\pi\mu R U}{\epsilon}}$$

Low Reynolds number viscous flows:



$$\nabla \cdot \underline{u} = 0$$

$$-\nabla p + \mu \nabla^2 \underline{u} = 0$$

$$\nabla^2 p = 0$$

$$\nabla^2 \underline{u}^{(g)} = 0$$

$$\underline{u} = \underline{u}^{(g)} + \frac{1}{2\mu} p \underline{x}$$

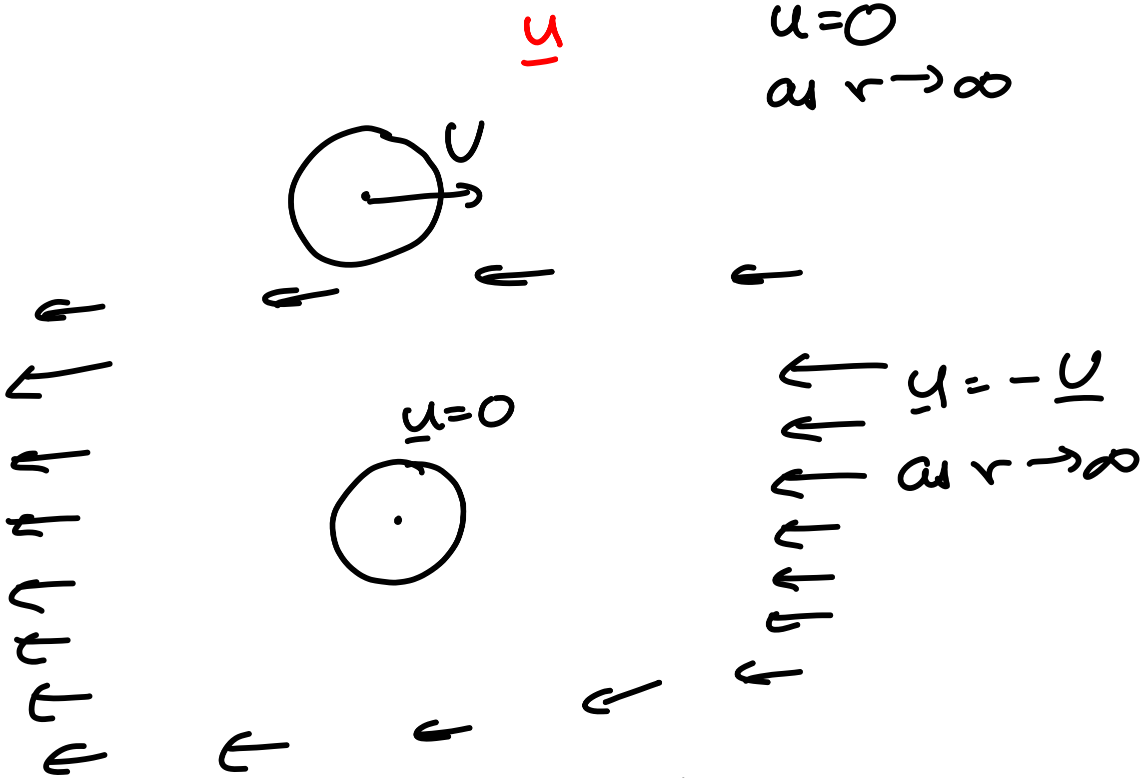
$$F = \frac{6\pi\mu R U}{\epsilon}$$

$$u_x = \frac{\partial \psi}{\partial y} \qquad u_y = -\frac{\partial \psi}{\partial x}$$

$$\nabla^2(\nabla^2 \psi) = 0$$

$$\nabla^4 \psi = 0$$

Inertial corrections to Stokes flow:



$$\underline{u} = -\underline{U} + \underline{u}'(\underline{x})$$

$$\nabla\cdot\underline{u} = 0$$

$$\rho\left[\underline{u}\cdot\nabla\underline{u}\right] = -\nabla p + \mu\nabla^2\underline{u}$$

$$\rho\left[(-\underline{U}+\underline{u}')\cdot\nabla\underline{u}'\right] = -\nabla p + \mu\nabla^2\underline{u}'$$

$$\underline{u}' \propto \frac{1}{r} \quad \text{as} \quad r \to \infty \qquad \underline{u}'^* \propto \left(\frac{R}{r}\right)$$

$$\nabla \underline{u}' \propto \frac{1}{r^2} \qquad \nabla^* \underline{u}'^* \propto (R/r)^2$$

$$\nabla^2 \underline{u}' \propto \frac{1}{r^3} \qquad \nabla^{*2} u'^* \propto (R/r)^3$$

$$r^* = (r/R) \; ; \; \underline{u}'^* = (\underline{u}'/U) \qquad \underline{U}^* = (\underline{U}/U) \; ; \; \nabla^* = R^{-1}\nabla$$

$$Re\left[(-\underline{U}^* + u'^*) \cdot \nabla^* \underline{u}'^*\right] = -\nabla^* p^* + \nabla^{*2} \underline{u}'^*$$

$$Re = \left(\frac{\rho U R}{\mu}\right)$$

$$Re\left[(\underbrace{-\underline{U}^*}_{O(1)} + \underbrace{\underline{u}'^*}_{(R/r)}) \cdot \underbrace{\nabla^* \underline{u}'^*}_{(R/r)^2}\right] = -\nabla^* p^* + \underbrace{\nabla^{*2} u'^*}_{(R/r)^3}$$

$$Re \left(\frac{R}{r}\right)^2 \qquad \left(R/r\right)^3$$

Inertial terms become important

for $\frac{r}{R} \propto Re^{-1} \Longrightarrow \frac{r}{R} \propto \frac{\mu}{\rho U R}$

or $r \propto \frac{\mu}{\rho U}$

$$Re\left[\left(\underline{U}^* + \underline{u}^{*\prime}\right)\cdot\nabla^*\underline{u}^{*\prime}\right] = -\nabla^* p^* + \nabla^{*2}\underline{u}^{\prime *}$$

$O(\underline{U}) \quad O\left(\frac{\underline{U}R}{r}\right)$

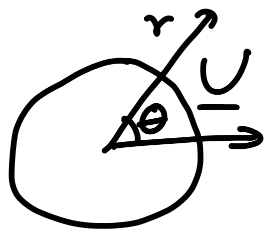
For $\frac{r}{R} \ll Re^{-1}$

For $\frac{r}{R} \gg Re^{-1}$ $\quad \underline{u}^{*\prime} \ll \underline{U}^*$

$$-Re\,\underline{U}^*\cdot\nabla^*(\underline{u}^{\prime *}) = -\nabla^* p^* + \nabla^{*2}\underline{u}^{\prime *}$$

Oseen equation

$$-\rho\,\underline{U}\cdot\nabla\underline{u}^{\prime} = -\nabla p + \mu\nabla^2\underline{u}^{\prime}$$



$$u_r = \frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta} \qquad u_\theta = \frac{-1}{r \sin\theta} \frac{\partial \psi}{\partial r}$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 u_r \right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \, u_\theta \right) = 0$$

$$\psi = UR^2 \left[\frac{-1}{4} \frac{R}{r} \sin^2\theta + 3(1-\cos\theta) \left[\frac{1 - \exp\left(-\frac{1}{8} Re (1+\cos\theta) \frac{r}{R}\right)}{2 \, Re} \right] \right]$$

$$Re = \frac{\rho U R}{\mu} \qquad \psi = UR^2 \sin^2\theta \left[-\frac{1}{4} \frac{R}{r} + \frac{3}{4} \frac{r}{R} \right]$$

$$F_i = 6 \pi \mu R U_i \left(1 + \frac{3}{8} Re \right)$$ 'Oseen correction'

High Reynolds number 'potential flow':

$$\nabla \cdot \underline{u} = 0 \qquad \nabla \cdot (\nabla \phi) = 0 \qquad \nabla^2 \phi = 0$$

$$\rho\left[\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u}\right] = -\nabla p + \boxed{\mu \nabla^2 \underline{u}} + \underline{f}$$

① Inviscid

② Irrotational $\underline{\omega} = \nabla \times \underline{u} = 0$

|| Potential

$$\underline{u} = \nabla \phi \Rightarrow \nabla \times \nabla \phi = 0$$

ϕ = Velocity potential

$$\rho\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)=-\frac{\partial p}{\partial x_i}+f_i$$

$$\rho\left(\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial x_i}\right)+\frac{\partial}{\partial x_i}\left(\frac{1}{2}u_j^2\right)\right)=-\frac{\partial p}{\partial x_i}+f_i$$

$$\frac{\partial}{\partial x_i}\left[\rho\frac{\partial\phi}{\partial t}+\frac{1}{2}\rho u_j^2\right]=-\frac{\partial p}{\partial x_i}+f_i$$

$$f_i=-\frac{\partial V}{\partial x_i}$$

$$\frac{\partial}{\partial x_i}\left[p+V+\rho\frac{\partial\phi}{\partial t}+\frac{1}{2}\rho u_j^2\right]=0$$

$$p+V+\rho\frac{\partial\phi}{\partial t}+\frac{1}{2}\rho u_j^2=p_0$$

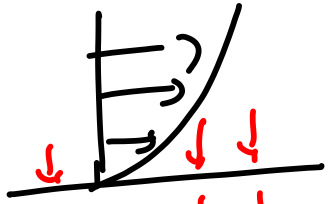
'Bernoulli equation'

$$p+\rho g z+\rho\frac{\partial\phi}{\partial t}+\frac{1}{2}\rho u_j^2=p_0$$

Potential flow equations

$$\nabla^2 \phi = 0$$

$$u_i = \frac{\partial \phi}{\partial x_i}$$



$$\underline{u} \cdot \underline{n} = \underline{U} \cdot \underline{n}$$

$$p + \frac{1}{2} \rho u_i^2 + \rho \frac{\partial \phi}{\partial t} + V = p_0$$

$$T_{ij} = -p\,\delta_{ij}$$

High Reynolds number potential flow:

Inviscid $\qquad \mu = 0$

Irrotational $\qquad \underline{\omega} = \nabla \times \underline{u} = 0 \qquad \underline{u} = \nabla \phi$

$f = -\nabla V$

$\nabla \cdot \underline{u} = 0 \qquad \nabla^2 \phi = 0$

$$\rho\left[\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right] = -\nabla p + \mu \nabla^2 \underline{u} + \underline{f}$$

$$\frac{\partial}{\partial x_i}\left[p + \rho\frac{\partial \phi}{\partial t} + \frac{1}{2}\rho u_j^2 + V\right] = 0$$

$$p + \rho\frac{\partial \phi}{\partial t} + \frac{1}{2}\rho u_j^2 + V = p_0$$

$$T_{ij} = -p\,\delta_{ij}$$

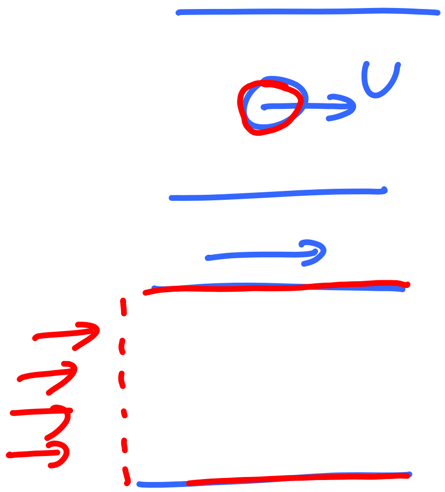
$$\frac{\partial c}{\partial t} + \underline{u}\cdot\nabla c = D\nabla^2 c$$

$$D\nabla^2 c = 0$$

$$\frac{\partial c}{\partial t} + \underline{u}\cdot\nabla c = 0$$

$$\underline{u}\cdot\nabla c = 0$$

If there is no <u>normal</u> velocity at boundaries, the fluid velocity is zero everywhere.



Kinetic energy $= \int dV \left(\frac{1}{2}\rho u_i^2\right) \geq 0$

$$= \frac{1}{2}\rho\int dV\, u_i^2 = \frac{1}{2}\rho\int dV\, u_i \times u_i$$

$$= \frac{1}{2}\rho\int dV\, u_i \times \frac{\partial \phi}{\partial x_i}$$

$$= \frac{1}{2}\rho\int dV \left(\frac{\partial}{\partial x_i}(u_i \phi) - \phi \cancel{\frac{\partial u_i}{\partial x_i}}\right)$$

$$= \frac{1}{2}\rho\int dV\, \frac{\partial}{\partial x_i}(u_i \phi) = \frac{1}{2}\rho\int dS\, (n_i u_i \phi)$$

Uniqueness: For specified normal velocity conditions at boundaries, the potential flow solution is unique.

Assume two solutions (u_i, u_i^*) satisfy the normal velocity conditions $u_i n_i = u_i^* n_i$ at the surface.

$$I = \int dV\,(u_i^* - u_i)(u_i^* - u_i) \geq 0 = 0$$

$$= \int dV\,(u_i^* - u_i)\frac{\partial}{\partial x_i}(\phi^* - \phi)$$

$$= \int dV \left\{ \frac{\partial}{\partial x_i}\left[(u_i^* - u_i)(\phi^* - \phi)\right] - \underbrace{(\phi^* - \phi)\frac{\partial}{\partial x_i}(u_i^* - u_i)}_{0} \right]$$

$$= \int dV\,\frac{\partial}{\partial x_i}\left((u_i^* - u_i)(\phi^* - \phi)\right) = \int dS\, n_i(u_i^* - u_i)(\phi^* - \phi)$$

Minimum kinetic energy principle:

For a given set of zero normal velocity conditions at surface, the total kinetic energy of a potential flow is smaller than that of any other flow.

$(\underline{u}, \underline{u}^*)$ $\quad \underline{u} = \nabla \phi$ $\quad \nabla \cdot \underline{u}^* = \nabla \cdot \underline{u} = 0$

$$KE = \int dV \left(\tfrac{1}{2}\rho u^2\right) \leq \int dV \tfrac{1}{2}\rho u^{*2}$$

$$KE^* - KE = \tfrac{1}{2}\rho \int dV \left(u_i^{*2} - u_i^2\right)$$

$$= \tfrac{1}{2}\rho \int dV \left[\left(u_i^* - u_i\right)^2 + 2u_i\left(u_i^* - u_i\right)\right]$$

$$= \tfrac{1}{2}\rho \int dV \left(u_i^* - u_i\right)^2 + \rho \int dV \left[u_i\left(u_i^* - u_i\right)\right]$$

$$= \tfrac{1}{2}\rho \int dV \left(u_i^* - u_i\right)^2 + \rho \int dV \left(\frac{\partial \phi}{\partial x_i}\right)\left(u_i^* - u_i\right)$$

$$= \tfrac{1}{2}\rho \int dV \left(u_i^* - u_i\right)^2 + \rho \int dV \left[\frac{\partial}{\partial x_i}\left(\phi\left(u_i^* - u_i\right)\right) - \cancel{\phi \frac{\partial}{\partial x_i}\left(u_i^* - u_i\right)}^{\,0}\right]$$

$$= \frac{1}{2}\rho\int dV\,(u_i^* - u_i)^2 + \rho\int dV\,\frac{\partial}{\partial x_i}\left(\Phi(u_i^* - u_i)\right)$$

$$KE^* - KE = \frac{1}{2}\rho\int dV\,(u_i^* - u_i)^2 + \rho\int dS\,n_i(u_i^* - u_i)\,\Phi$$

$$KE = \frac{1}{2}\rho\int dV\,u_i^2 = \frac{1}{2}\rho\int dS\,n_i\,\Phi u_i$$

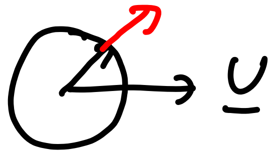
Flow around a sphere:

$$\nabla^2 \phi = 0$$

$$\phi = A\, U_j\, \Phi_j^{(1)} = \frac{A U_j x_j}{r^3}$$

$$u_i = \frac{\partial \phi}{\partial x_i} = A U_j \left[\frac{\delta_{ij}}{r^3} - \frac{3 x_i x_j}{r^5} \right]$$

As $r \to \infty$, $u_i = 0$



At $r=R$ $\quad u_i n_i = U_i n_i$

$$n_i \frac{\partial \phi}{\partial x_i} = U_i n_i$$

$$n_i = \frac{x_i}{r}$$

$$u_i n_i = U_i n_i$$

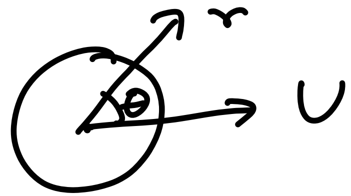
$$A U_j \left[\frac{\delta_{ij}}{r^3} - \frac{3 x_i x_j}{r^5} \right] \frac{x_i}{r} = \frac{U_i x_i}{r} \quad \text{at } r = R$$

$$A \frac{U_i x_i}{r^4} - \frac{3 A U_j x_i^2 x_j}{r^6} = \frac{U_i x_i}{r}$$

$$-\frac{2 A U_j x_j}{r^4} = \frac{U_i x_i}{r} \quad \text{at } r = R$$

$$A = -\frac{R^3}{2}$$

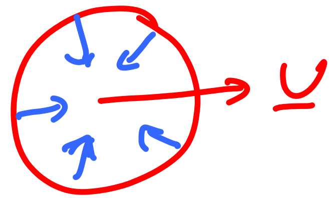
$$\phi = -\frac{R^3}{2}\frac{U_j x_j}{r^3} = -\frac{R^3}{2}\frac{U\cos\theta}{r^2}$$



$$u_i = -\frac{R^3}{2}\left[\frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}\right]$$

$$u_r = \frac{\partial\phi}{\partial r} = -\frac{R^3 U\cos\theta}{r^3}$$

$u_i n_i = U_i n_i$



$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{R^3 U\sin\theta}{2r^3}$$

$$KE = \frac{1}{2}\rho\int ds\, \phi\, n_j u_j$$

$$= -\frac{1}{2}\rho\int ds\, \phi\, u_j\left(\frac{x_j}{r}\right)$$

$$= -\frac{1}{2}\rho\int ds\, \phi\, \frac{U_j x_j}{r}$$

$$\phi = -\frac{R^3}{2}\frac{U_i x_i}{r^3}$$

$$KE = \frac{R^3}{4}\rho\int ds\left(\frac{U_i x_i}{r^3}\right)\left(\frac{U_j x_j}{r}\right)$$

$$= \frac{\rho}{4R}\, U_i U_j \int ds\, x_i x_j$$

$$\int ds\, x_i x_j = A\,\delta_{ij} \qquad A = \frac{4\pi R^4}{3}$$

$$= \frac{4\pi R^4}{3}\delta_{ij}$$

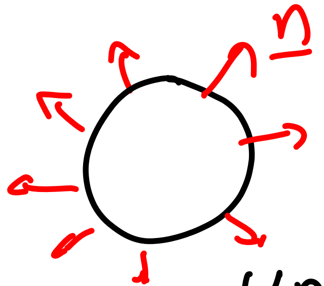
$$KE = \frac{8\pi R^3}{3} U_j^2 = \frac{1}{2}\rho U_j^2\left(\frac{2}{3}\pi R^3\right)$$

$$= \frac{1}{2} M_a U_j^2$$

M_a = Added mass = $\rho\left(\frac{2}{3}\pi R^3\right)$ =

= $\frac{1}{2}$ of mass of fluid displaced by the sphere.

Potential flows:



$\underline{u}\cdot\underline{n} = \underline{U}\cdot\underline{n}$

Inviscid

Irrotational $\nabla \times \underline{u} = \underline{\omega} = 0$

$$\underline{u} = \nabla \phi \quad \text{'Velocity potential'}$$

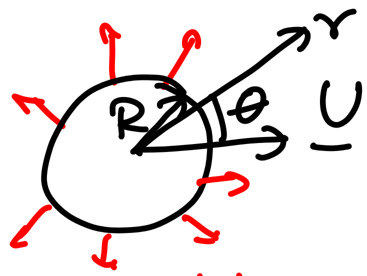
$$\nabla \cdot \underline{u} = 0 \Longrightarrow \nabla^2 \phi = 0$$

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \underline{f} \qquad \underline{f} = -\nabla V$$

$$\nabla\left(p + \tfrac{1}{2}\rho u^2 + \rho\frac{\partial \phi}{\partial t} + V\right) = 0 \qquad T_{ij} = -p\,\delta_{ij}$$

$$p + \tfrac{1}{2}\rho u^2 + \rho\frac{\partial \phi}{\partial t} + V = p_0$$

$$\nabla^2 \phi = 0$$



$u_i n_i = U_i n_i$ at $r = R$

$u_i \to 0$ as $r \to \infty$

$$\phi = \text{Linear}(U_i, \Phi^{(n)})$$

$$= A\, U_j\, \Phi_j^{(1)} = \frac{A\, U_j x_j}{r^3}$$

$$n_i = \frac{x_i}{r} = \frac{x_i}{R}$$

$$u_i n_i = U_i n_i$$

$$u_i = \frac{\partial \phi}{\partial x_i} = A U_j \left(\frac{\delta_{ij}}{r^3} - \frac{3 x_i x_j}{r^5} \right)$$

$$A U_j \left(\frac{\delta_{ij}}{r^3} - \frac{3 x_i x_j}{r^5} \right) \frac{x_i}{r} = \frac{U_i x_i}{r} \text{ at } r = R$$

$$A = -\frac{R^3}{2} \Rightarrow \phi = -\frac{R^3}{2} \frac{U_j x_j}{r^3} \;;\; u_i = -\frac{R^3}{2} \left(\frac{\delta_{ij}}{r^3} - \frac{3 x_i x_j}{r^5} \right)$$

$$KE = \frac{1}{2}\rho\int_V dV\, u_i^2 = \frac{1}{2}\rho\int ds\, u_i n_i \phi$$

$$= \frac{1}{2}\rho\int_{S_\infty} ds\, \cancelto{0}{u_i n_i \phi} +$$

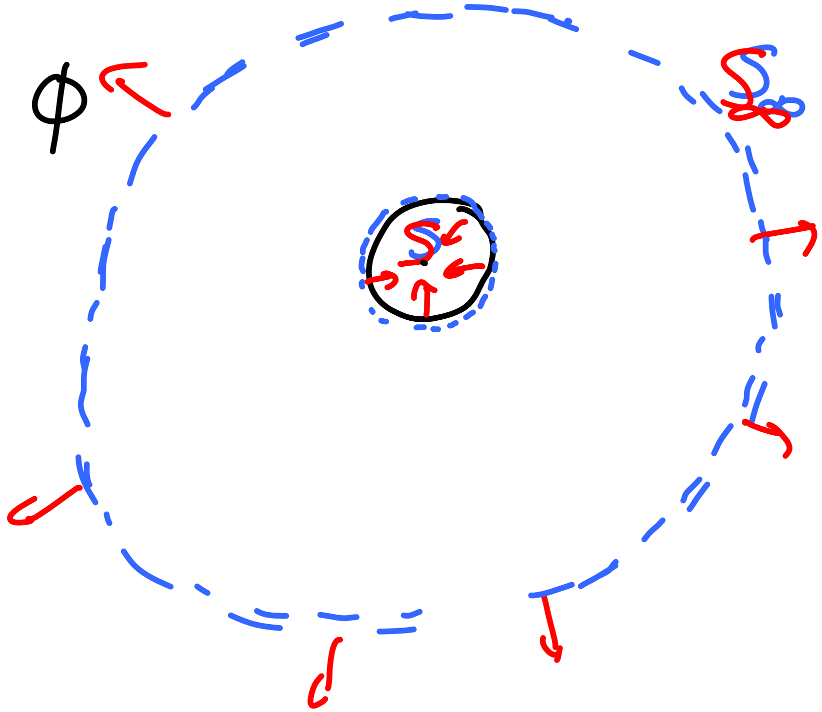
$$\frac{1}{2}\rho\int_S ds\, u_i \left(\frac{-x_i}{r}\right)\phi$$

$$= -\frac{1}{2}\rho\int_S ds\, u_i \frac{x_i}{r}\phi$$

$$= -\frac{1}{2}\rho\int_S ds\, \frac{U_i x_i}{r}\phi$$

$$= -\frac{1}{2}\rho\int_S ds\, \frac{U_i x_i}{r}\left(-\frac{R^3}{2}\frac{U_j x_j}{r^3}\right)$$

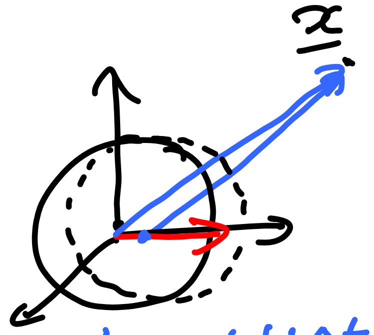
$$= \frac{\rho U_i U_j}{4R}\int ds\, x_i x_j = \frac{1}{2} M_a U_i^2$$



$$M_a = \left(\frac{2}{3}\pi R^3 \rho\right)$$

$$f_i = T_{ij} n_j = -p\delta_{ij} n_j = -p n_i$$

$$F_i = \int ds\,(-p n_i)$$



$$x_i' = x_0 + U\Delta t$$

$$p + \tfrac{1}{2}\rho u_i^2 + \rho\frac{\partial\phi}{\partial t} + V = p_0$$

$$\phi = -\frac{R^3}{2}\,\frac{U_j x_j}{r^3}$$

$$p = p_0 - \tfrac{1}{2}\rho u_i^2 - \rho\frac{\partial\phi}{\partial t}$$

$$\frac{\partial\phi}{\partial t} = -\frac{R^3 x_j}{2r^3}\frac{dU_j}{dt} - U\cdot u$$

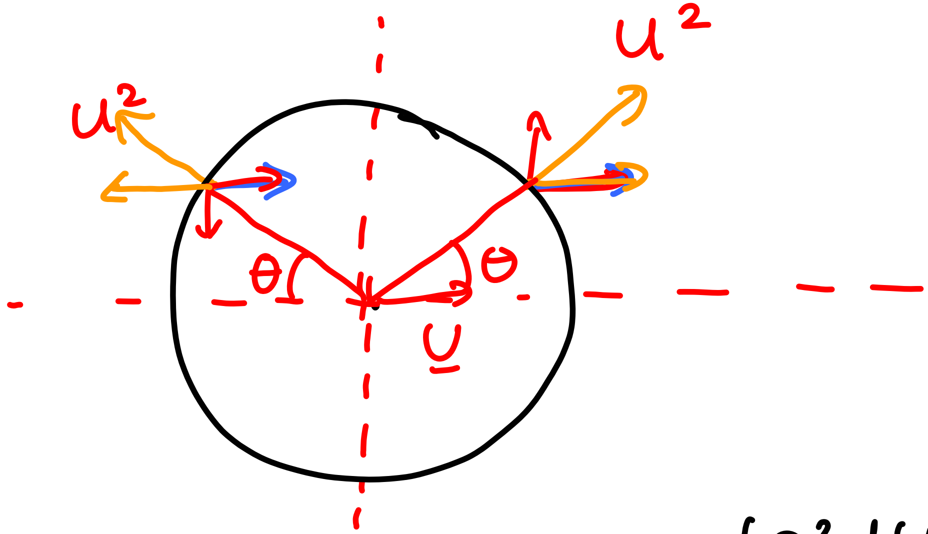
$$p = p_0 - \tfrac{1}{2}\rho u_i^2 \boxed{-\frac{\rho R^3 x_j}{2r^3}\frac{dU_j}{dt}} + \rho U_j u_j$$

For steady flow

$$p = p_0 - \tfrac{1}{2}\rho u_i^2 + \rho U_i u_i$$

$$F_i = \int ds\, n_i\left[p_0 - \tfrac{1}{2}\rho u_j^2 + \rho U_j u_j\right]$$

$F_i = 0$ 'd'Alembert's paradox'



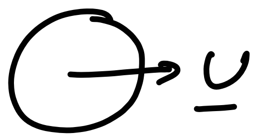
$$F_i = \int ds\, n_i(-p) = \int ds\, n_i \rho \left[\frac{R^3}{2}\frac{dU_j}{dt}\frac{x_j}{r^3}\right]$$

$$= \frac{\rho}{2}\frac{dU_j}{dt}\int ds\, \frac{x_i}{r}\, x_j$$

$$= M_a \frac{dU_j}{dt}$$

where $M_a = \left(\frac{2}{3}\pi R^3 \rho\right)$

Potential flow:



$$u_i = \frac{\partial \phi}{\partial x_i}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \Longrightarrow \nabla^2 \phi = 0$$

$$p + \frac{1}{2}\rho u_i^2 + \rho \frac{\partial \phi}{\partial t} = p_0$$

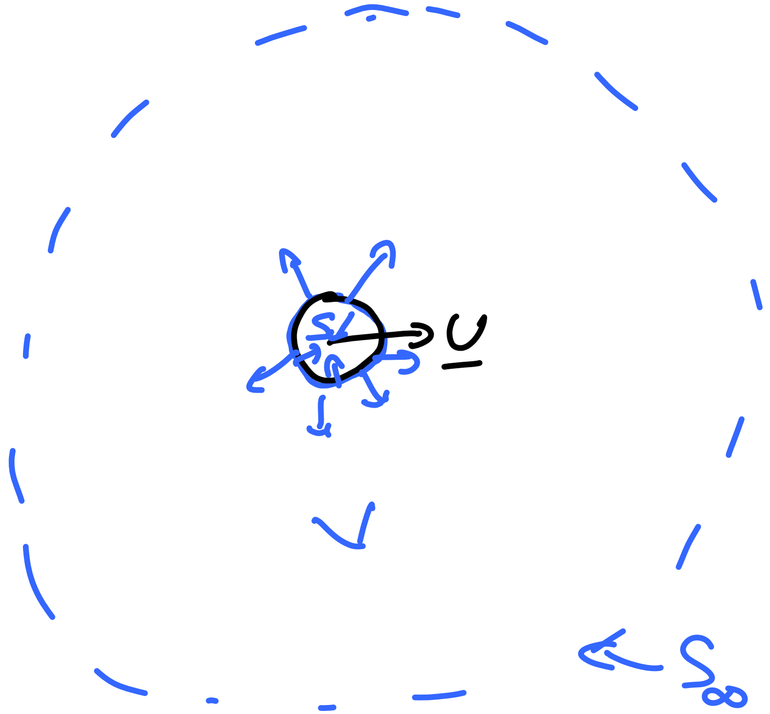
$$\left.\begin{aligned} F_i &= M_a \frac{dU_i}{dt} \\ KE &= \frac{1}{2} M_a U_i^2 \end{aligned}\right] M_a = \rho\left(\frac{2}{3}\pi R^3\right)$$

$$F_i = \int ds\,(-p\,n_i)$$

$$= -\int ds\left(p_0 - \frac{1}{2}\rho u_j^2 - \rho\frac{\partial\phi}{\partial t}\right)n_i$$

$$= -\int ds\left(p_0 - \tfrac{1}{2}\rho u_j^2 + \rho U_j n_j\right)n_i$$

$$= \int ds\left(\tfrac{1}{2}\rho u_j^2 - \rho U_j u_j\right)n_i$$



$$\int_V dV\,\frac{\partial}{\partial x_i}\left(\frac{\rho u_j^2}{2} - \rho U_j u_j\right) = \int_{S_\infty} ds\,n_i\left(\frac{\rho u_j^2}{2} - \rho U_j u_j\right) - \int_S ds\,n_i\left(\frac{\rho u_j^2}{2} - \rho U_j u_j\right)$$

$$= \int_{S_\infty} ds\,n_i\left(\frac{\rho u_j^2}{2} - \rho U_j u_j\right) - F_i$$

$$F_i = \int_{S_\infty} ds\,n_i\left(\frac{\rho u_j^2}{2} - \rho U_j u_j\right) - \int_V dV\,\frac{\partial}{\partial x_i}\left(\frac{\rho u_j^2}{2} - \rho u_j U_j\right)$$

$$= \int_{S_\infty} ds\, n_i \left(\frac{\rho u_j^2}{2} - \rho U_j u_j \right) - \int dV \left[\rho u_j \frac{\partial u_j}{\partial x_i} - \rho U_j \frac{\partial u_j}{\partial x_i} \right]$$

$$F_i = \int_{S_\infty} ds\, n_i \left(\frac{\rho u_j^2}{2} - \rho u_j U_j \right) - \int dV \left[\rho u_j \frac{\partial u_i}{\partial x_j} - \rho U_j \frac{\partial u_i}{\partial x_j} \right]$$

$$\rho u_j \frac{\partial u_i}{\partial x_j} \equiv \rho \frac{\partial}{\partial x_j}(u_i u_j) - \cancel{\rho u_i \frac{\partial u_j}{\partial x_j}}$$

$$\rho U_j \frac{\partial u_i}{\partial x_j} \equiv \rho \frac{\partial}{\partial x_j}(u_i U_j)$$

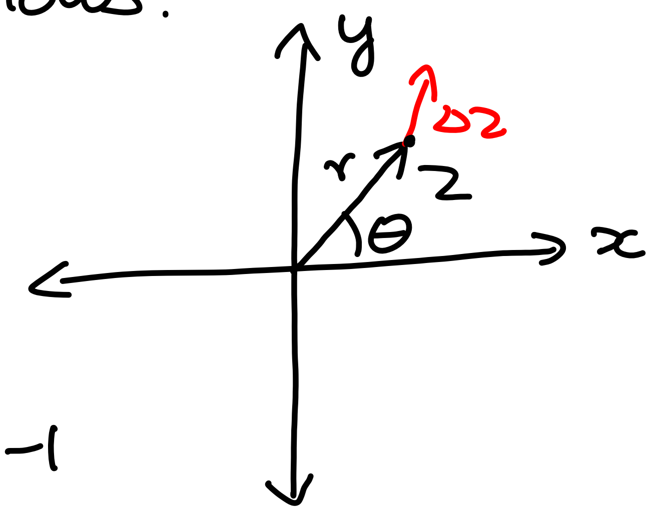
$$F_i = \int_{S_\infty} ds \left[n_i \left(\frac{\rho u_j^2}{2} - \rho u_j U_j \right) \right] - \int dV \left[\rho \frac{\partial}{\partial x_j}(u_i u_j) - \rho \frac{\partial}{\partial x_j}(u_i U_j) \right]$$

$$= \int_{S_\infty} ds \left[n_i \left(\frac{\rho u_j^2}{2} - \rho u_j U_j \right) \right] - \left[\int_{S_\infty} ds\, n_j \left(\rho u_i u_j - \rho u_i U_j \right) - \int_{S} ds\, n_j \left(\rho u_i u_j - \rho u_i U_j \right) \right]$$

$$F_i = \int_{S_\infty} ds\, n_i \left(\frac{\rho u_j^2}{2} - \rho u_j V_j \right) - \int_{S_\infty} ds\, n_j \left(\rho u_i u_j - \rho u_i V_j \right)$$

$$+ \int_{S} ds\, (\rho u_i)\left(n_j u_j - n_j V_j \right)$$

Two-dimensional potential flows:



$$\nabla^2\phi = 0 \qquad \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

$$z = x + iy = re^{i\theta}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$i^2 = -1$$

$$F(z) = \phi(x,y) + i\psi(x,y)$$

Analytic function:

$$F(z+\Delta z) - F(z) = \Delta F = \left(\frac{dF}{dz}\right)\Delta z = \frac{dF}{dz}(\Delta x + i\Delta y)$$

$$\Delta F = \Delta x\left(\frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x}\right) + \Delta y\left(\frac{\partial\phi}{\partial y} + i\frac{\partial\psi}{\partial y}\right)$$

$$= \Delta x\left(\frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x}\right) + i\Delta y\left(-i\frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial y}\right)$$

$$\frac{\partial}{\partial y}\left[\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}\right]$$

$$\frac{\partial}{\partial x}\left[\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}\right]$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

For the potential flow

$$F = \phi(x,y) + i\,\psi(x,y)$$

Potential function = ϕ

$$u_x = \frac{\partial \phi}{\partial x} \qquad u_y = \frac{\partial \phi}{\partial y}$$

$$\Delta F = (\Delta x + i\Delta y)\left(\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}\right)$$

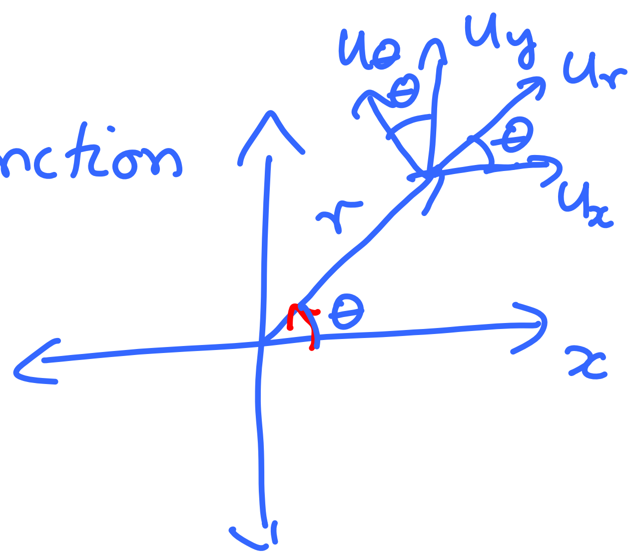
$$\Delta F = \Delta z\left(\frac{\partial \phi}{\partial x} - i\,\frac{\partial \phi}{\partial y}\right) = \Delta z\,(u_x - i u_y)$$

$$W = \frac{dF}{dz} = u_x - i u_y$$

$$\frac{\partial \phi}{\partial x} = u_x = \frac{\partial \psi}{\partial y}$$

$$\frac{\partial \phi}{\partial y} = u_y = -\frac{\partial \psi}{\partial x}$$

ψ = Stream function



$$F(z) = \phi(x,y) + i\,\psi(x,y)$$

$$\frac{dF}{dz} = u_x - i u_y = W$$

$$u_x = u_r \cos\theta - u_\theta \sin\theta$$

$$u_y = u_r \sin\theta + u_\theta \cos\theta$$

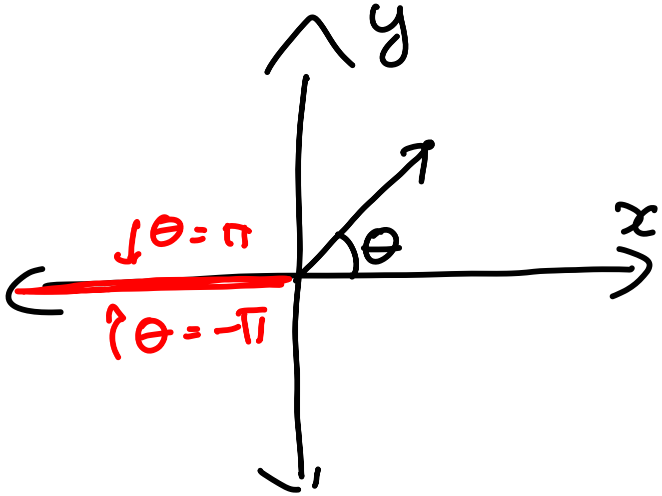
$$\frac{dF}{dz} = (u_r \cos\theta - u_\theta \sin\theta) - i(u_r \sin\theta + u_\theta \cos\theta)$$

$$= (u_r - i u_\theta)(\cos\theta - i\sin\theta) = (u_r - i u_\theta)e^{-i\theta}$$

$$F(z) = \phi(x,y) + i\,\psi(x,y)$$

$$W = \frac{dF}{dz} = u_x - i u_y$$

$$= (u_r - i u_\theta) e^{-i\theta}$$



$\theta \to \theta + 2\pi$

$F(z)$ which are analytic?

$= z, z^2, \dots z^n, e^z, \sin z, \cos z,$

$\frac{1}{z}, \frac{1}{z^2} \dots$

$z = x + iy$

$\bar{z} = x - iy$

$\log z$

$$F(z) = \log(z) = \log(re^{i\theta})$$

$$= \log r + i\theta$$

Potential flow:

Inviscid

Irrotational $\underline{\omega} = \nabla \times \underline{u} = 0$

$$\underline{u} = \nabla \phi \qquad \tau_{ij} = -p\,\delta_{ij}$$

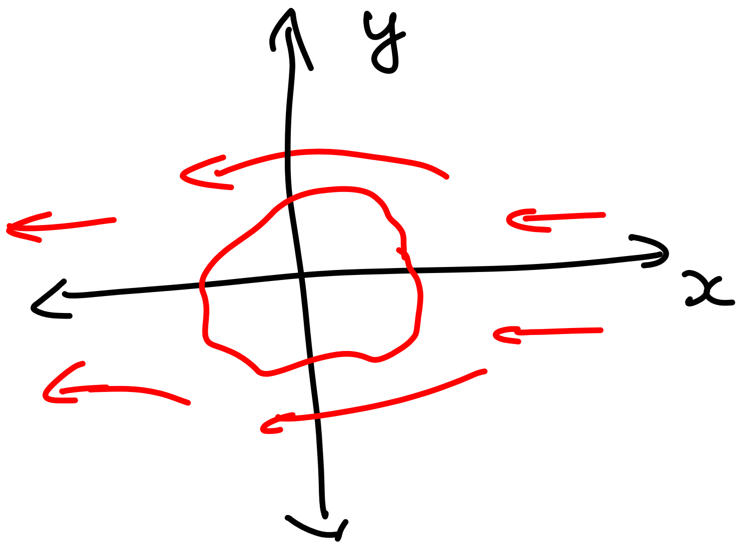
$$\nabla \cdot \underline{u} = 0 \implies \nabla^2 \phi = 0$$

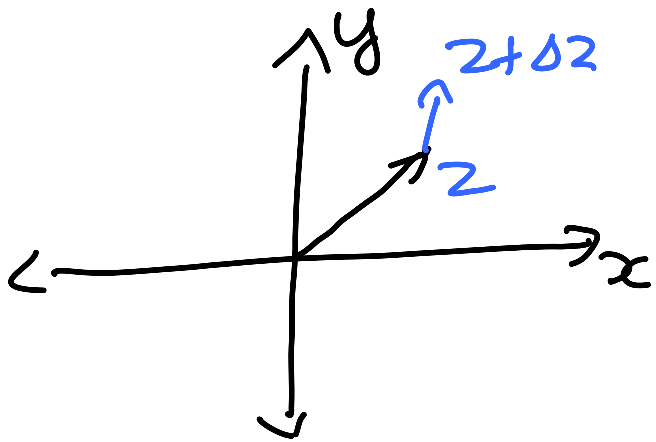
$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u}\right) = -\nabla p + \underline{f} \qquad \underline{f} = -\nabla V$$

$$\nabla\left(p + \tfrac{1}{2}\rho u^2 + \rho \frac{\partial \phi}{\partial t}\right) = 0$$

$$p + \tfrac{1}{2}\rho u^2 + \rho \frac{\partial \phi}{\partial t} = p_0$$

Two dimensional potential flow:





F(z) is analytic if

$$\lim_{\Delta z \to 0} F(z+\Delta z) - F(z) = \frac{dF}{dz}\Delta z$$

$$F(z) = \phi(x,y) + i\psi(x,y)$$

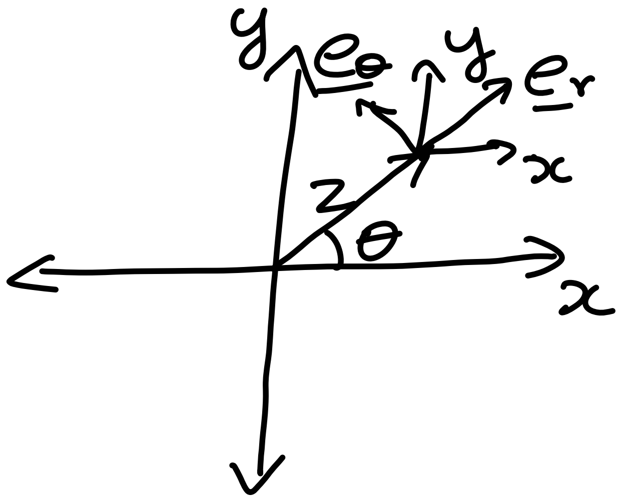
$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad \Longrightarrow \qquad \nabla^2 \phi = 0$$

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \qquad \nabla^2 \psi = 0$$

$F = \text{Complex Potential} = \phi + i\psi$

$\phi = \text{Velocity potential}$

$u_x = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad u_y = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$



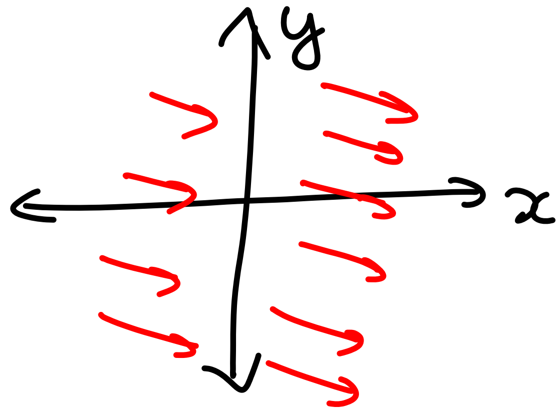
$\psi = \text{Stream function}$

$w = \frac{dF}{dz} = u_x - iu_y = (u_r - iu_\theta)e^{-i\theta}$

$z = re^{i\theta} = x + iy$

$$F = Uz \qquad W = U = (u_x - i u_y)$$

$$u_x = U, \quad u_y = 0$$



$$= U e^{i\alpha} z \qquad W = U e^{i\alpha}$$

$$= U(\cos\alpha + i \sin\alpha)$$

$$= u_x - i u_y$$

$$u_x = U\cos\alpha \qquad u_y = -U\sin\alpha$$

$$F = Az^2 \Rightarrow W = 2Az = u_x - iu_y$$

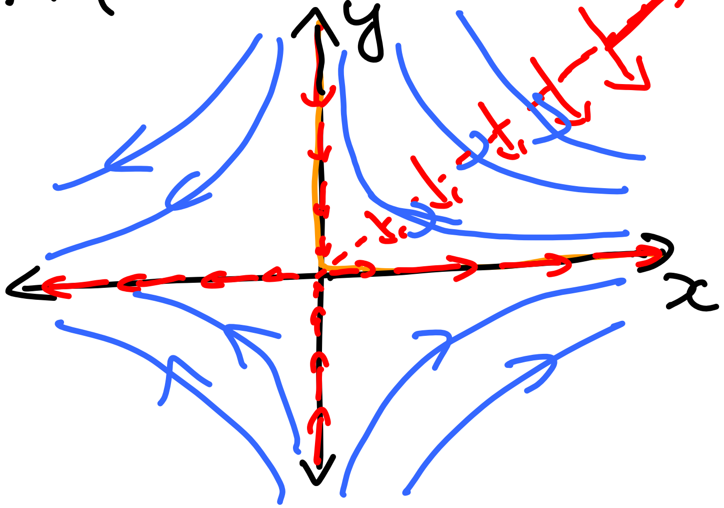
$$= 2Are^{i\theta} = (u_r - iu_\theta)e^{-i\theta}$$

$$u_r - iu_\theta = 2Are^{2i\theta}$$

$$= 2Ar(\cos 2\theta + i\sin 2\theta)$$

$$u_r = 2Ar\cos 2\theta$$

$$u_\theta = -2Ar\sin 2\theta$$



$$F = Az^m \qquad W = \frac{dF}{dz} = mAz^{m-1} = mAr^{m-1}e^{i(m-1)\theta}$$

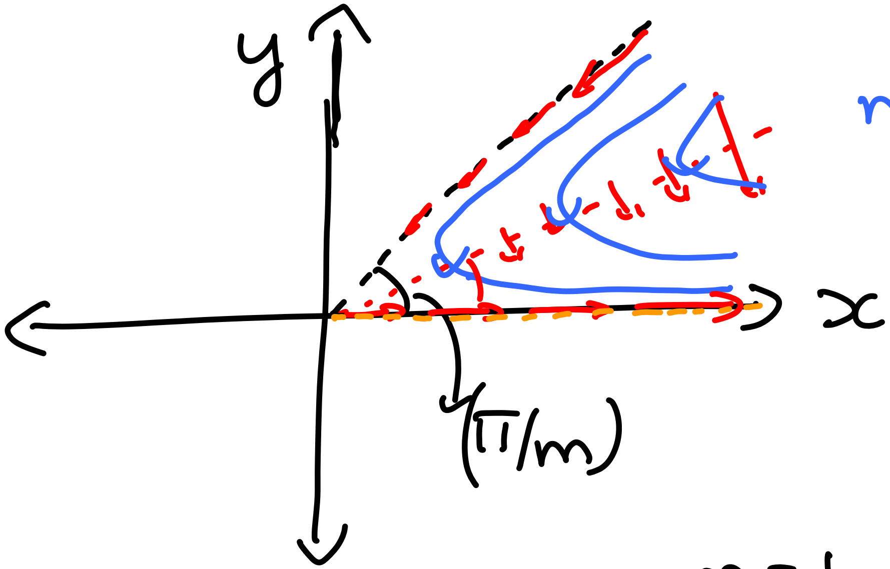
$$mAr^{m-1}e^{i(m-1)\theta} = (u_r - iu_\theta)e^{-i\theta}$$

$$mAr^{m-1}e^{im\theta} = u_r - iu_\theta$$

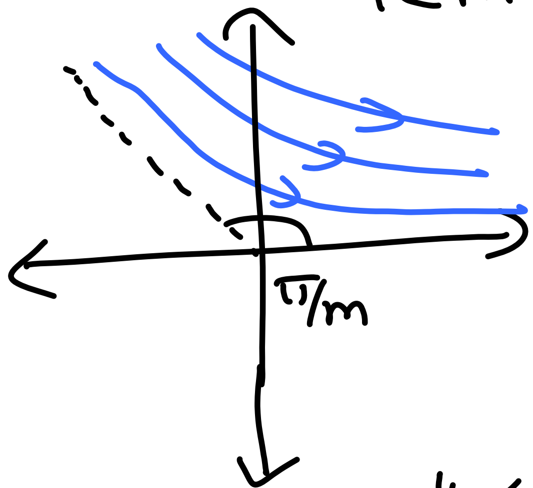
$$u_r = mAr^{m-1}\cos(m\theta)$$

$$u_\theta = -mAr^{m-1}\sin(m\theta)$$

$m > 2$

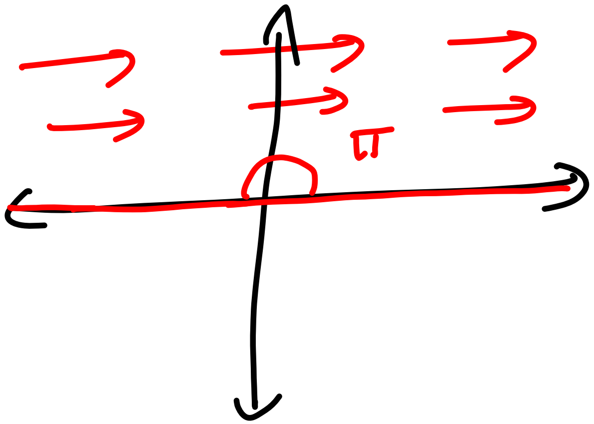


$1 < m < 2$

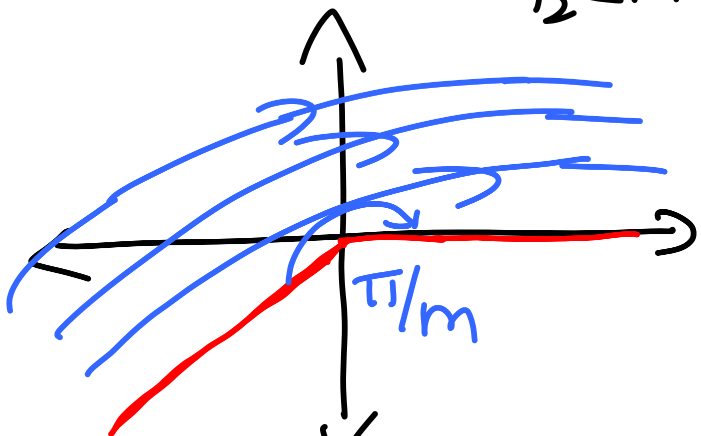


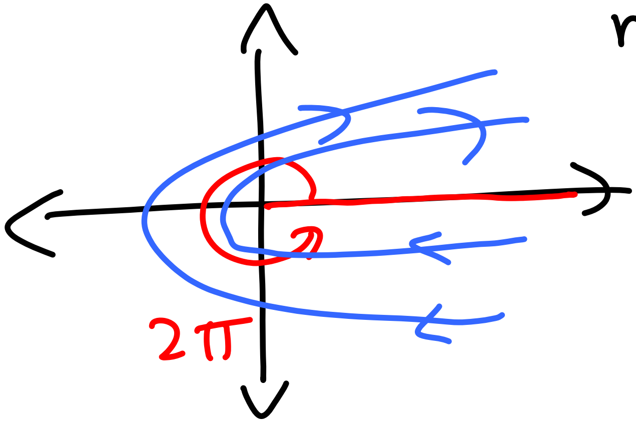
$m = 1$

$F = Az$



$\frac{1}{2} < m < 1$





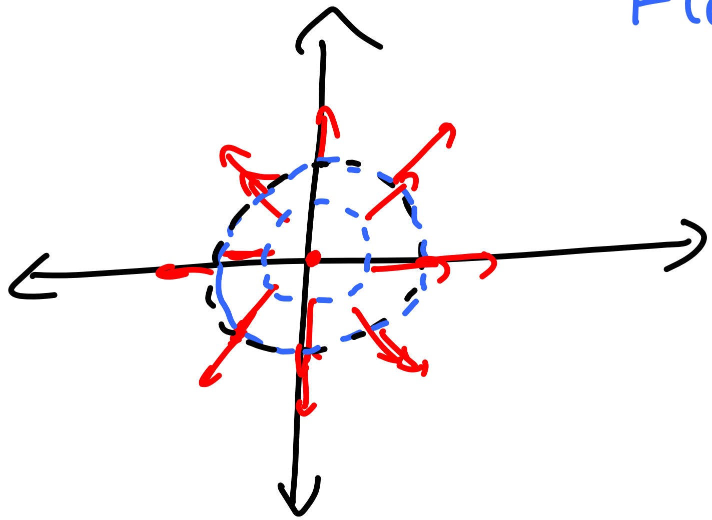
$m = 1/2$

$$F = \frac{m}{2\pi}\log z$$

$$W = \frac{m}{2\pi z} = \left(\frac{m}{2\pi r}\right)e^{-i\theta} = (u_r - i u_\theta)e^{-i\theta}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 0$$

$$u_r = \frac{m}{2\pi r} \qquad u_\theta = 0$$



$$\text{Flux} = \int_{2\pi} ds\, u_r$$

$$= \int_0^{2\pi} r\, d\theta \left(\frac{m}{2\pi r}\right)$$

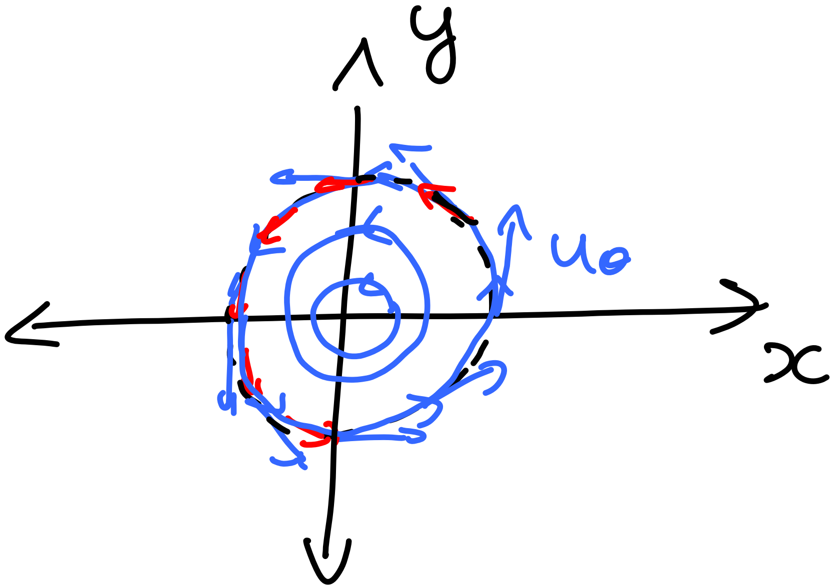
$$= m$$

$$F = \frac{-i\Gamma}{2\pi}\log z \Rightarrow w = -\frac{i\Gamma}{2\pi z} = \frac{-i\Gamma}{2\pi r}e^{-i\theta} = (u_r - iu_\theta)e^{-i\theta}$$

$u_r = 0;\ u_\theta = \dfrac{\Gamma}{2\pi r}$ Circulation $= \oint d\underline{x}\cdot\underline{u}$

$= \oint (r\,d\theta)\,u_\theta$

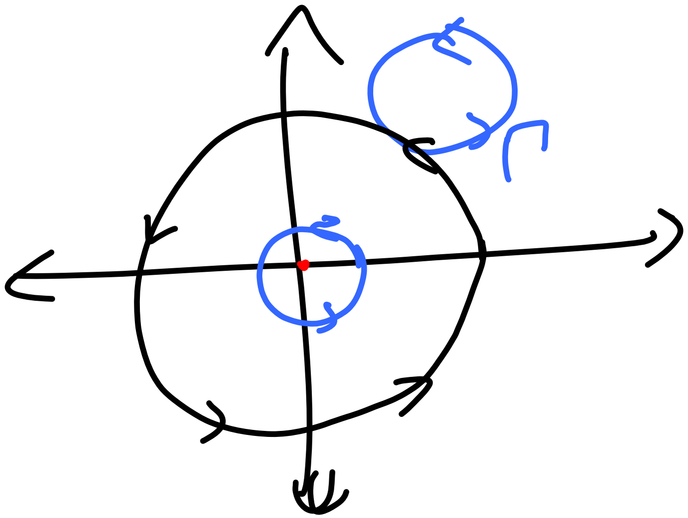
$= \oint r\,d\theta \dfrac{\Gamma}{2\pi r} = \Gamma$

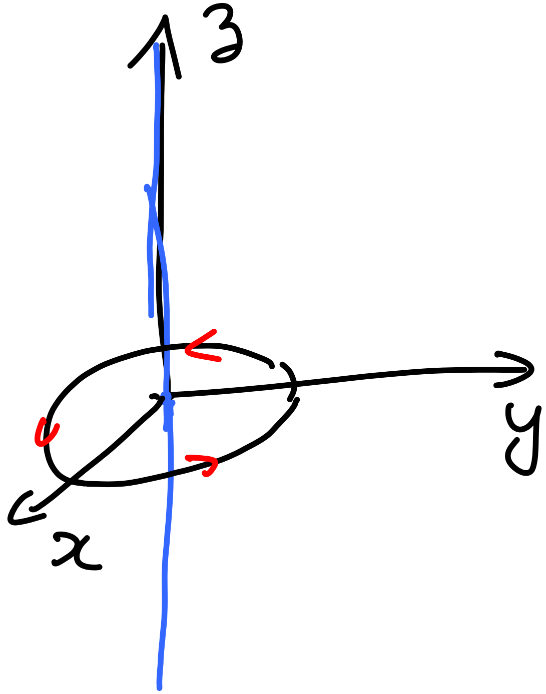


'Line vortex'

$$\int dS\, \underline{n}\cdot(\nabla\times\underline{A}) = \oint d\underline{x}\cdot\underline{A}$$

$$\int dS\, \underline{n}\cdot\underline{\omega} = \oint d\underline{x}\cdot\underline{u} = \Gamma$$

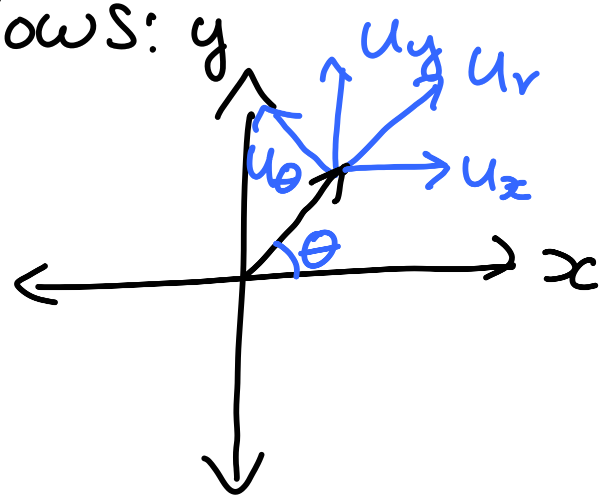




$$\underline{\omega} = \Gamma\,\delta(\underline{x}) \qquad u_\theta = \frac{\Gamma}{2\pi r}$$

$$F(z) = -\frac{i\Gamma}{2\pi}\log z$$

Two-dimensional potential flows:



$$F(z) = \phi(x,y) + i\psi(x,y)$$

'Analytic' $\Delta F = \left(\frac{dF}{dz}\right)\Delta z$

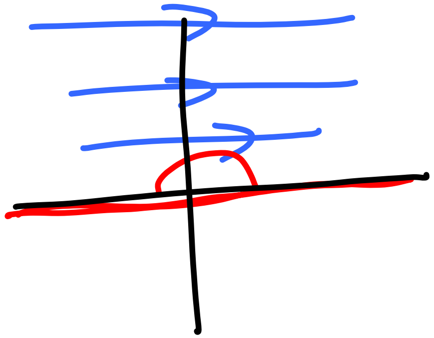
$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \qquad \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

ϕ = Velocity potential $\qquad \psi$ = Stream fn.

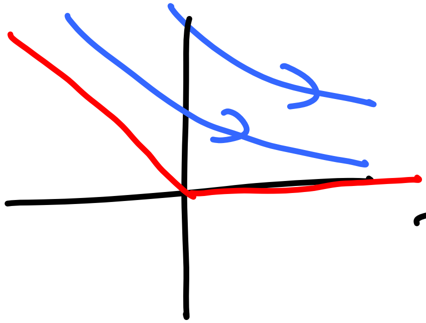
$$W(z) = \frac{dF}{dz} = u_x - i u_y$$
$$= (u_r - i u_\theta) e^{-i\theta}$$

$F = Az^n$

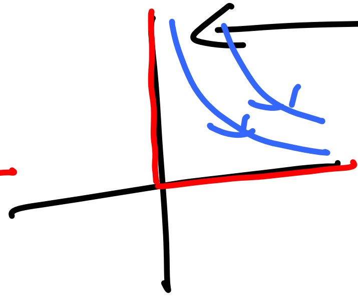
$n = 1$

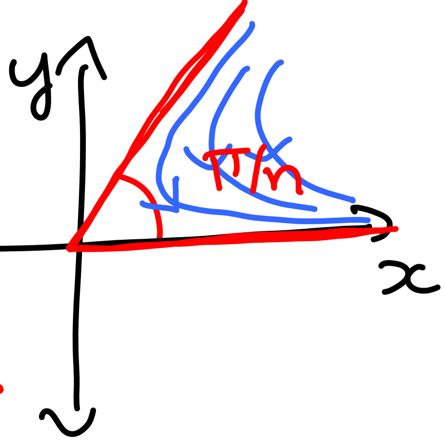


$1 < n < 2$

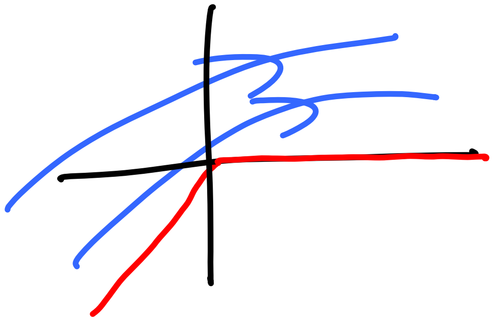


$n = 2$

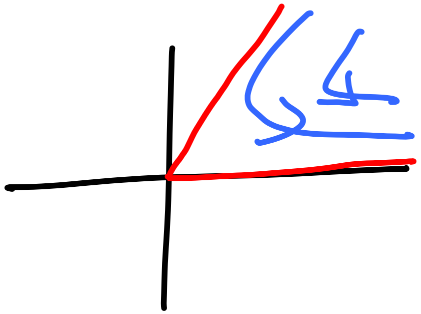




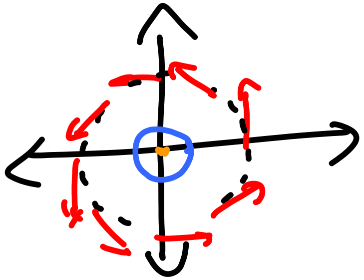
$\frac{1}{2} < n < 1$

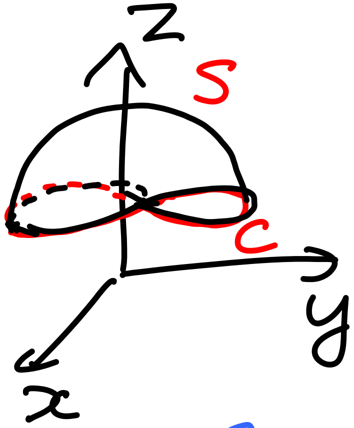


$n > 2$



$$\int_S ds\, \underline{n}\cdot(\nabla\times\underline{u}) = \oint_C d\underline{x}\cdot\underline{u}$$





$$\int_S ds\, \underline{n}\cdot\underline{\omega} = \oint_C d\underline{x}\cdot\underline{u}$$

$$= \Gamma$$

$$\int_S ds\, \omega_z$$

$$\omega_z = \Gamma\,\delta(\underline{x})$$

$$\int_S ds\, \delta(z) = 1$$

$$\boxed{F(z) = \frac{-i\Gamma}{2\pi}\log z} \qquad F(z) = \frac{-i\Gamma}{2\pi}\log(z-z_0)$$

$$w = \frac{-i\Gamma}{2\pi z}$$

$$u_r = 0 \qquad u_\theta = \left(\frac{\Gamma}{2\pi r}\right)$$

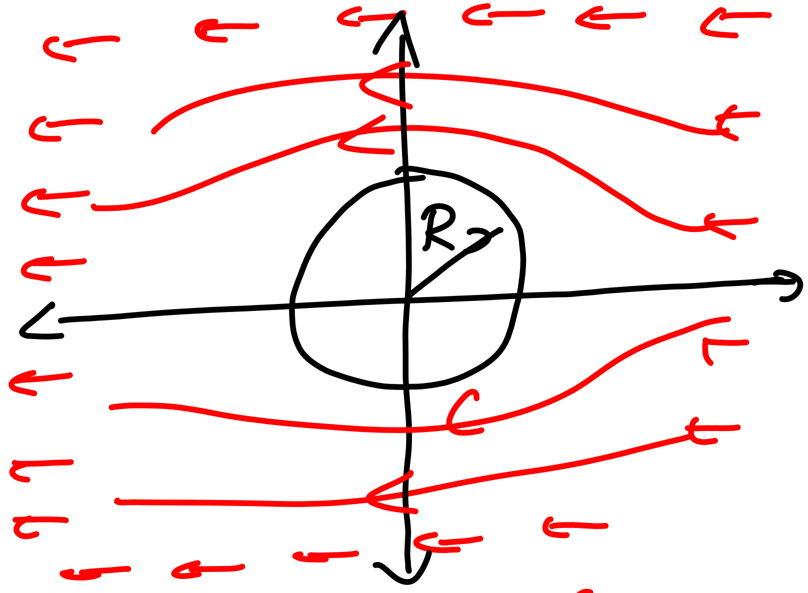
$$\Gamma = \oint d\underline{x}\cdot\underline{u}$$

$$F = -Uz - \frac{UR^2}{z} - \frac{i\Gamma}{2\pi}\log z$$

$$W = \left[U + \frac{UR^2}{z^2} - \frac{i\Gamma}{2\pi z}\right]$$

$$= \left[-U + \frac{UR^2}{r^2}e^{-2i\theta} - \frac{i\Gamma}{2\pi r}e^{-i\theta}\right]$$

$$= \left[-Ue^{i\theta} + \frac{UR^2}{r^2}e^{-i\theta} - \frac{i\Gamma}{2\pi r}\right]e^{-i\theta}$$



$\underline{u}.\underline{n} = \underline{U}.\underline{n} = 0$
at $r = R$

$$u_r = -U\cos\theta + \frac{UR^2}{r^2}\cos\theta = U\cos\theta\left[-1 + \frac{R^2}{r^2}\right]$$

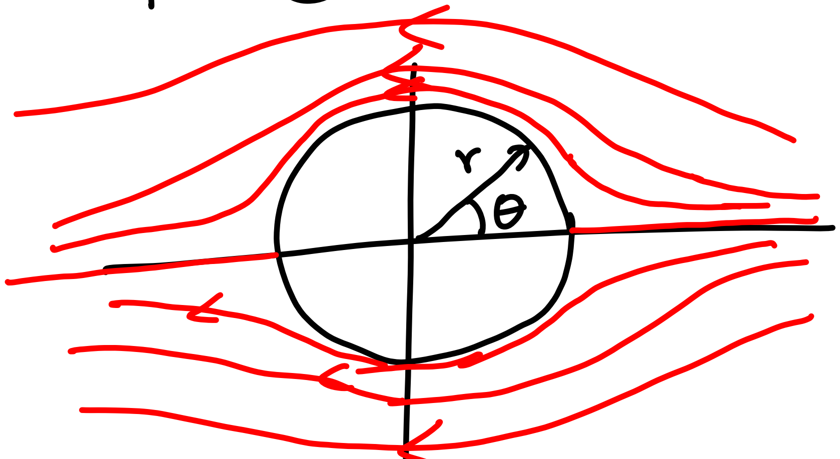
$$u_\theta = +U\sin\theta + \frac{UR^2}{r^2}\sin\theta + \frac{\Gamma}{2\pi r}$$

At $r = R$, $u_r = 0$, $u_\theta = 2U\sin\theta + \dfrac{\Gamma}{2\pi r}$

Flow around a cylinder:

$$u_r = U\cos\theta\left[-1+\frac{R^2}{r^2}\right] \qquad u_\theta = U\sin\theta\left[1+\frac{R^2}{r^2}\right]+\frac{\Gamma}{2\pi r}$$

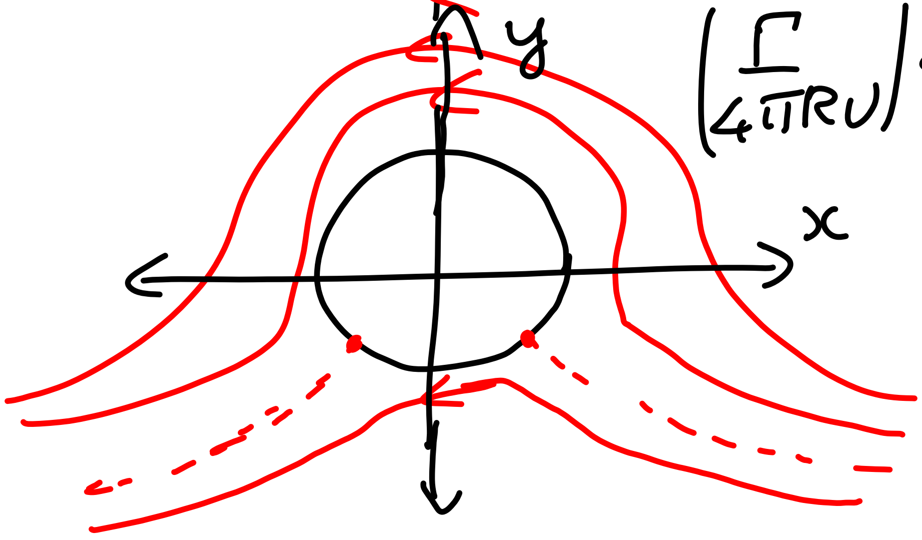
$\Gamma = 0$



At $r = R$, $u_\theta = 2U\sin\theta + \frac{\Gamma}{2\pi R}$

$$\sin\theta = -\frac{\Gamma}{4\pi R U}$$

$$\left|\frac{\Gamma}{4\pi R U}\right| < 1$$



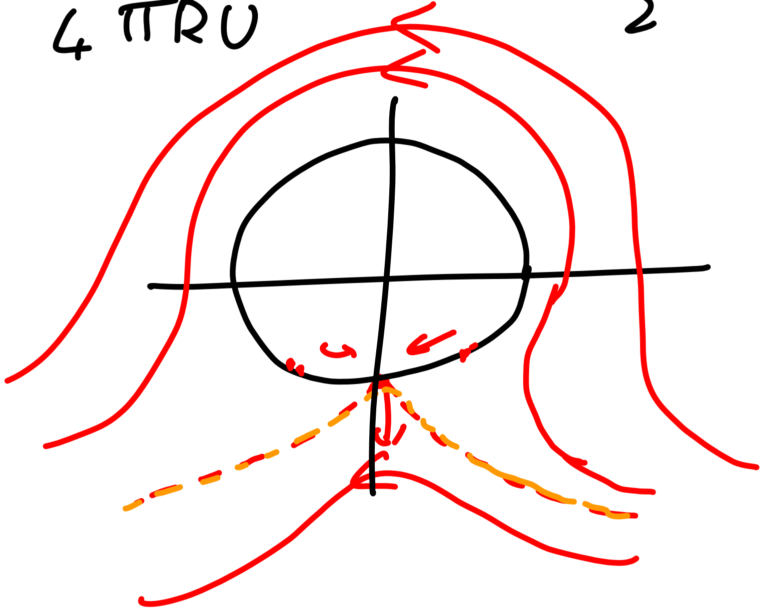
$u_r = U\cos\theta\left[-1+\frac{R^2}{r^2}\right]$ $\qquad u_\theta = U\sin\theta\left(1+\frac{R^2}{r^2}\right)+\frac{\Gamma}{2\pi r}$

$\frac{\Gamma}{4\pi RU} = 1 \Rightarrow \theta = \frac{3\pi}{2}$

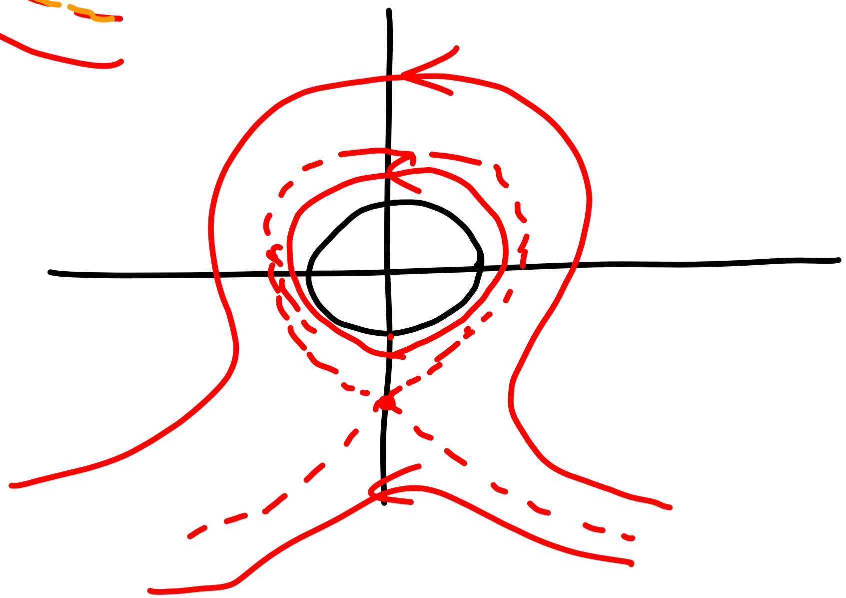
At $r=R$, $u_\theta = 2U\sin\theta + \frac{\Gamma}{2\pi R}$

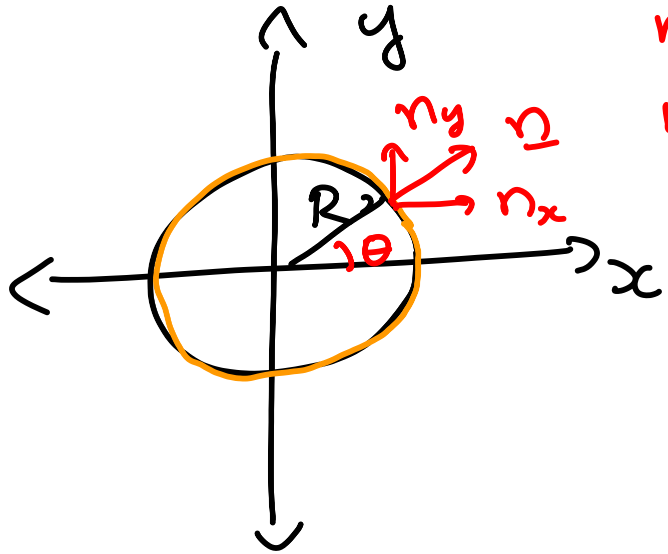
For $\Gamma/4\pi RU > 1$, $\sin\theta = -1$

$u_\theta = 0$ at $-U\left[1+\frac{R^2}{r^2}\right]+\frac{\Gamma}{2\pi r} = 0$



$\frac{\Gamma}{4\pi RU} > 1$



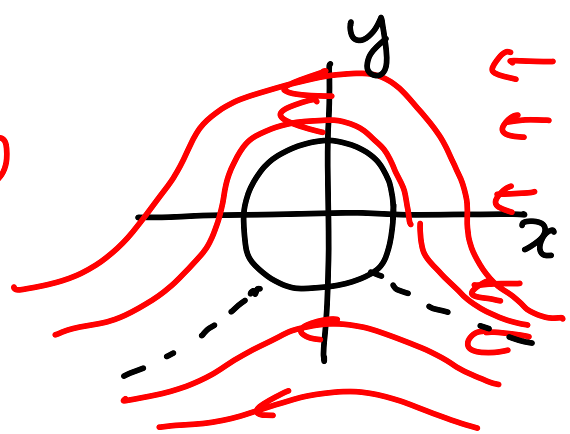


$n_x = \cos\theta$

$n_y = \sin\theta$

$$p = \cancel{p_0} - \tfrac{1}{2}\rho u^2 - \rho \cancel{\frac{\partial \phi}{\partial t}}$$

$$= -\tfrac{1}{2}\rho\left(u_r^2 + u_\theta^2\right)$$



At $r = R$, $u_r = 0$, $u_\theta = 2U\sin\theta + \dfrac{\Gamma}{2\pi r}$

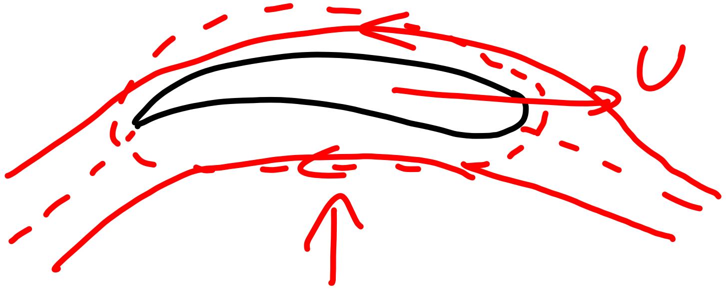
$$F_i = \int ds\,(-p n_i) = \int_0^{2\pi} R\,d\theta \left[\tfrac{1}{2}\rho\left(2U\sin\theta + \frac{\Gamma}{2\pi R}\right)^2\right] n_i$$

$$= \tfrac{1}{2}\rho R \int d\theta \left[4U^2\sin^2\theta + \left(\frac{\Gamma}{2\pi R}\right)^2 + \frac{2U\sin\theta\,\Gamma}{\pi R}\right] n_i$$

$$F_x = \tfrac{1}{2}\rho R \int d\theta \left[4U^2\sin^2\theta + \left(\frac{\Gamma}{2\pi R}\right)^2 + \frac{2U\sin\theta\,\Gamma}{\pi R}\right]\cos\theta = 0$$

$$F_y = \tfrac{1}{2}\rho R \int_0^{2\pi} d\theta \left[4U^2\sin^2\theta + \left(\frac{\Gamma}{2\pi R}\right)^2 + \frac{2U\sin\theta\,\Gamma}{\pi R}\right]\sin\theta$$

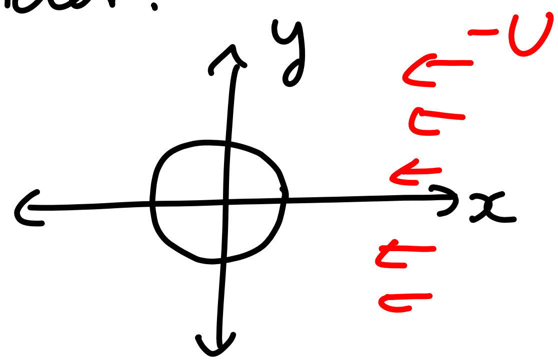
$= \rho U \Gamma$ 'Lift force'



$$\oint d\underline{x}\cdot\underline{u} = \Gamma$$

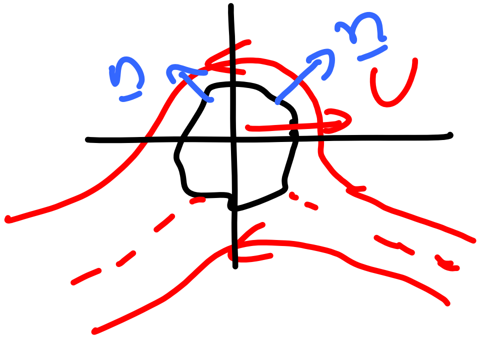
Potential flow around a cylinder:

$$F(z) = -Uz - \frac{UR^2}{z} - \frac{i\Gamma}{2\pi}\log z$$



$$F_x = 0 \qquad F_y = \rho U \Gamma$$

$$F(z) = -\frac{UR^2}{z} - \frac{i\Gamma}{2\pi}\log z$$



$$\oint d\underline{x}\cdot\underline{u} = \Gamma$$

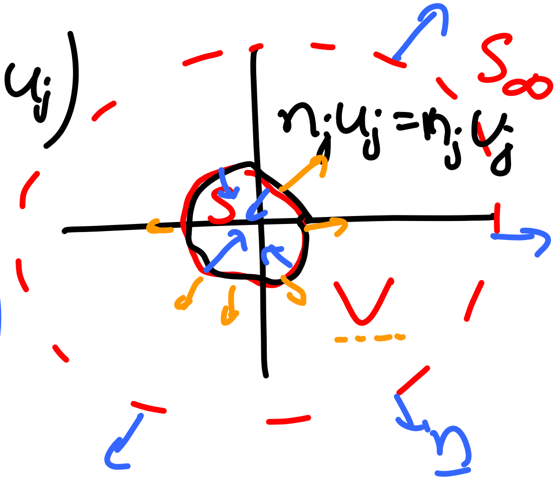
$$p = p_0 - \tfrac{1}{2}\rho u_j^2 - \rho\frac{\partial\phi}{\partial t}$$

$$= \cancel{p_0} - \tfrac{1}{2}\rho u_j^2 + \rho u_j U_j$$

$$F_i = \int ds\,(-p n_i) = \int ds\left(\tfrac{1}{2}\rho u_j^2 - \rho u_j U_j\right) n_i$$

$$= \rho\int ds\left(\tfrac{1}{2}u_j^2 - U_j u_j\right) n_i$$

$$\rho \int_V dV \frac{\partial}{\partial x_i}\left(\frac{1}{2} u_j^2 - U_j u_j\right) = \rho \oint_{S_\infty} ds\, n_i\left(\tfrac{1}{2} u_j^2 - U_j u_j\right)$$



$$+\rho \oint_S ds\, n_i\left(\tfrac{1}{2} u_j^2 - U_j u_j\right)$$

$$= \rho \oint_{S_\infty} ds\, n_i\left(\tfrac{1}{2} u_j^2 - U_j u_j\right) - F_i$$

$$F_i = \rho \oint_{S_\infty} ds\, n_i\left(\tfrac{1}{2} u_j^2 - U_j u_j\right) - \rho \int_V dV \frac{\partial}{\partial x_i}\left(\tfrac{1}{2} u_j^2 - U_j u_j\right)$$

$$-\rho \int_V dV \left(u_j \frac{\partial u_j}{\partial x_i} - U_j \frac{\partial u_j}{\partial x_i}\right)$$

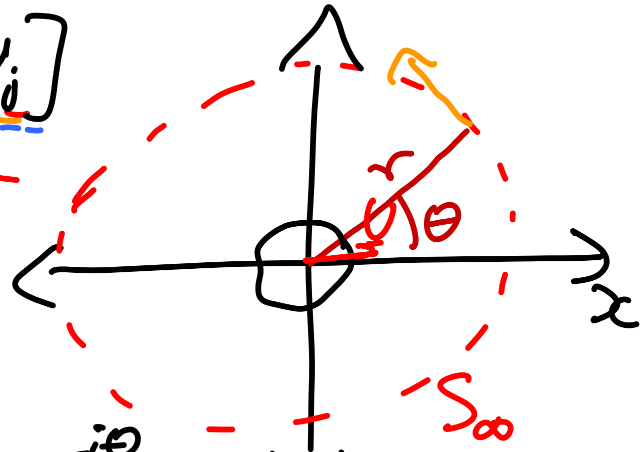
$$-\rho \int_V dV \left(u_j \frac{\partial u_i}{\partial x_j} - U_j \frac{\partial u_i}{\partial x_j}\right)$$

$$-\rho \int_V dV \left[\frac{\partial}{\partial x_j}(u_i u_j) - u_i \cancel{\frac{\partial u_j}{\partial x_j}} - \frac{\partial}{\partial x_j}(u_i U_j)\right]$$

$$F_i = \rho \int_{S_\infty} ds\, n_i\left(\tfrac{1}{2} u_j^2 - U_j u_j\right) - \underline{\rho \int_V dV \frac{\partial}{\partial x_j}\left[u_i u_j - u_i U_j\right]}$$

$$F_i = \rho\int_{S_\infty} ds\, n_i\left(\tfrac{1}{2}u_j^2 - U_j u_j\right) - \left[\rho\int_{S_\infty} ds\, n_j\left[u_i u_j - u_i U_j\right] - \rho\int_{S} ds\, u_i\left(\cancel{n_j u_j - n_j U_j}\right)\right.$$

$$= \rho\int_{S_\infty} ds\left[n_i\left(\tfrac{1}{2}u_j^2 - U_j u_j\right) - n_j\left[u_i u_j - u_i U_j\right]\right]$$



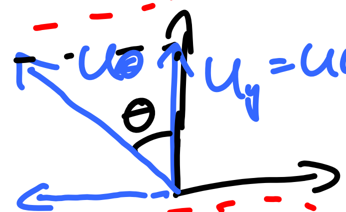
$$F(z) = -\frac{UR^2}{z} - \frac{i\Gamma}{2\pi}\log z$$

$$W = \frac{UR^2}{z^2} - \frac{i\Gamma}{2\pi z} = \frac{UR^2}{r^2}e^{-2i\theta} - \frac{i\Gamma}{2\pi r}e^{-i\theta}$$

$n_x = \cos\theta$

$n_y = \sin\theta$

$$u_r = \frac{UR^2}{r^2}\cos\theta \qquad u_\theta = \frac{UR^2}{r^2}\sin\theta + \frac{\Gamma}{2\pi r}$$



$u_x = -u_\theta\sin\theta$

$u_y = u_\theta\cos\theta$

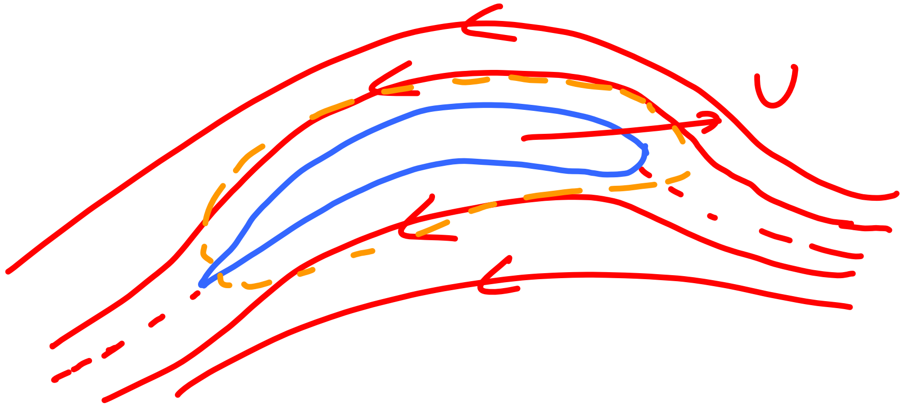
$$F_i = \rho\int_{S_\infty} ds\, U_j\left(n_j u_i - n_i u_j\right)$$

$$F_x = 0$$

$$F_y = \rho\int r\,d\theta\, U_x\left[n_x u_y - n_y u_x\right]$$

$$= \rho \int r\,d\theta\; U_x \left[\cos\theta\left(\cos\theta\, u_\theta\right) - \sin\theta\left(-\sin\theta\, u_\theta\right)\right]$$

$$= \rho \int_0^{2\pi} r\,d\theta\; U_x \left(\frac{\Gamma}{2\pi r}\right) = \boxed{\rho U \Gamma}$$



$$\oint \underline{dx} \cdot \underline{y} = \Gamma$$

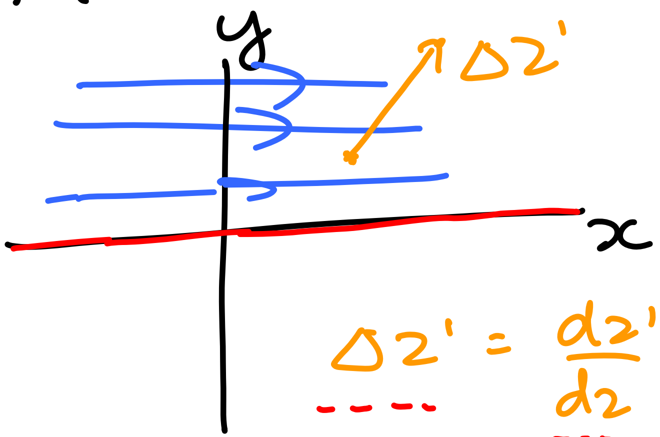
Conformal mappings:

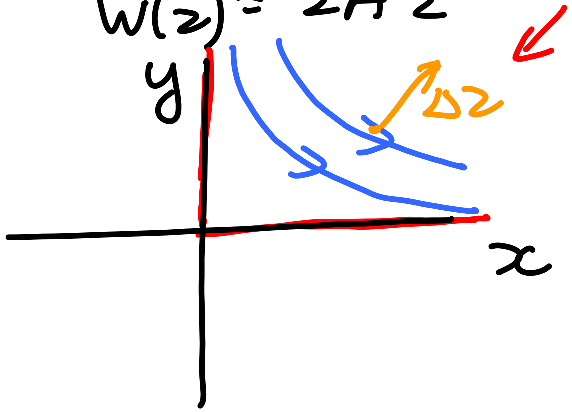
$F(z) = Az^2$

$W(z) = 2Az$

$\boxed{z' = z^2}$

$F(z') = Az'$





$$\Delta z' = \frac{dz'}{dz}\Delta z$$

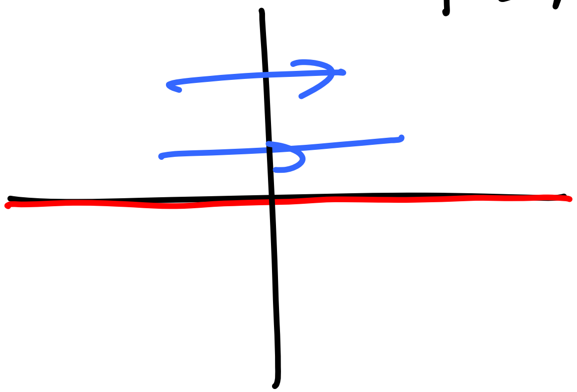
$$W(z) = \frac{dF}{dz}$$

$$W(z') = \frac{dF}{dz'} = \frac{dF}{dz}\left(\frac{dz}{dz'}\right)$$

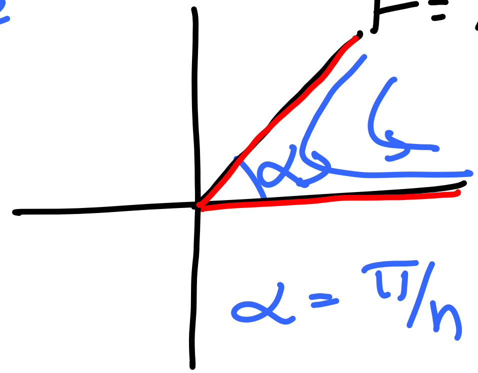
$$W(z') = W(z)\left(\frac{dz}{dz'}\right)$$

$$\Gamma = \oint dx \cdot u \qquad KE = \int ds\left(\tfrac{1}{2}\rho u^2\right)$$

$F = A z'$ $\quad z' = z^n$



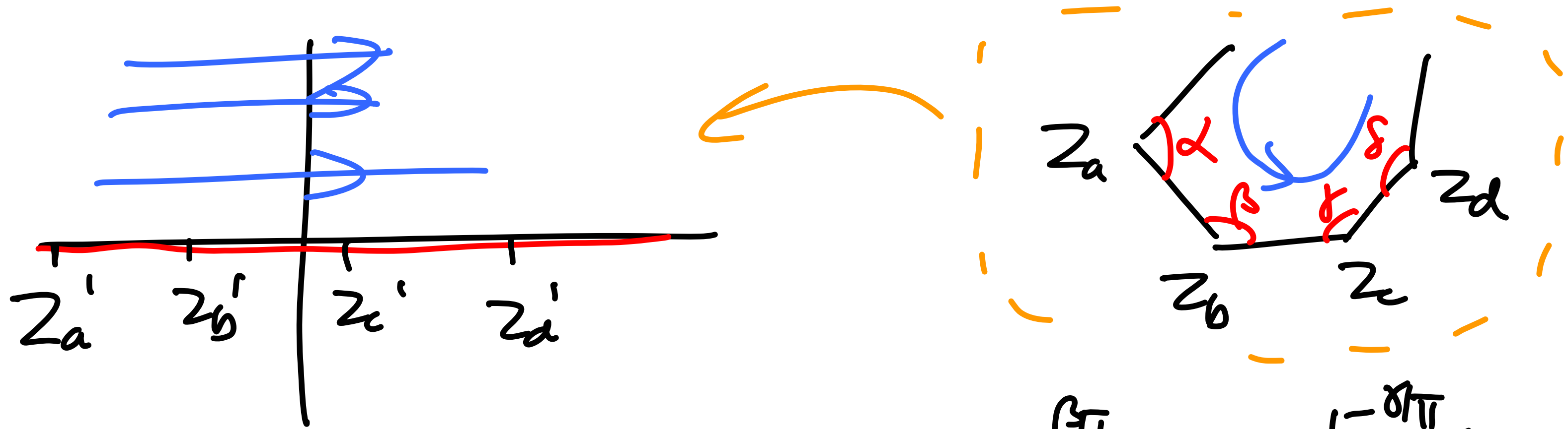
$F = A z^n$



$\alpha = \pi / n$

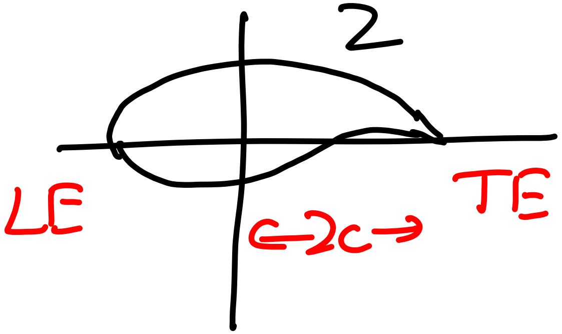
$$z' = z^n$$

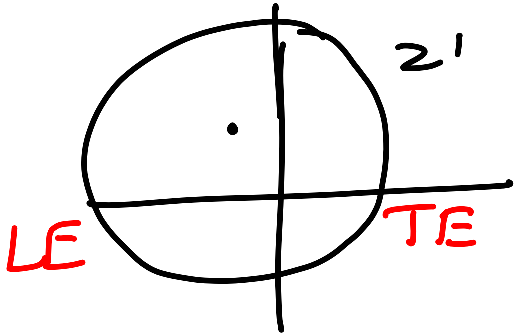
$$\frac{dz'}{dz} = n z^{n-1} = n (z')^{\frac{n-1}{n}} = n (z')^{1-\alpha/\pi}$$



$$\frac{dz'}{dz} = k(z'-z_a')^{1-\alpha/\pi}(z'-z_b')^{1-\beta/\pi}(z'-z_c')^{1-\gamma/\pi}(z'-z_d')^{1-\delta/\pi}$$

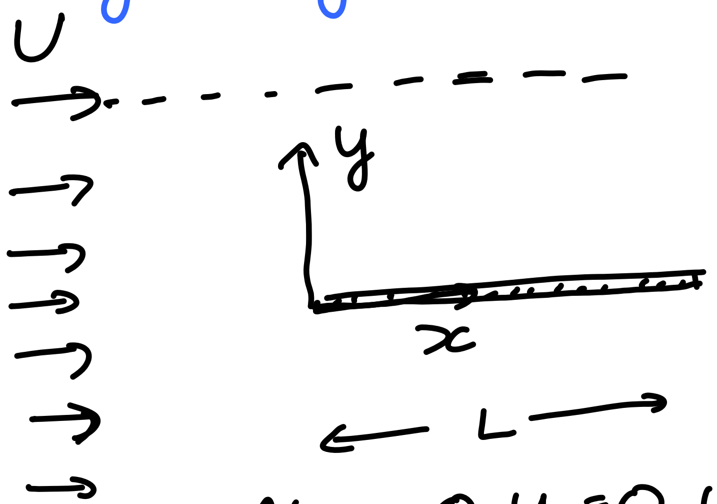
Schwarz-Christoffel transform





$$z = z' + \frac{c^2}{z'}$$

Boundary layer theory: High Reynolds number



$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\rho\left(u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2}$$

At $y=0$, $u_x=0$, $u_y=0$

for $x>0$

As $y \to \infty$, $u_x = U$

For $x \leq 0$, $u_x = U$ for all y

$$\rho\left(u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right)$$

$$\rho\left(u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} \right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} \right)$$

$$x^* = (x/L)\ ;\ y^* = (y/L)\ ;\ u_x^* = (u_x/U)\ ;\ u_y^* = (u_y/U)\ ;\ p^* = p/\rho U^2$$

$$\left(u_x^* \frac{\partial u_x^*}{\partial x^*} + u_y^* \frac{\partial u_x^*}{\partial y^*}\right) = -\frac{\partial p^*}{\partial x^*} + \boxed{Re^{-1}\left(\frac{\partial^2 u_x^*}{\partial x^{*2}} + \frac{\partial^2 u_x^*}{\partial y^{*2}}\right)}$$

$$\left(u_x^* \frac{\partial u_y^*}{\partial x^*} + u_y^* \frac{\partial u_y^*}{\partial y^*}\right) = -\frac{\partial p^*}{\partial y^*} + \boxed{Re^{-1}\left(\frac{\partial^2 u_y^*}{\partial x^{*2}} + \frac{\partial^2 u_y^*}{\partial y^{*2}}\right)}$$

$$Re = \frac{\rho U L}{\mu} \equiv \frac{UL}{\nu}$$

$$x_i \sim L \Rightarrow \frac{\partial}{\partial x_i} \sim \frac{1}{L}$$

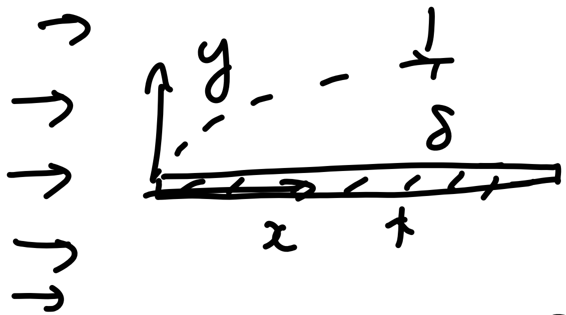
$$u_j^* \frac{\partial u_i^*}{\partial x_j^*} = -\frac{\partial p^*}{\partial x_i^*}$$

$$y \sim \delta \Rightarrow \frac{\partial}{\partial y} \sim \frac{1}{\delta}$$

$$\underline{u} \cdot \nabla c = 0$$

$$x^* = (x/L)\ ;\ y^* = (y/\delta) \quad u_x^* = (u_x/U) \quad u_y^* = u_y/(U\delta/L)$$

$$p^* = (p/\rho U^2)$$



$$\rho\left(u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right)$$

$$\rho\left(\frac{U^2}{L} u_x^* \frac{\partial u_x^*}{\partial x^*} + \left(\frac{U\delta}{L}\right)\left(\frac{U}{\delta}\right) u_y^* \frac{\partial u_x^*}{\partial y^*}\right) = -\frac{1}{L}\frac{\partial p}{\partial x^*} + \mu\left(\frac{U}{L^2}\frac{\partial^2 u_x^*}{\partial x^{*2}} + \frac{U}{\delta^2}\frac{\partial^2 u_x^*}{\partial y^{*2}}\right)$$

$$\frac{\rho U^2}{L}\left[u_x^* \frac{\partial u_x^*}{\partial x^*} + u_y^* \frac{\partial u_x^*}{\partial y^*}\right] = -\frac{1}{L}\frac{\partial p}{\partial x^*} + \frac{\mu U}{\delta^2}\left[\frac{\partial^2 u_x^*}{\partial y^{*2}} + \frac{\delta^2}{L^2}\frac{\partial^2 u_x^*}{\partial x^{*2}}\right]$$

$$u_x^* \frac{\partial u_x^*}{\partial x^*} + u_y^* \frac{\partial u_x^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{\mu L}{\rho U \delta^2}\left[\frac{\partial^2 u_x^*}{\partial y^{*2}} + \cancel{\left(\frac{\delta}{L}\right)^2 \frac{\partial^2 u_x^*}{\partial x^{*2}}}\right]$$

$$\frac{\rho U \delta^2}{\mu L} = O(1) \Rightarrow \left(\frac{\rho U L}{\mu}\right)\left(\frac{\delta}{L}\right)^2 = C$$

$$\left(\frac{\delta}{L}\right) = C\,Re^{-1/2} = Re^{-1/2}$$

$$\rho\left(u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\right)$$

$$u_x^* = (u_x/U)\ ; \quad u_y^* = u_y/(U\delta/L)\ ; \quad x^* = x/L\ ; \quad y^* = y/\delta\ ; \quad p^* = \frac{p}{\rho U^2}$$

$$\rho\left(U\left(\frac{U\delta}{L}\right)\left(\frac{1}{L}\right) u_x^* \frac{\partial u_y^*}{\partial x^*} + \left(\frac{U\delta}{L}\right)^2 \frac{1}{\delta} u_y^* \frac{\partial u_y^*}{\partial y^*}\right) = -\frac{\rho U^2}{\delta}\frac{\partial p^*}{\partial y^*}$$

$$+\mu\left(\frac{U\delta}{L}\right)\left[\frac{1}{L^2}\frac{\partial^2 u_y^*}{\partial x^{*2}} + \frac{1}{\delta^2}\frac{\partial^2 u_y^*}{\partial y^{*2}}\right]$$

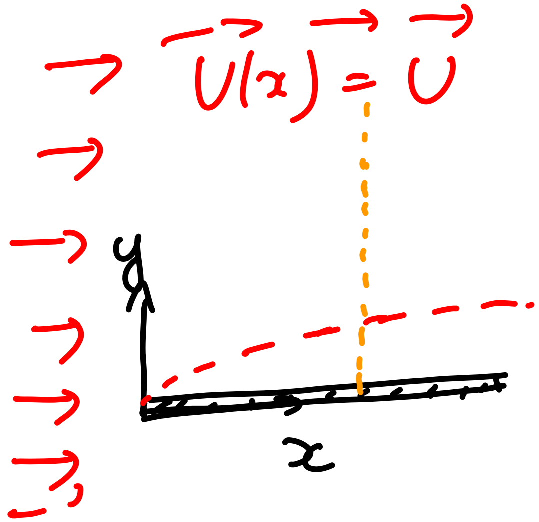
$$\frac{\rho U^2 \delta}{L^2}\left(u_x^* \frac{\partial u_y^*}{\partial x^*} + u_y^* \frac{\partial u_y^*}{\partial y^*}\right) = -\boxed{\frac{\rho U^2}{\delta}\frac{\partial p^*}{\partial y^*}}$$

$$+\mu\left(\frac{U\delta}{L}\right)\left(\frac{1}{\delta^2}\right)\left[\frac{\partial^2 u_y^*}{\partial y^{*2}} + \left(\frac{\delta}{L}\right)^2 \frac{\partial^2 u_y^*}{\partial x^{*2}}\right]$$

$$\boxed{\left(\frac{\delta^2}{L^2}\right)}\left[u_x^* \frac{\partial u_y^*}{\partial x^*} + u_y^* \frac{\partial u_y^*}{\partial y^*}\right] = -\frac{\partial p^*}{\partial y^*} + \boxed{\left(\frac{\mu}{\rho U L}\right)}\frac{\partial^2 u_y^*}{\partial y^{*2}}$$

$$\frac{\partial p^*}{\partial y^*} = 0$$

$$\rho\left(u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \boxed{\mu \frac{\partial^2 u_x}{\partial y^2}}$$



$$\boxed{\frac{\partial p}{\partial y} = 0}$$

Re^{-1}

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

Boundary layer equations

$$U \frac{\partial U}{\partial x} + \cancel{u_y \frac{\partial U}{\partial y}} = -\frac{\partial p}{\partial x}$$

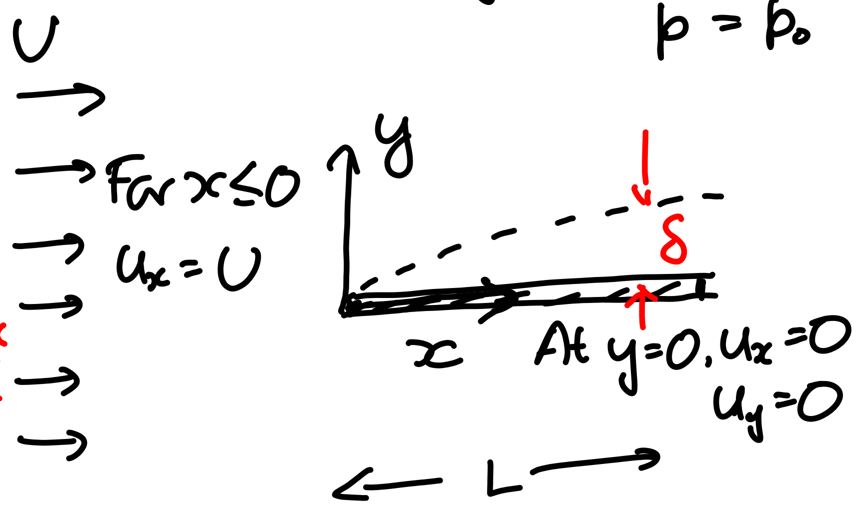
Outer

Inviscid flow

$$\rho\left(u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y}\right) = U \frac{\partial U}{\partial x} + \mu \frac{\partial^2 u_x}{\partial y^2}$$

Boundary layer theory:

As $y \to \infty$, $u_x = U$
$p = p_0$



$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

$$\rho\left(u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u_x}{\partial y^2}$$

$$\rho\left(u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y}\right) = -\frac{\partial p}{\partial y}$$

$$Re = \frac{\rho U L}{\mu} \gg 1$$

'Boundary layer logic'

$$x^* = (x/L)\ ;\quad y^* = (y/\delta)\ ;\quad u_x^* = (u_x/U)\ ;\quad u_y^* = u_y/(U\delta/L) \quad p^* = p/(\tfrac{1}{2}\rho U^2)$$

$$\left(u_x^* \frac{\partial u_x^*}{\partial x^*} + u_y^* \frac{\partial u_x^*}{\partial y^*}\right) = -\frac{\partial p^*}{\partial x^*} + \frac{\partial^2 u_x^*}{\partial y^{*2}} \qquad \delta = Re^{-1/2} L$$

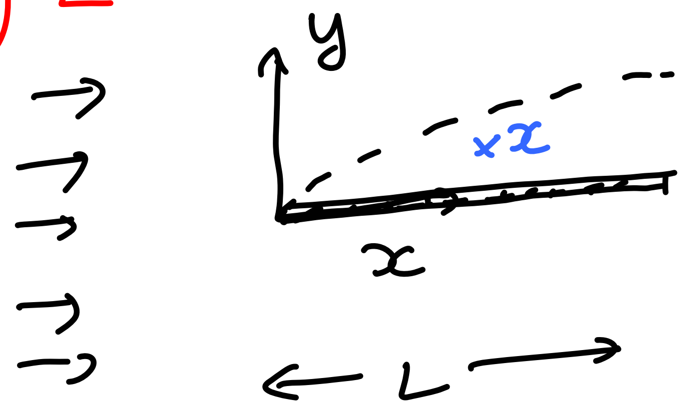
$$0 = -\frac{\partial p^*}{\partial y^*}$$

$$u_x^* \frac{\partial u_x^*}{\partial x^*} + u_y^* \frac{\partial u_x^*}{\partial y^*} = \frac{\partial^2 u_x^*}{\partial y^{*2}}$$

$$\delta = Re^{-1/2} L = \left(\frac{\nu L}{U}\right)^{1/2} = \left(\frac{\nu x}{U}\right)^{1/2}$$

$$= \left(\frac{\mu}{\rho U L}\right)^{1/2} L$$

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = \nu \frac{\partial^2 u_x}{\partial y^2}$$



$$\frac{\partial p}{\partial y} = 0$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

$$\boxed{y^* = y/(\nu x/U)^{1/2} = \eta}$$

$$u_x = \frac{\partial \psi}{\partial y} \qquad u_y = -\frac{\partial \psi}{\partial x}$$

$$u_x = \frac{\partial \psi}{\partial y} \qquad u_y = -\frac{\partial \psi}{\partial x} \qquad \eta = y\Big/\sqrt{\frac{\nu x}{U}} \qquad \psi = \sqrt{\nu x U}\, f(\eta)$$

$$= \sqrt{\nu x U}\, \frac{1}{\sqrt{\nu x / U}} \frac{df}{d\eta}$$

$$\frac{\partial \eta}{\partial y} = \frac{1}{\sqrt{\nu x / U}}$$

$$= U f'(\eta)$$

$$\frac{\partial \eta}{\partial x} = \frac{-y}{2x^{3/2}(\nu/U)^{1/2}} = \frac{-y}{2x(\nu x/U)^{1/2}} = \frac{-\eta}{2x}$$

$$u_y = -\frac{\partial \psi}{\partial x} = \frac{-\partial}{\partial x}\left[\sqrt{\nu x U}\, f(\eta)\right]$$

$$= -\frac{1}{2}\sqrt{\frac{\nu U}{x}}\, f(\eta) - \sqrt{\nu x U}\, f'(\eta)\frac{\partial \eta}{\partial x}$$

$$= -\frac{1}{2}\sqrt{\frac{\nu U}{x}}\, f(\eta) - \sqrt{\nu x U}\, f'(\eta)\left(\frac{-\eta}{2x}\right)$$

$$= \frac{1}{2}\sqrt{\frac{\nu U}{x}}\left[\eta f' - f\right]$$

$$u_x = Uf'(\eta) \qquad u_y = \frac{1}{2}\left(\frac{\nu U}{x}\right)^{1/2}(\eta f' - f)$$

$$\frac{\partial u_x}{\partial x} = Uf''(\eta)\frac{\partial \eta}{\partial x} = Uf''\left[\frac{-\eta}{2x}\right]$$

$$\boxed{\frac{\partial u_x}{\partial y} = Uf''\frac{\partial \eta}{\partial y} = Uf''\left[\frac{1}{\sqrt{\nu x/U}}\right]}$$

$$\frac{\partial^2 u_x}{\partial y^2} = \frac{Uf'''}{(\nu x/U)}$$

$$u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} = \nu\frac{\partial^2 u_x}{\partial y^2}$$

$$(Uf')\left[Uf''\left(\frac{-\eta}{2x}\right)\right] + \frac{1}{2}\left(\frac{\nu U}{x}\right)^{1/2}(\eta f' - f)\frac{Uf''}{\sqrt{\nu x/U}} = \frac{\nu}{(\nu x/U)}Uf'''$$

$$\frac{U^2}{\cancel{x}}\left[-\frac{1}{2}\cancel{\eta f'f''} + \frac{1}{2}\cancel{\eta f'f''} - \frac{1}{2}ff''\right] = \frac{\cancel{U^2}}{\cancel{x}}f'''$$

$$f''' + \frac{1}{2}ff'' = 0$$

Blasius boundary layer eqn.

$$\psi = (\nu x U)^{1/2} f(\eta)$$

$$u_x = U f'(\eta) = U \frac{df}{d\eta}$$

$$u_y = \frac{1}{2}\left(\frac{\nu U}{x}\right)^{1/2}(\eta f' - f)$$

As $y \to \infty$, $\frac{u_x}{U} = 1 \Rightarrow \frac{df}{d\eta} = 1$

$\eta \to \infty$

At $y = 0$, $u_x = 0$ $\quad \frac{df}{d\eta} = 0$

$\eta = 0$ $\quad u_y = 0$ $\quad f = 0$

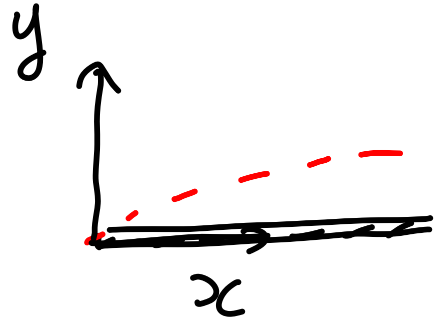
For $x \leq 0$, $u_x = U \Rightarrow \frac{df}{d\eta} = 1$

$\eta \to \infty$

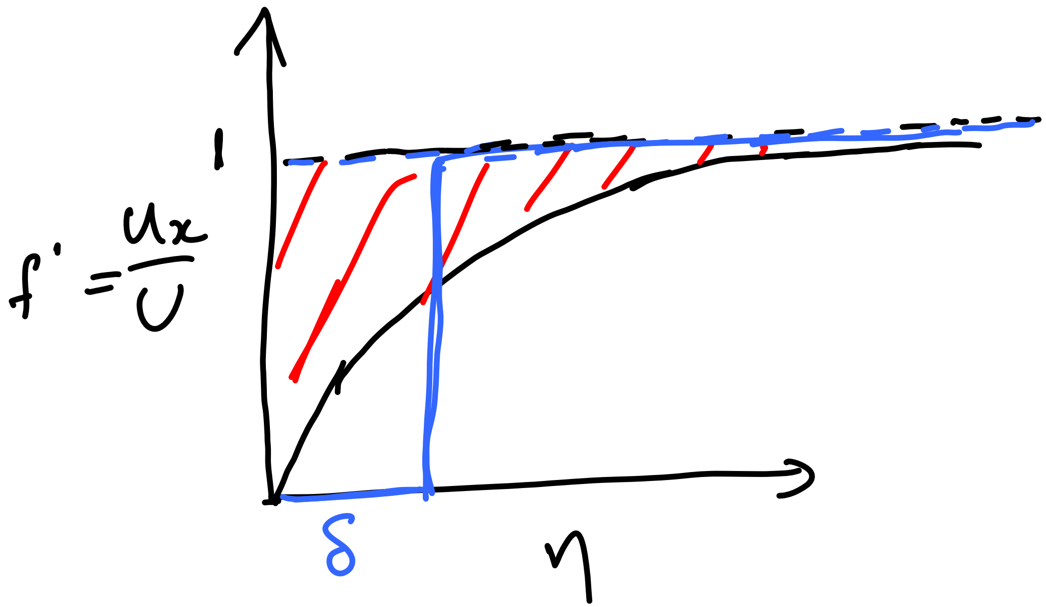
$$\delta = \left(\frac{\nu x}{U}\right)^{1/2}$$

As $y \to \infty$, $u_x = U$

$$\eta = \frac{y}{\sqrt{\nu x / U}}$$



$$f''' + \frac{1}{2} f f'' = 0$$



$$\eta = y / \sqrt{\nu x / U}$$

$$\psi = (\nu x U)^{1/2} f(\eta)$$

$$u_x = U f'(\eta)$$

$$\delta_{0.99} = 4.9 \left(\frac{\nu x}{U}\right)^{1/2}$$

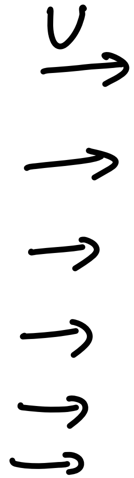
$$\text{Total flow rate} = \int_0^\infty dy\, \rho u_x$$

$$\text{Potential flow rate} = \int_\delta^\infty dy\, \rho U$$

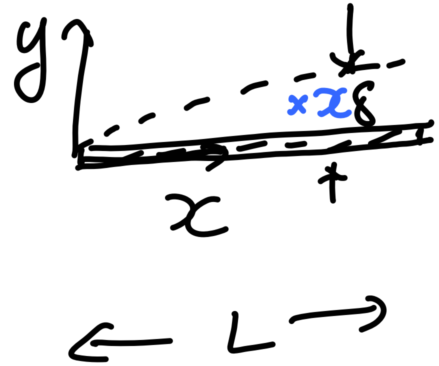
$$\rho \delta U = \int_0^\infty dy\, \rho U - \int_0^\infty dy\, \rho u_x$$

$$\delta = \int_0^\infty dy \left(1 - \frac{u_x}{U}\right) = 1.72 \left(\frac{\nu x}{U}\right)^{1/2}$$

Flow past flat plate:



$\delta = Re^{-1/2} L$



$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = \qquad + \boxed{\nu \frac{\partial^2 u_x}{\partial y^2}}$$

$$\frac{\partial p}{\partial y} = 0$$

$Re = \left(\frac{UL}{\nu}\right) >> 1$

$$\delta = (\nu x / U)^{1/2} \qquad \eta = y/\delta = y\Big/\sqrt{\frac{\nu x}{U}}$$

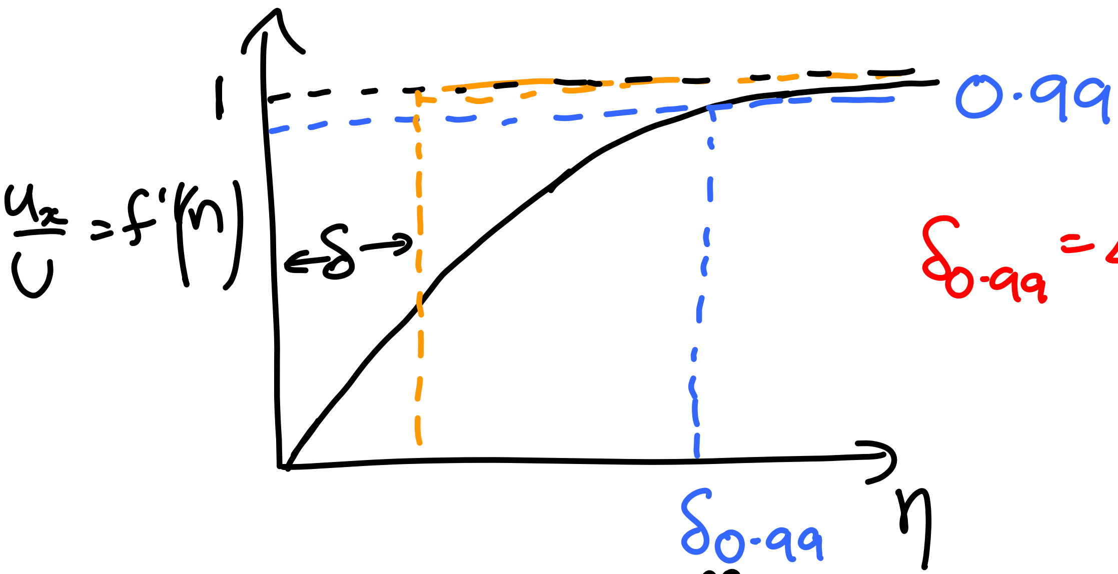
$$u_x = \frac{\partial \psi}{\partial y} \qquad u_y = -\frac{\partial \psi}{\partial x} \qquad \psi = (\nu x U)^{1/2} f(\eta)$$

$$f''' + \frac{1}{2} f f'' = 0$$

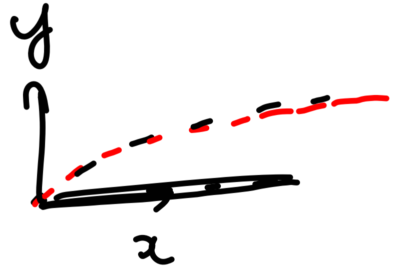
At $y = 0$, $u_x = 0$ $(f'(\eta) = 0)$ $u_y = 0$ $(f(\eta) = 0)$ $(\eta = 0)$

As $y \to \infty$ $u_x = U$ $(f'(\eta) = 1)$

As $x \to 0$, $u_x = U$ $(f'(\eta) = 1)$ $\Big]$ $\eta \to \infty$



$$\delta_{0.99} = 4.9\left(\frac{\nu x}{U}\right)^{1/2}$$



$$\eta = y/\sqrt{\nu x/U}$$

$$\text{Total flow rate} = \int_0^\infty dy\, u_x$$

$$\text{Potential flow} = \int_0^\infty dy\, U$$

$$\text{Corrected flow rate} = \int_\delta^\infty dy\, U = \int_0^\infty dy\, u_x$$

$$\int_0^\infty dy\, U - \int_0^\delta dy\, U = \int_0^\infty dy\, u_x$$

$$\int_0^\infty dy\,(U - u_x) = \int_0^\delta dy\, U = U\delta \Rightarrow \delta = \int_0^\infty dy\left(\frac{U - u_x}{U}\right)$$

$$= 1.72\sqrt{\frac{\nu x}{U}}$$

van-Karman momentum thickness:

$$\Theta = \int_0^\infty dy \left(\frac{u_x}{U}\right)\left(1-\frac{u_x}{U}\right) = 0.664\sqrt{\frac{\nu x}{U}}$$

$$\tau_{xy} = \mu \left.\frac{du_x}{dy}\right|_{y=0} = \frac{\mu U}{\sqrt{\nu x/U}} f''(\eta)\Big|_{\eta=0} = \frac{\mu U}{\sqrt{\nu x/U}}\, 0.332$$

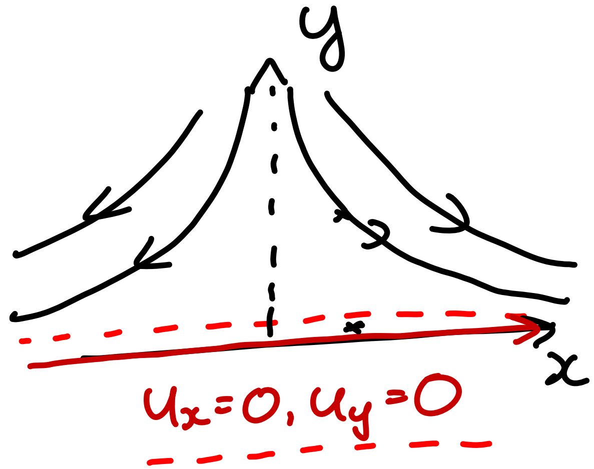
$$C_f = \frac{\tau_w}{\tfrac{1}{2}\rho U^2} = 2 f''(\eta)\Big|_{\eta=0} Re_x^{-1/2} = 0.664\, Re_x^{-1/2}$$

$$Re_x = \left(\frac{Ux}{\nu}\right)^{1/2}$$

$$F_x = \int_0^L dx\, \tau_{xy} = \int_0^L dx\, (0.332)\frac{\mu U}{\sqrt{\nu x/U}}$$

$$C_D = \frac{F_x}{\tfrac{1}{2}\rho U^2 L} = 1.338\, Re_L^{-1/2}$$

Stagnation point flow



$$F(z) = Az^2 \Rightarrow W = 2Az$$

$$= \phi(x,y) + i\psi(x,y) \qquad = 2A(x+iy)$$

$$= u_x - iu_y$$

$$= A(x^2 - y^2 + 2xyi) \qquad u_x = 2Ax = \boxed{kx}$$

$$u_y = -2Ay = -ky$$

$$\psi = kxy$$

$$p = p_0 - \tfrac{1}{2}\rho u^2 = p_0 - \tfrac{1}{2}\rho k^2(x^2+y^2) \qquad \frac{\partial p}{\partial x} = (-\rho k^2 x)$$

$$\delta = \left(\frac{\nu x}{U}\right)^{1/2} = \left(\frac{\nu}{k}\right)^{1/2}$$

$$u_x = \frac{\partial \psi}{\partial y} = kx$$

$$u_x^* = \frac{u_x}{kx} = \frac{1}{(\nu/k)^{1/2}} \frac{\partial \psi}{\partial \eta} \qquad \psi = kx\left(\frac{\nu}{k}\right)^{1/2} f(\eta)$$

$$\psi = Kx\left(\frac{\nu}{k}\right)^{1/2} f(\eta) \qquad \eta = y/(\nu/k)^{1/2}$$

$$u_x = \frac{\partial \psi}{\partial y} = Kx\frac{df}{d\eta} = kx(\nu/k)^{1/2}\frac{1}{(\nu/k)^{1/2}}\frac{df}{d\eta}$$

$$u_y = -\frac{\partial \psi}{\partial x} = -K\left(\frac{\nu}{k}\right)^{1/2} f(\eta)$$

$$\frac{\partial u_x}{\partial x} = kf'(\eta) \qquad \frac{\partial u_x}{\partial y} = \frac{Kx}{(\nu/k)^{1/2}} f''(\eta) \qquad \frac{\partial^2 u_x}{\partial y^2} = \frac{kx}{(\nu/k)} f'''(\eta)$$

$$\frac{\partial p}{\partial x} = -\rho k^2 x$$

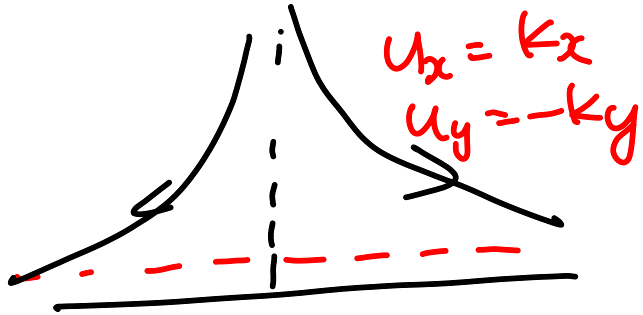
$$u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u_x}{\partial y^2}$$

$$\frac{\partial p}{\partial y} = 0$$

$$(Kxf')(kf') + \left(-k\cancel{\left(\frac{\nu}{k}\right)^{1/2}}f\right)\left(\frac{kx}{\cancel{(\nu/k)^{1/2}}}f''\right) = k^2x + \frac{\cancel{\nu}\,kx}{(\cancel{\nu}/k)}f'''$$

$$\cancel{K^2 x}\left[f'^2 - ff''\right] = \cancel{K^2 x}\left[1 + f'''\right]$$

$$\eta = y\left(\nu/k\right)^{1/2}$$



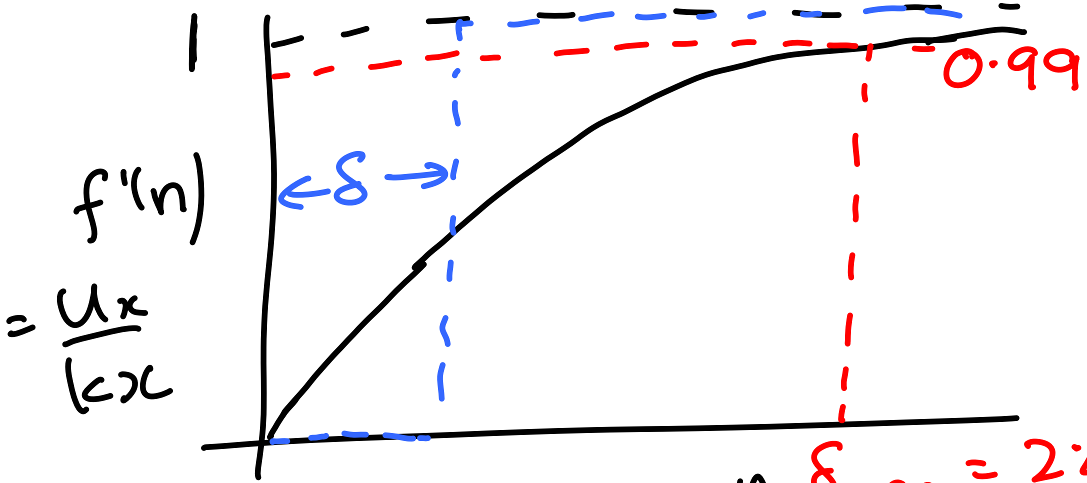
$$f''' + ff'' + (1 - f'^2) = 0$$

At $y = 0$, $u_x = 0 \Rightarrow \dfrac{df}{d\eta} = 0$

$\eta = 0$ $\quad u_y = 0 \Rightarrow f = 0$

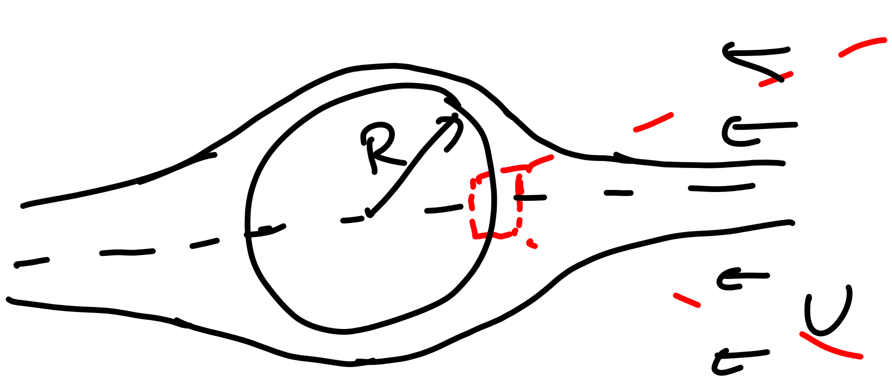
As $y \to \infty$, $u_x = Kx = Kx f'(\eta)$

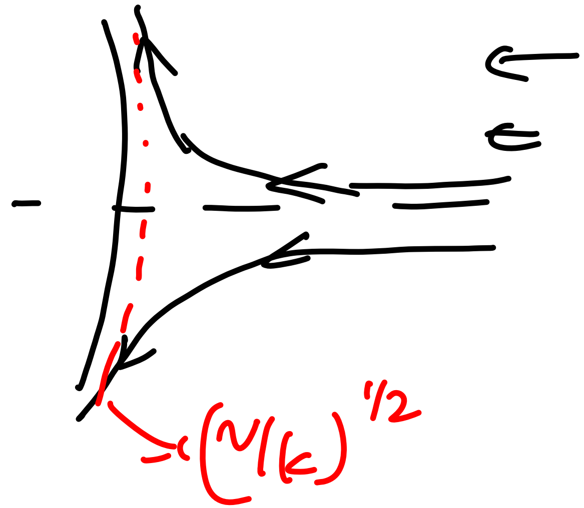
$\eta \to \infty$ $\quad f'(\eta) \to 1$



$\delta_{0.99} = 2.4\left(\dfrac{\nu}{k}\right)^{1/2}$

Displacement thickness $= 0.65\left(\nu/k\right)^{1/2}$





Stagnation point flow

$$\left(\frac{\nu}{k}\right)^{1/2} \ll R \qquad u_x = kx$$

$$u_y = -ky$$

$$k \propto \left(\frac{U}{R}\right) \qquad \left(\frac{\nu}{U/R}\right)^{1/2} \ll R \quad \text{or} \quad \left(\frac{\nu}{UR}\right)^{1/2} \ll 1$$

$$Re^{-1/2} \ll 1$$

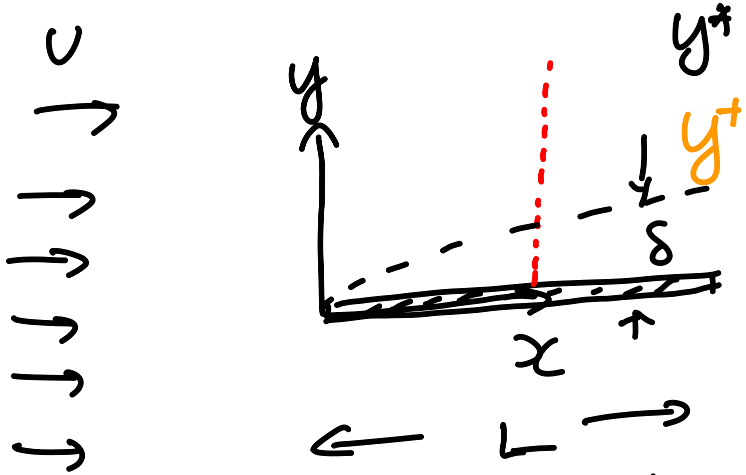
Boundary layer theory:

At $y=0$, $\boxed{u_x=0}$, $\boxed{u_y=0}$

$y^* = (y/L)$

$y^+ = (y/\delta)$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$



$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = -\frac{1}{\rho}\boxed{\frac{\partial p}{\partial x}} + \nu \frac{\partial^2 u_x}{\partial y^2}$$

In boundary layer

$y^* \propto (\delta/L) \to 0$

$$\frac{\partial p}{\partial y} = 0$$

Approach Potential flow from BL

$y^+ \propto (H/\delta) \to \infty$

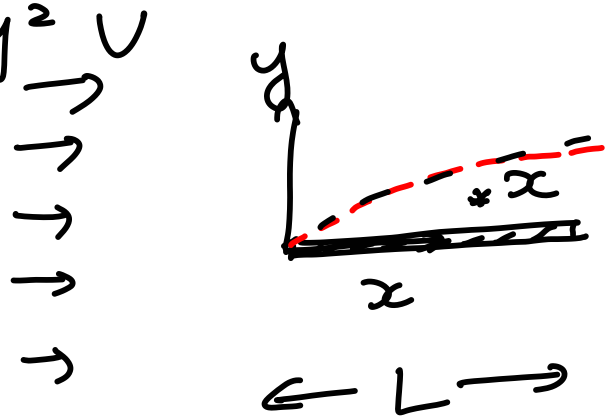
$\eta \to \infty$

$$p = p_0 - \tfrac{1}{2}\rho u^2 = p_0 - \tfrac{1}{2}\rho\left(u_x^2 + u_y^2\right)$$

$$\frac{\partial p}{\partial x} = -\rho\left(u_x \frac{\partial u_x}{\partial x} + \boxed{u_y \frac{\partial u_y}{\partial x}}\right) = -\rho u_x \frac{\partial u_x}{\partial x}$$

$$= -\rho U \frac{\partial U}{\partial x}\bigg|_{y^* = (y/L) \to 0}$$

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = U \frac{\partial U}{\partial x} + \nu \frac{\partial^2 u_x}{\partial y^2}$$



$$\boxed{\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0}$$

$$\eta = y/\delta(x) \quad \text{where } \delta(x) = \left(\frac{\nu x}{U}\right)^{1/2}$$

$$\frac{\delta(x)}{x} = \left(\frac{\nu}{xU}\right)^{1/2} = Re_x^{-1/2}$$

$$u_x = U \frac{df}{d\eta}$$

$$\psi = (\nu x U)^{1/2} f(\eta)$$

$$u_x = \frac{\partial \psi}{\partial y} \qquad u_y = -\frac{\partial \psi}{\partial x}$$

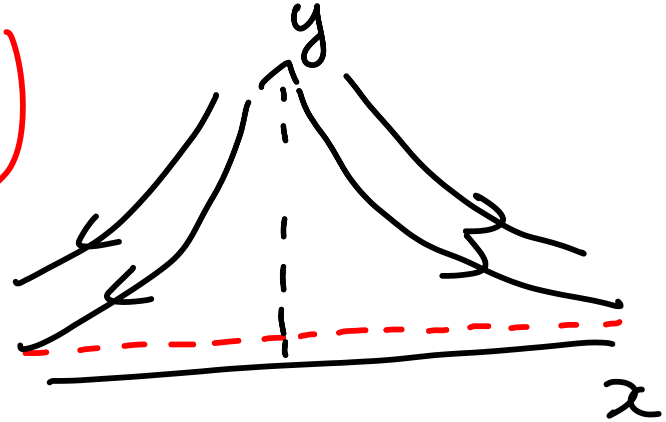
$$\frac{d^3 f}{d\eta^3} + \frac{1}{2} f \frac{d^2 f}{d\eta^2} = 0$$

At $y=0$, $f=0$, $\frac{df}{d\eta}=0$

$\eta=0$

As $y \to \infty$ $(\eta \to \infty)$ $\frac{df}{d\eta} = 1$

$$\delta(x) = \left(\frac{\nu x}{U}\right)^{1/2} \equiv \left(\frac{\nu}{k}\right)^{1/2} \qquad \eta = \left(\frac{y}{\delta(x)}\right)$$



$$\psi = (\nu x U)^{1/2} f(\eta)$$

$$= (\nu k)^{1/2} x f(\eta)$$

$$u_x = k x f'(\eta)$$

$$f''' + f f'' + (1 - f'^2) = 0$$

Potential flow solution

$u_x = kx$, $u_y = -ky$

$$\psi = kxy$$

$$p = -\tfrac{1}{2}k^2x^2 + p_0$$

General velocity profile $U(x)$

$$\delta = \left(\frac{\nu x}{U(x)}\right)^{1/2} = \left(\frac{\nu x}{U}\right)^{1/2} = \delta(x) \qquad \eta = y/\delta(x)$$

$$\psi = \boxed{U(x)\,\delta(x)}\,\boxed{f(\eta)}$$

$$u_x = \frac{\partial \psi}{\partial y} = U(x)\frac{df}{d\eta}$$

$$u_y = -\frac{\partial \psi}{\partial x} = -\frac{d}{dx}(U\delta)\,f + (U\delta)\left(\frac{\eta}{\delta}\frac{d\delta}{dx}\right)\frac{df}{d\eta}$$

$$= -\frac{d}{dx}(U\delta)\,f + U\eta\frac{d\delta}{dx}\frac{df}{d\eta}$$

$$\frac{\partial}{\partial x} \equiv \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = -\frac{y}{\delta^2}\frac{d\delta}{dx}\frac{d}{d\eta}$$

$$= \boxed{-\frac{\eta}{\delta}\frac{d\delta}{dx}\frac{d}{d\eta}}$$

$$\frac{\partial}{\partial y} = \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} = \frac{1}{\delta}\frac{\partial}{\partial \eta}$$

$$\frac{\partial u_x}{\partial x} = \left(\frac{dU}{dx}\right)\frac{df}{d\eta} - \frac{\eta}{\delta}\frac{d\delta}{dx}U(x)\frac{d^2 f}{d\eta^2}$$

$$\frac{\partial u_x}{\partial y} = \frac{U(x)}{\delta}\frac{d^2 f}{d\eta^2} \quad ; \quad \frac{\partial^2 u_x}{\partial y^2} = \frac{U(x)}{\delta^2}\frac{d^3 f}{d\eta^3}$$

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = U\frac{dU}{dx} + \nu \frac{\partial^2 u_x}{\partial y^2}$$

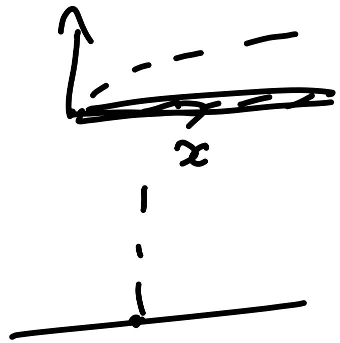
$$\left(U\frac{df}{d\eta}\right)\left(\frac{dU}{dx}\frac{df}{d\eta} - \cancel{\frac{U\eta}{\delta}\frac{d\delta}{dx}\frac{d^2 f}{d\eta^2}}\right) + \left[-\frac{d}{dx}(U\delta)\, f(\eta) + \cancel{\eta U \frac{d\delta}{dx}\frac{df}{d\eta}}\right]\left[\frac{U}{\delta}\frac{d^2 f}{d\eta^2}\right] = U\frac{dU}{dx} + \frac{\nu U}{\delta^2}\frac{d^3 f}{d\eta^3}$$

$$U\frac{dU}{dx}\left(\frac{df}{d\eta}\right)^2 - \frac{U}{\delta}\frac{d}{dx}(U\delta)\, f \frac{d^2 f}{d\eta^2} = U\frac{dU}{dx} + \frac{\nu U}{\delta^2}\frac{d^3 f}{d\eta^3}$$

$$\frac{d^3 f}{d\eta^3} + \frac{\delta^2}{\nu}\frac{dU}{dx}\left[1 - \frac{d^2 f}{d\eta^2}\right] + \frac{\delta}{\nu}\frac{d}{dx}(U\delta)\left(f\frac{d^2 f}{d\eta^2}\right) = 0$$

$$\frac{\delta}{\nu}\frac{d}{dx}(U\delta) = \alpha \qquad \frac{\delta^2}{\nu}\frac{dU}{dx} = \beta$$

$$\frac{\delta^2}{\nu}\frac{dU}{dx} + \frac{\delta U}{\nu}\frac{d\delta}{dx} = \alpha \implies \frac{\delta U}{\nu}\frac{d\delta}{dx} = \alpha - \beta$$



$$\frac{1}{\nu}\frac{d}{dx}\left(\delta^2 U\right) = \frac{2\delta U}{\nu}\frac{d\delta}{dx} + \frac{\delta^2}{\nu}\frac{dU}{dx}$$

$$= (2\alpha - \beta)$$

$$\delta = \left(\frac{\nu x}{U}\right)^{1/2}$$

$$\frac{d}{dx}\left(\delta^2 U\right) = \nu(2\alpha - \beta)$$

$$\delta^2 U = \nu(2\alpha - \beta)x + \cancel{C}$$

$$\delta^2 U = \nu(2\alpha - \beta)x = \left(\frac{\nu(2\alpha-\beta)x}{U}\right)^{1/2}$$

$$= \nu x$$

$$\frac{\delta^2}{\nu}\frac{dU}{dx} = \beta \qquad \frac{1}{\nu}\left(\frac{\nu x}{U}\right)\frac{dU}{dx} = \beta$$

$$\frac{x}{U}\frac{dU}{dx} = \beta \implies \frac{dU}{U} = \beta\frac{dx}{x}$$

$$\implies U = K x^{\beta}$$

Boundary layer theory:

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u_x}{\partial y^2}$$

$$\frac{\partial p}{\partial y} = 0$$

Potential flow for $y^* \to 0$

$$p = p_0 - \tfrac{1}{2}\rho(u_x^2 + u_y^2) = p_0 - \tfrac{1}{2}\rho U^2$$

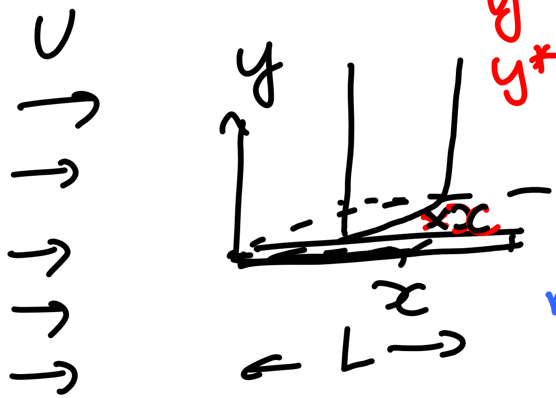
$$\frac{\partial p}{\partial x} = -\rho U \frac{dU}{dx}$$

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = U\frac{dU}{dx} + \nu \frac{\partial^2 u_x}{\partial y^2}$$

$u_x = U(x)$ for $y^* \to 0$

$y \sim L$

$y^* = (y/L)$



$y \sim \delta$

$\eta = (y/\delta)$

$$\boxed{\delta = \left(\frac{\nu x}{U}\right)^{1/2}}$$

$$\frac{\delta}{x} = Re_x^{-1/2}$$

$$\psi = U\delta f(\eta) \qquad \eta = \left(y/\delta(x)\right)$$

$$\delta(x) = \left(\frac{\nu x}{U}\right)^{1/2}$$

$$u_x = \frac{\partial \psi}{\partial y} \qquad u_y = -\frac{\partial \psi}{\partial x}$$

$$= U\left(\frac{df}{d\eta}\right)$$

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = U\frac{dU}{dx} + \nu \frac{\partial^2 u_x}{\partial y^2}$$

$$\frac{d^3 f}{d\eta^3} + \frac{\delta^2}{\nu}\frac{dU}{dx}\left(1 - \left(\frac{df}{d\eta}\right)^2\right) + \frac{\delta}{\nu}\frac{d(U\delta)}{dx}\left[f\frac{d^2 f}{d\eta^2}\right] = 0$$

① $\dfrac{\delta^2}{\nu}\dfrac{dU}{dx} = \beta$ ② $\dfrac{\delta}{\nu}\dfrac{d}{dx}(U\delta) = \alpha$

③ $\dfrac{1}{\nu}\dfrac{d}{dx}(\delta^2 U) = (2\alpha - \beta)$

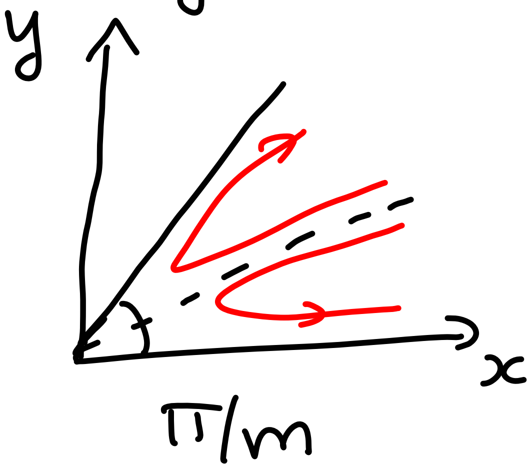
$$\delta^2 U = \nu(2\alpha - \beta)x \implies \delta = \left[\frac{(2\alpha - \beta)\nu x}{U}\right]^{1/2}$$

$$= \left(\frac{\nu x}{U}\right)^{1/2}$$

$$\frac{\delta^2}{\nu}\frac{dU}{dx} = \beta \Rightarrow \left(\frac{\nu x}{U}\right)\frac{1}{\nu}\frac{dU}{dx} = \beta$$

$$\frac{1}{U}\frac{dU}{dx} = \frac{\beta}{x} \Rightarrow U = Kx^{\beta} \Rightarrow \begin{array}{l} K = mA \\ \beta = m-1 \end{array}$$

$$U(x) = \lim_{y^* \to 0} u_x(x)$$

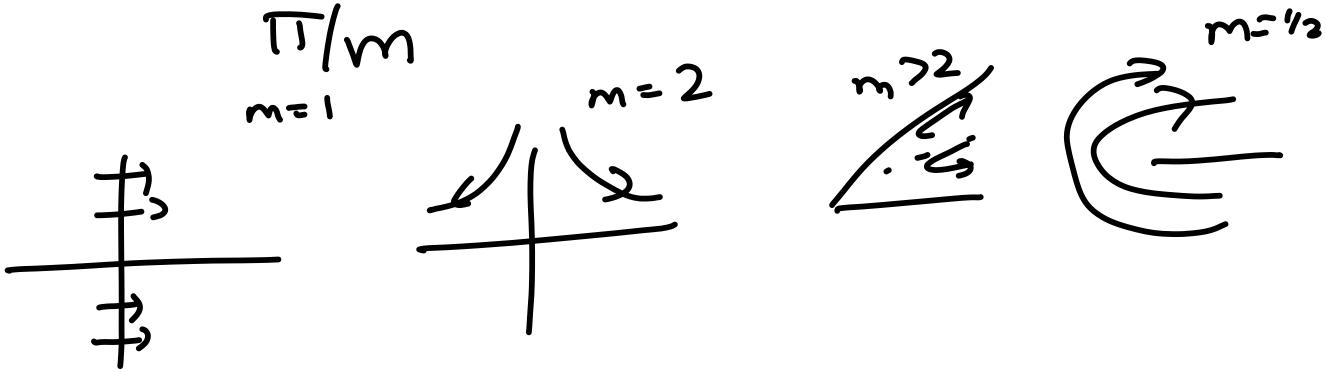


As $y \to 0$

$$F = Az^m$$

$$W = mAz^{m-1} = mA(x+iy)^{m-1} = mAx^{m-1}$$

$$= (u_x - iu_y) \qquad = U(x)$$



$$\frac{d^3f}{d\eta^3} + \frac{\delta^2}{\nu}\frac{dU}{dx}\left(1 - \left(\frac{df}{d\eta}\right)^2\right) + \frac{\delta}{\nu}\frac{d}{dx}(U\delta)\left(f\frac{d^2f}{d\eta^2}\right) = 0$$

$$\frac{\delta^2}{\nu}\frac{dU}{dx} = \beta = m-1$$

$2\alpha - \beta = 1 \Longrightarrow \delta(x) = \left(\frac{\nu x}{U}\right)^{1/2}$

$$\frac{\delta}{\nu}\frac{d}{dx}(U\delta) = \frac{1}{2}(1+\beta) = \frac{m}{2}$$

'Falkner-Skan equation'

$$\frac{d^3f}{d\eta^3} + \beta\left(1 - \frac{df}{d\eta^2}\right) + \frac{1}{2}(1+\beta) f\frac{d^2f}{d\eta^2} = 0$$

Flat plate $\beta = 0$, $m = 1$ $\quad f''' + \frac{1}{2}ff'' = 0$

Stagnation point $\beta = 1$ $m = 2$ $\quad f''' + (1 - f'^2) + ff'' = 0$

$$\delta = \left(\frac{\nu x}{U}\right)^{1/2} = \left(\frac{\nu x}{k x^{\beta}}\right)^{1/2} = \left(\frac{\nu}{k}\right)^{1/2} x^{(1-\beta)/2}$$

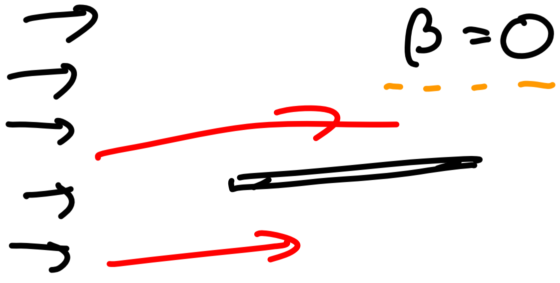
$$\tau_{xy} = \mu \frac{du_x}{dy} = \frac{\mu U}{\delta}\left(\frac{d^2 f}{d\eta^2}\right)$$

$$= \frac{\mu U}{(\nu x/U)^{1/2}} \frac{d^2 f}{d\eta^2} = \frac{\mu U^{3/2}}{x^{1/2}\nu^{1/2}} f''$$

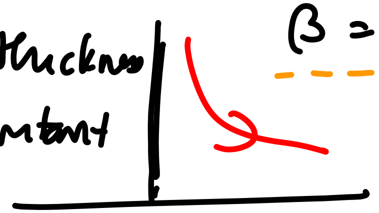
$$= \frac{\mu k^{3/2} x^{\left(\frac{3\beta}{2} - \frac{1}{2}\right)}}{\nu^{1/2}} f''$$

$$\tau_{xy} \propto x^{\left(\frac{3\beta-1}{2}\right)}$$

$m=1$
$\beta=0$



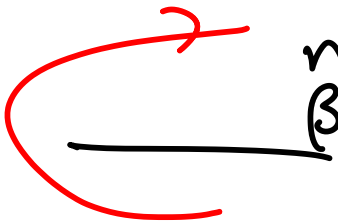
BL thickness constant



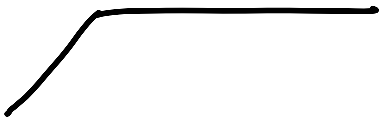
$m=2$
$\beta=1$

BL thickness decreasing

$m>2$
$\beta>1$

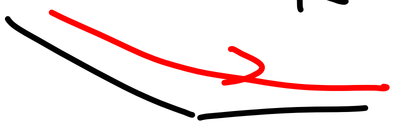


$m=1/2$
$\beta=-1/2$



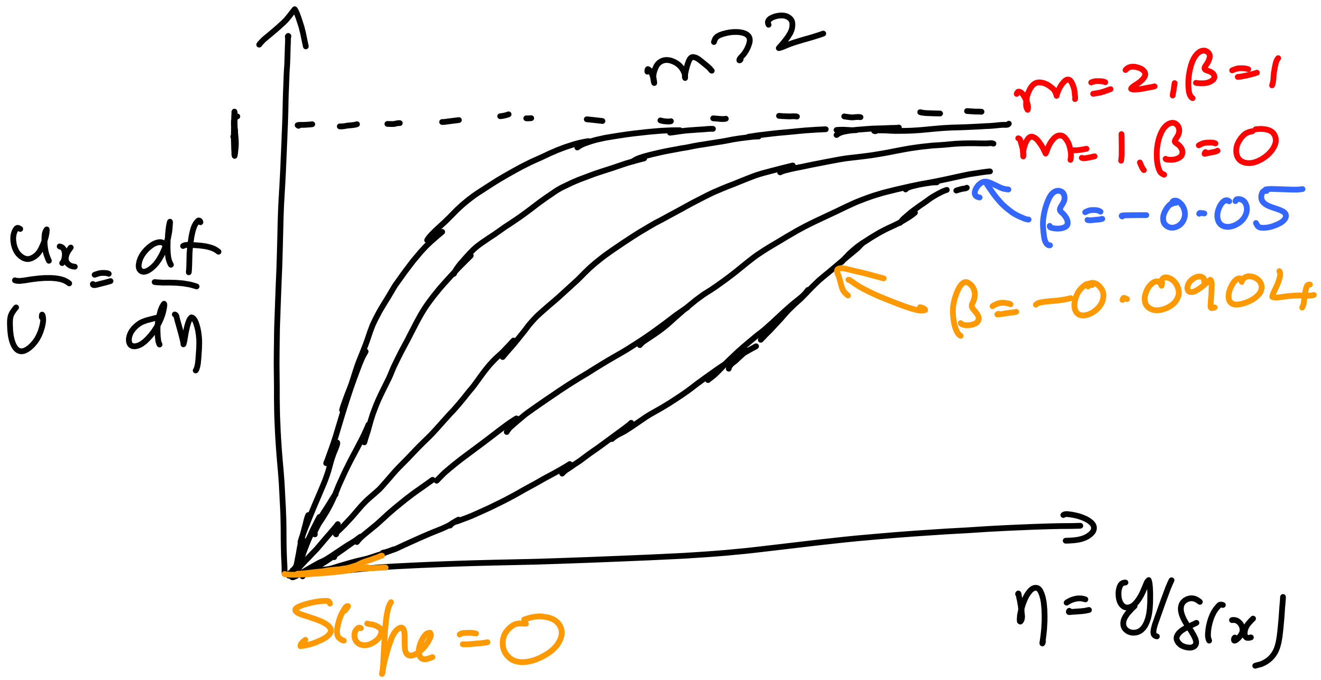
$m<1$
$\beta<0$

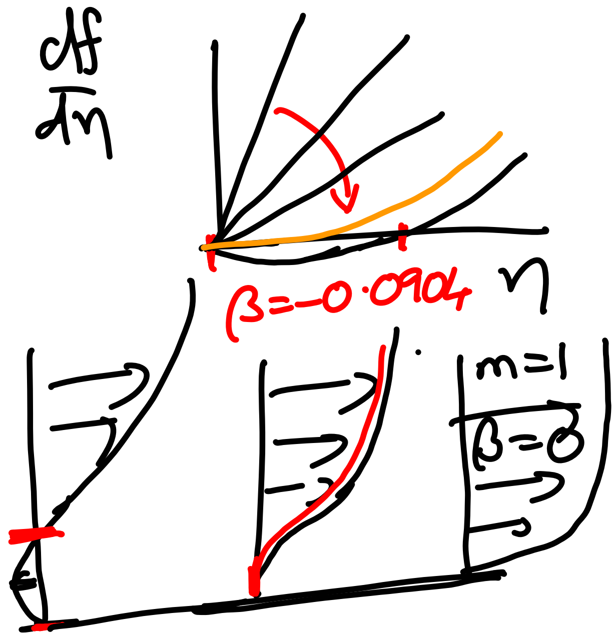
$1<m<2$



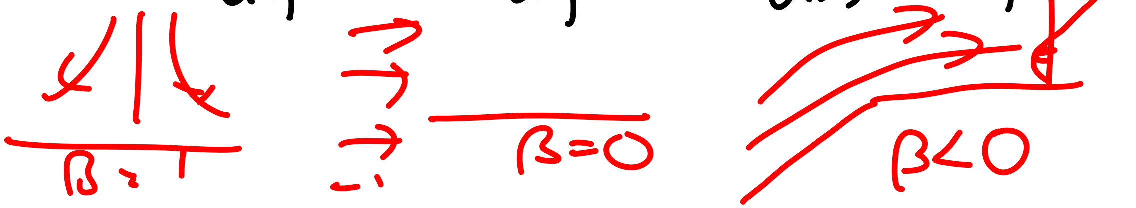
$0<\beta<1$

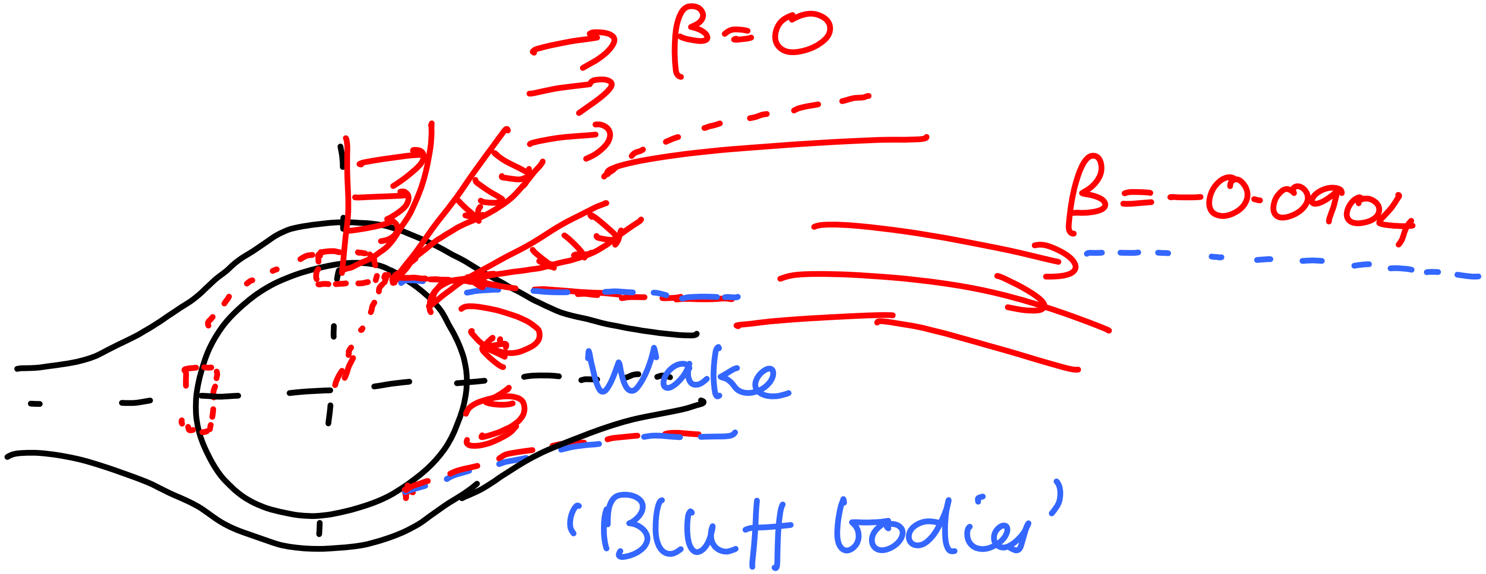
Falkner-Skan solutions

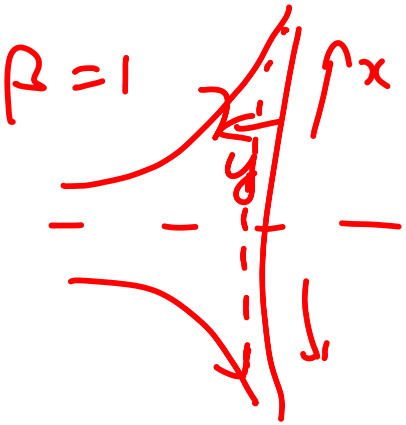




$$u_x = U\frac{df}{d\eta} = U(x)\frac{df}{d\eta} \Longrightarrow \frac{u_x}{U(x)} = \frac{df}{d\eta}$$



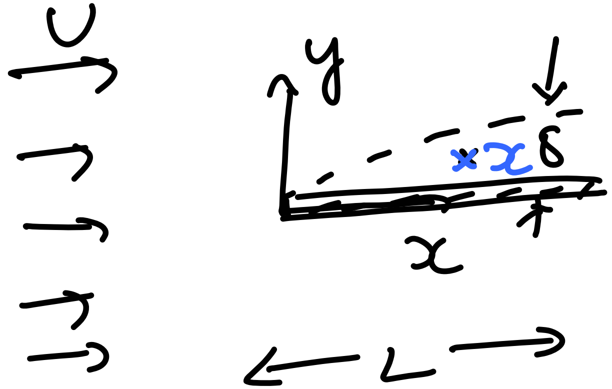

β = 0
β = −0.0904
Wake
'Bluff bodies'


β = 1
x
y

Boundary layer:

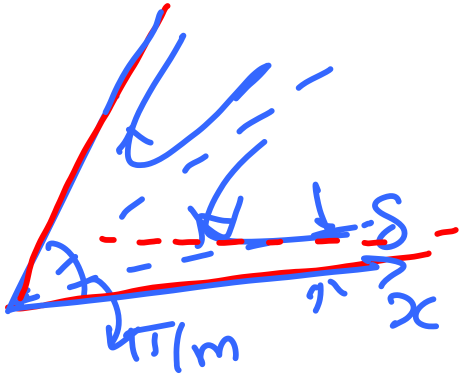
$$U \propto x^{\beta}$$

$$\frac{\delta}{L} \propto Re_L^{-1/2}$$



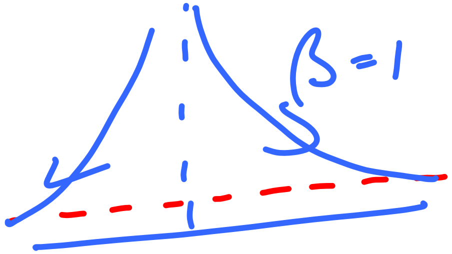
$$\delta \propto x^{1/2}$$

Falkner-Skan solutions



$$\delta = \left(\frac{\nu x}{U(x)}\right)^{1/2}$$

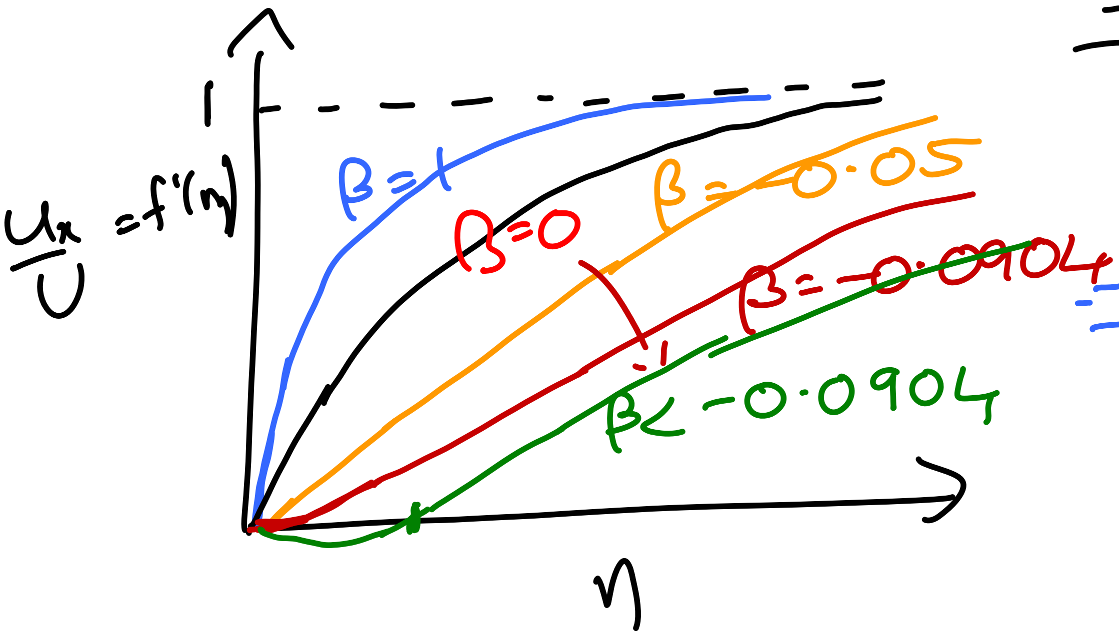
$$u_x = kx$$

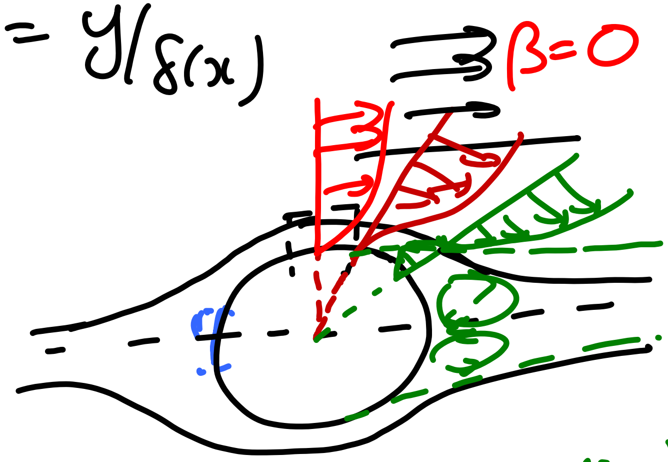


where $\beta = m - 1$

$$\psi = (\delta U)\, f(\eta) \quad \text{where } \eta = y/\delta(x)$$

$$u_x = U(x)\, f'(\eta)$$





'Bluff bodies'

'Slender bodies'

Vorticity dynamics:

$$\omega_i = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j} \qquad \underline{\omega} = \nabla\times\underline{u}$$

$$\oint_C d\underline{x}\cdot\underline{u} = \int ds\,\underline{n}\cdot(\nabla\times\underline{u}) = \int ds\,\underline{n}\cdot\underline{\omega}$$

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_j^2}$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_i}\left(\tfrac{1}{2}u_j^2\right) - \epsilon_{ijk}u_j\omega_k = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - \nu\,\epsilon_{ijk}\frac{\partial}{\partial x_j}\omega_k$$

$$\epsilon_{ijk}\frac{\partial}{\partial x_j}\left[\frac{\partial u_k}{\partial t} + \frac{\partial}{\partial x_k}\left(\tfrac{1}{2}u_l^2\right) - \epsilon_{klm}u_l\omega_m = -\frac{1}{\rho}\frac{\partial p}{\partial x_k} + \nu\frac{\partial^2 u_k}{\partial x_l^2}\right]$$

$$\epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\partial u_k}{\partial t}\right) \equiv \frac{\partial}{\partial t}\left(\epsilon_{ijk}\frac{\partial u_k}{\partial x_j}\right) = \frac{\partial \omega_i}{\partial t}$$

$$\epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_k}\left(\tfrac{1}{2}u_l^2\right)\right) = 0$$

$$-\epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\epsilon_{klm}\, u_l w_m\right) = -\epsilon_{ijk}\epsilon_{klm}\frac{\partial}{\partial x_j}\left(u_l w_m\right)$$

$$= -\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\frac{\partial}{\partial x_j}\left(u_l w_m\right)$$

$$= -\frac{\partial}{\partial x_j}\left(u_i w_j\right) + \frac{\partial}{\partial x_j}\left(u_j w_i\right)$$

$$= -w_j\frac{\partial u_i}{\partial x_j} - \boxed{\cancel{u_i\frac{\partial w_j}{\partial x_j}}} + u_j\frac{\partial w_i}{\partial x_j} + w_i\cancel{\frac{\partial u_j}{\partial x_j}}$$

$$= -w_j\frac{\partial u_i}{\partial x_j} + u_j\frac{\partial w_i}{\partial x_j}$$

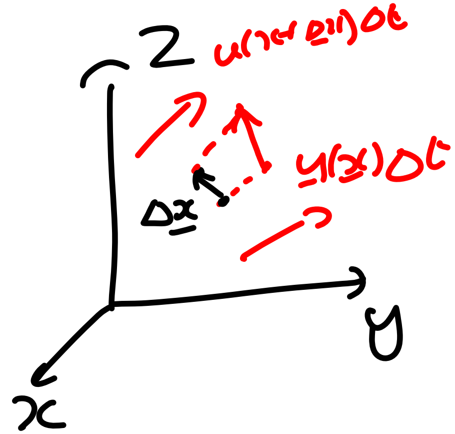
$$-\epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\partial p}{\partial x_k}\right) = 0$$

$$\nu\,\epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\partial^2 u_k}{\partial x_l^2}\right) = \nu\frac{\partial^2}{\partial x_l^2}\left(\epsilon_{ijk}\frac{\partial}{\partial x_j}u_k\right)$$

$$= \nu\frac{\partial^2 w_i}{\partial x_l^2}$$

$$\boxed{\frac{\partial w_i}{\partial t} + u_j\frac{\partial w_i}{\partial x_j}} - \boxed{w_j\frac{\partial u_i}{\partial x_j}} = \nu\frac{\partial^2 w_i}{\partial x_j^2}$$

$$\boxed{\frac{D\omega_i}{Dt}} = \boxed{\omega_j \frac{\partial u_i}{\partial x_j}} + \nu \frac{\partial^2 \omega_i}{\partial x_j^2}$$



$$\left(\frac{d \Delta x_i}{dt}\right) = \Delta \underline{x} \cdot \nabla \underline{u}$$

$$= \Delta x_j \frac{\partial u_i}{\partial x_j}$$

$$\frac{D\omega_i}{Dt} = \omega_j \frac{\partial u_i}{\partial x_j}$$

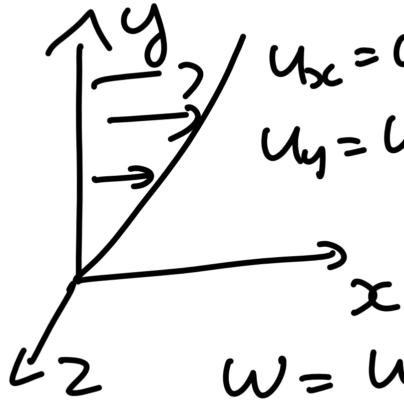
$$\omega_j \frac{\partial u_i}{\partial x_j} = \omega_j \left(S_{ij} + A_{ij}\right)$$

$$= \omega_j \left(S_{ij} - \frac{1}{2} \epsilon_{ijk} \omega_k\right)$$

$$= \omega_j S_{ij} - \frac{1}{2} \cancel{\epsilon_{ijk} \omega_k} \omega_j$$

$$= \omega_j S_{ij}$$

$$\frac{D\omega_i}{Dt} = \omega_j S_{ij} + \nu \frac{\partial^2 \omega_i}{\partial x_j^2}$$

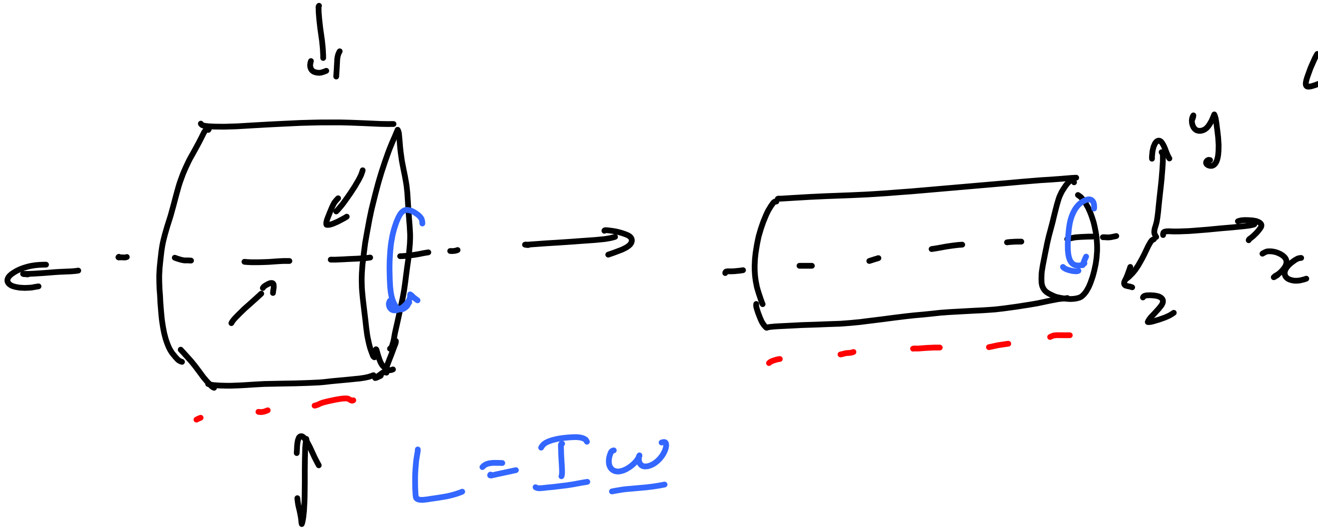


$u_x = u_x(x,y)$

$u_y = u_y(x,y)$

$\underline{\omega} = \omega_z \underline{e}_z$

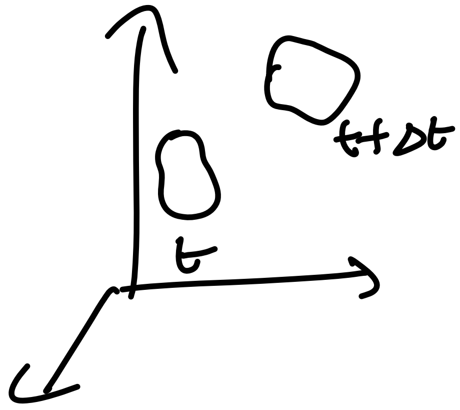
$\omega_j S_{ij} = 0$



$L = I\underline{\omega}$

Vorticity $\underline{\omega} = \omega_x \underline{e}_x$

Kelvin's circulation theorem



$$\frac{d}{dt}\oint \boxed{d\underline{x}}\cdot\boxed{\underline{u}} = \frac{d}{dt}\oint dx_i u_i$$

$$= \oint dx_i \frac{Du_i}{Dt} + \int u_i \, dx_j \left(\frac{\partial u_i}{\partial x_j}\right)$$

$$\frac{d}{dt}\Delta x_i = \Delta x_j \frac{\partial u_i}{\partial x_j}$$

$$\boxed{\frac{d}{dt}\oint d\underline{x}\cdot\underline{u}} = \oint dx_i \left(\cancel{-\frac{1}{\rho}\frac{\partial p}{\partial x_i}} + \nu \frac{\partial^2 u_i}{\partial x_j^2}\right)$$

$$+ \oint dx_j \left(\cancel{\frac{\partial}{\partial x_j}}\left(\tfrac{1}{2}u_i^2\right)\right)$$

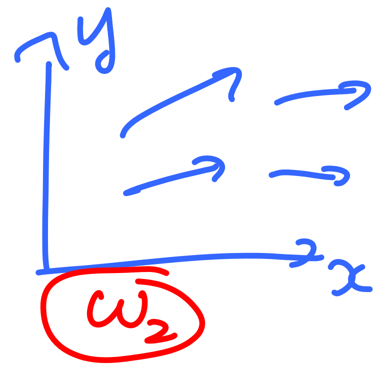
$$= \oint dx_i \left(\nu \frac{\partial^2 u_i}{\partial x_j^2}\right)$$

$$= \oint dx_i \left(-\nu \epsilon_{ijk}\frac{\partial \omega_k}{\partial x_j}\right)$$

Vorticity dynamics:

$$\underline{\omega} = \nabla \times \underline{u} \implies \omega_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_i}\left(\tfrac{1}{2} u_j^2\right) - \epsilon_{ijk} u_j \omega_k = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - \nu \epsilon_{ijk}\frac{\partial}{\partial x_j}\omega_k$$

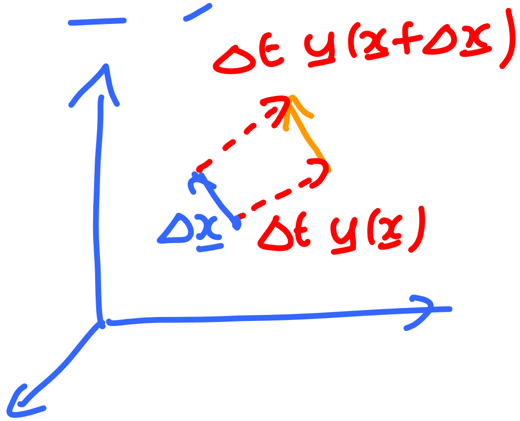


$$\boxed{\frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j}} = \omega_j \frac{\partial u_i}{\partial x_j} + \nu \frac{\partial^2 \omega_i}{\partial x_j^2}$$

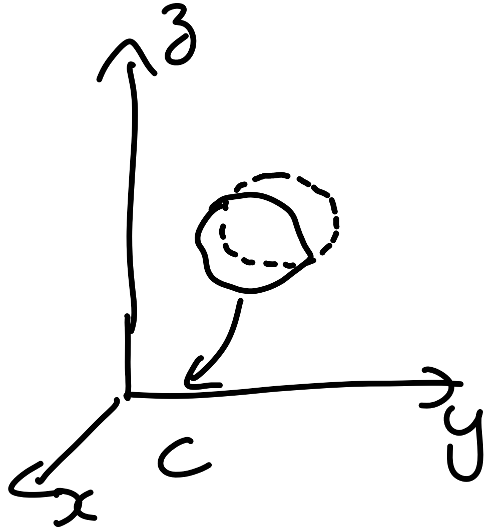
$$\boxed{\frac{D\omega_i}{Dt} = \boxed{\omega_j \frac{\partial u_i}{\partial x_j}} + \boxed{\nu\frac{\partial^2\omega_i}{\partial x_j^2}}}$$

$$\frac{d(\Delta \underline{x})}{dt} = \Delta\underline{x}\cdot\nabla\underline{u}$$

$$\boxed{\frac{d(\Delta x_i)}{dt} = \Delta x_j \frac{\partial}{\partial x_j}(u_i)}$$



Kelvin's theorem:



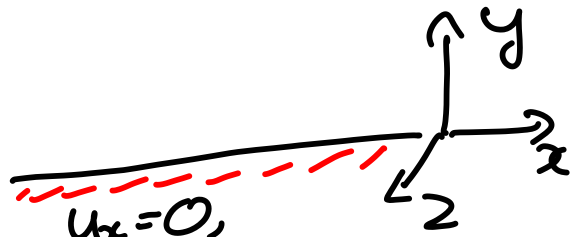
$$\Gamma = \oint_C d\underline{x} \cdot \underline{u}$$

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\left[\oint_c d\underline{x} \cdot \underline{u}\right]$$

$$= \nu \oint dx_i \left[\frac{\partial^2 u_i}{\partial x_j^2}\right]$$

$$= -\nu \oint dx_i \, \epsilon_{ijk} \frac{\partial}{\partial x_j}(\omega_k)$$

Shear stress at the wall



$u_x = 0$,

$u_y = 0 \Rightarrow \frac{\partial u_y}{\partial x} = 0$

$u_z = 0$

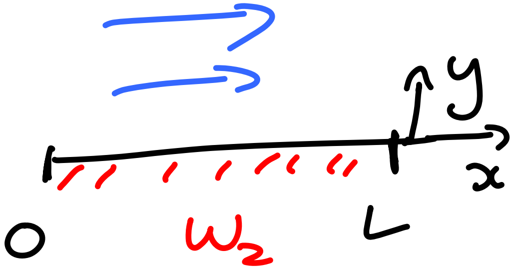
$$\tau_{xy} = \mu\left(\frac{\partial u_x}{\partial y}\right) = \mu\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right)$$

$$= -\mu \omega_z$$

$$F_x = \tau_{xy} n_y = -\mu n_y \omega_z$$

$$E_{visc} = -\mu\, \underline{n} \times \underline{\omega}$$

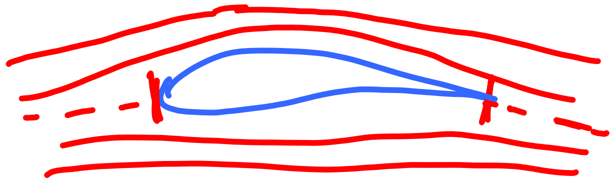
$$\int dx\, j_y^{\omega} =$$



$$j^{\omega} = -\nu\, \underline{n} \cdot \nabla \underline{\omega}$$

$$j_y^{\omega} = -\nu \frac{\partial \omega_z}{\partial y}$$

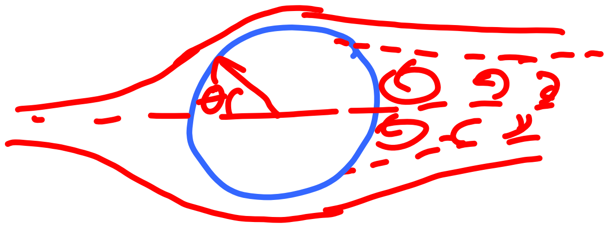
$$\rho\left(\cancel{\frac{\partial u_i}{\partial t}}\right) + \frac{\partial}{\partial x_i}\left(\tfrac{1}{2}\rho u_i^2\right) - \rho\, \epsilon_{ijk} \cancel{u_j \omega_k} = -\frac{\partial p}{\partial x_i} - \mu\, \epsilon_{ijk} \frac{\partial}{\partial x_j} \omega_k$$

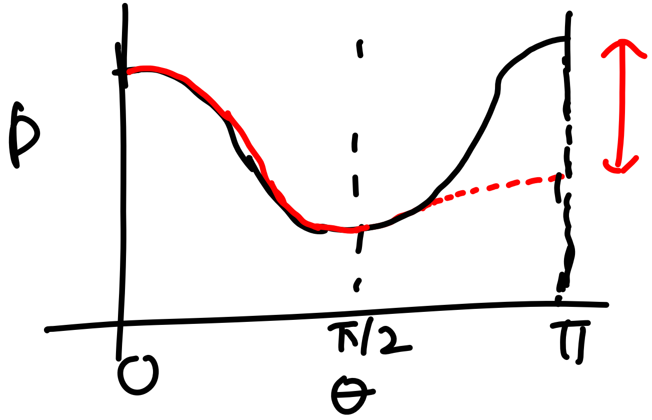


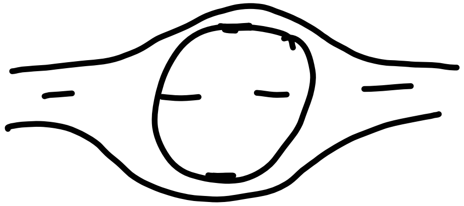
$$\frac{\partial p}{\partial x_i} = -\epsilon_{ijk}\, \mu \frac{\partial \omega_k}{\partial x_j}$$

$$\frac{\partial p}{\partial x} = -\mu \frac{\partial \omega_z}{\partial y} = +\frac{\mu}{\nu} j_y^{\omega} = \rho\, j_y^{\omega}$$

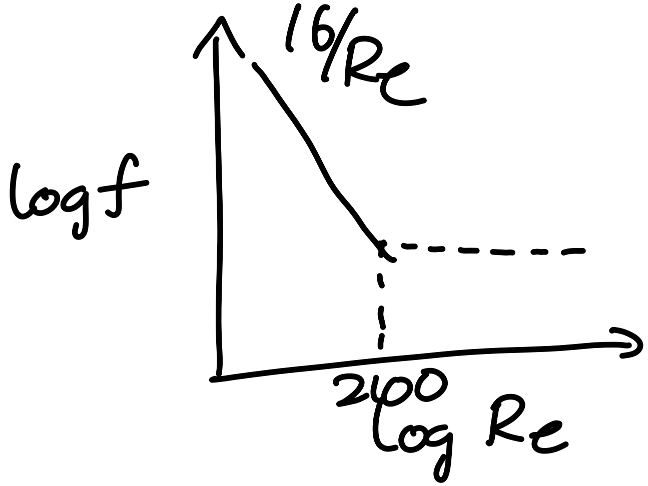
$$\int_0^L dx\, j_y^{\omega} = \int_0^L dx \left(\frac{1}{\rho}\frac{\partial p}{\partial x}\right) = \frac{1}{\rho}\Big[p(x=L) - p(x=0)\Big]$$

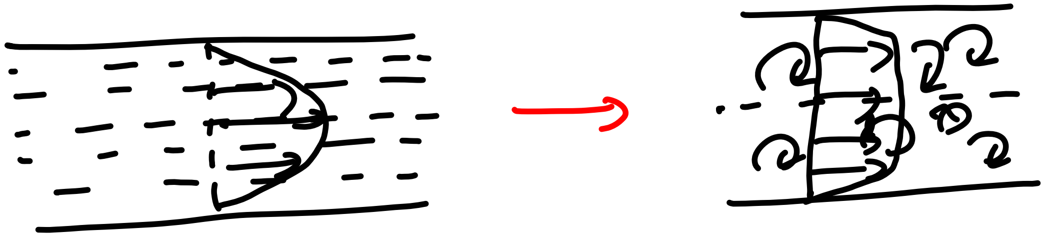


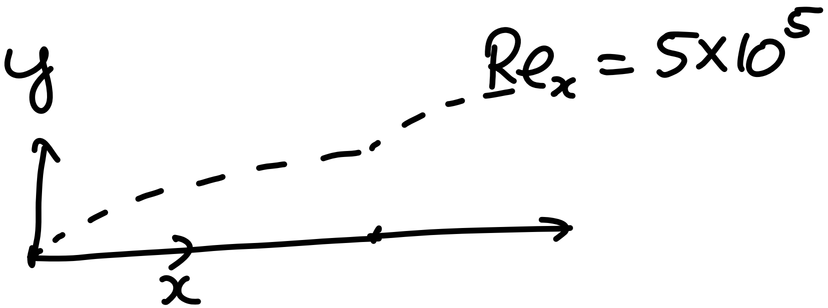




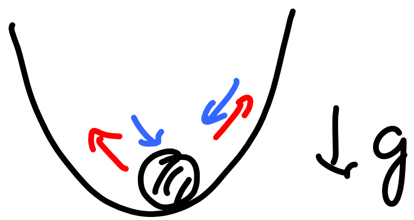
Turbulence:

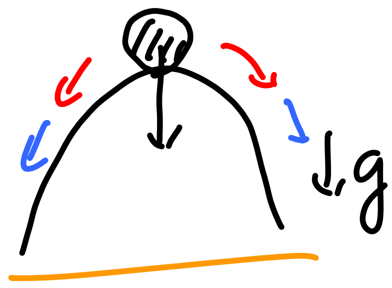


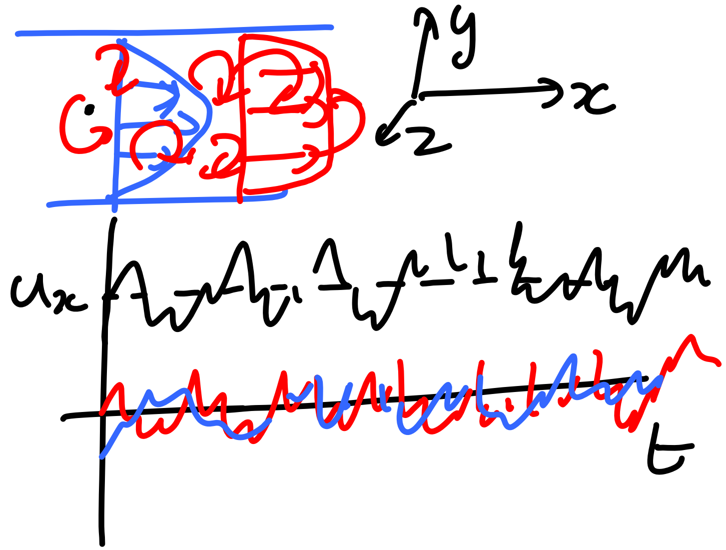




Laminar profile goes unstable:







Turbulence:

① High Reynolds number

② Continuum

③ Irregular

④ Three-dimensional

⑤ Diffusive ⑥ Dissipation

$$D \propto \lambda \left(\frac{kT}{m}\right)^{1/2}$$

$$D_T \propto l u'$$

$$u_i = U_i + u_i'$$

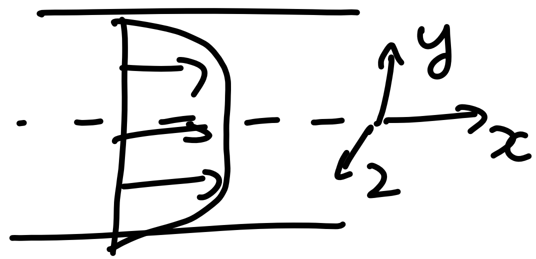
$$U_i = \frac{1}{T}\int_0^T dt\, u_i = \langle u_i \rangle$$

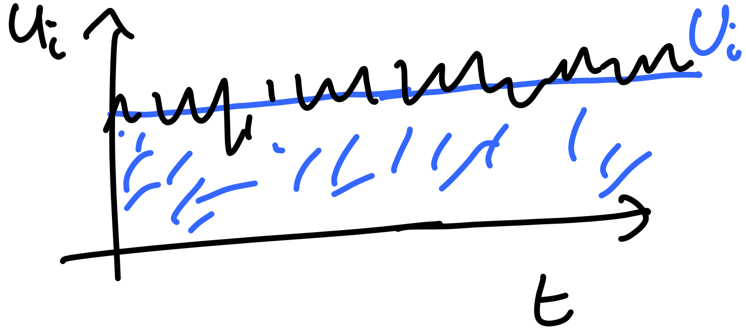
$$u_i' = u_i - U_i$$

$$\langle u_i' \rangle = \frac{1}{T}\int_0^T dt\, u_i'$$

$$= \frac{1}{T}\int_0^T dt\,(u_i - U_i) = \left[\frac{1}{T}\int_0^T dt\, u_i\right] - U_i$$

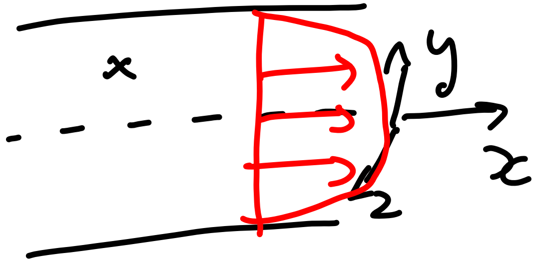
$$= 0$$



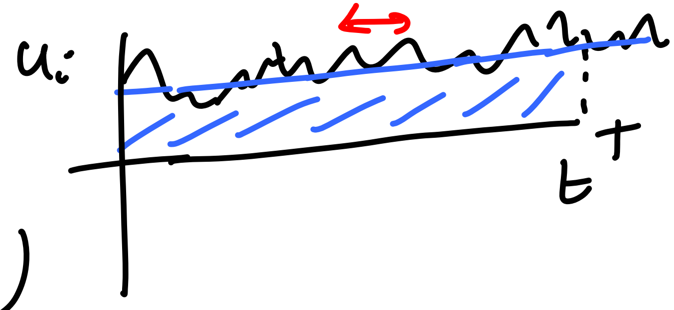


Turbulence:

$$u_i = U_i + u_i' \qquad p = P + p'$$



$$U_i = \frac{1}{T}\int_0^T dt\, u_i = \langle u_i \rangle \qquad P = \frac{1}{T}\int_0^T dt\, p$$



$$u_i' = u_i - U_i$$

$$\langle u_i' \rangle = \frac{1}{T}\int_0^T dt\,(u_i - U_i)$$

$$= 0$$

Mass conservation eqn:

$$\frac{\partial u_i}{\partial x_i} = 0 \implies \frac{\partial}{\partial x_i}(U_i + u_i') = 0$$

$$\frac{1}{T}\int_0^T dt\left(\frac{\partial U_i}{\partial x_i} + \frac{\partial u_i'}{\partial x_i}\right) = 0 \implies \frac{\partial}{\partial x_i}\left[\frac{1}{T}\int_0^T dt\, U_i\right] + \frac{\partial}{\partial x_i}\left[\frac{1}{T}\int_0^T dt\, u_i'\right] = 0$$

$$\frac{\partial U_i}{\partial x_i} = 0$$

$$\frac{\partial}{\partial x_i}(U_i + u_i') = 0 \Longrightarrow \frac{\partial u_i'}{\partial x_i} = 0$$

$$\frac{\partial U_i}{\partial x_i} = 0 \qquad \frac{\partial u_i'}{\partial x_i} = 0$$

$$\rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \tau_{ij}$$

$$\tau_{ij} = 2\mu e_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\langle \tau_{ij} \rangle = \mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \qquad \tau_{ij}' = \mu\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)$$

$$\rho(U_j + u_j')\frac{\partial}{\partial x_j}(U_i + u_i') = -\frac{\partial}{\partial x_i}(P + p') + \frac{\partial}{\partial x_j}\left(\langle \tau_{ij} \rangle + \tau_{ij}'\right)$$

$$\rho\left(U_j \frac{\partial U_i}{\partial x_j} + U_j \frac{\partial u_i'}{\partial x_j} + u_j' \frac{\partial U_i}{\partial x_j} + u_j' \frac{\partial u_i'}{\partial x_j}\right)$$

$$= -\frac{\partial}{\partial x_i}(P + p') + \frac{\partial}{\partial x_j}\left(\langle \tau_{ij} \rangle + \tau_{ij}'\right)$$

$$u_j' \frac{\partial u_i'}{\partial x_j} = \frac{\partial}{\partial x_j}\left(u_i' u_j'\right) - u_i' \boxed{\frac{\partial u_j'}{\partial x_j}}$$

$$\left[\rho\left[U_j \frac{\partial U_i}{\partial x_j} + U_j \frac{\partial u_i'}{\partial x_j} + u_j' \frac{\partial U_i}{\partial x_j} + \boxed{\frac{\partial}{\partial x_j}\left(u_i' u_j'\right)}\right] = -\frac{\partial P}{\partial x_i} - \frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\langle \tau_{ij} \rangle\right) + \frac{\partial}{\partial x_j}\left(\tau_{ij}'\right)\right] \frac{1}{T}\int_0^T dt$$

$$\rho U_j \frac{\partial U_i}{\partial x_j} + \rho \frac{\partial}{\partial x_j} \langle u_i' u_j' \rangle = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\langle \tau_{ij} \rangle\right)$$

$$\langle u_i' u_j' \rangle = \frac{1}{T}\int_0^T dt \; u_i' u_j'$$

$$\rho U_j \frac{\partial U_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\langle \tau_{ij} \rangle\right) - \frac{\partial}{\partial x_j}\left(\rho \langle u_i' u_j' \rangle\right)$$

$$= -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\langle \tau_{ij} \rangle\right) + \frac{\partial}{\partial x_j}\left(T_{ij}^R\right)$$

where $T_{ij}^R = -\rho \langle u_i' u_j' \rangle$ 'Reynolds stress'

$$T_{xy} = -\rho\langle u_x' u_y' \rangle$$

Momentum balance $\times\, U_i$

$$\rho U_i U_j \frac{\partial U_i}{\partial x_j} = -U_i \frac{\partial P}{\partial x_i} + U_i \frac{\partial}{\partial x_j}\left(\langle \tau_{ij} \rangle\right) + U_i \frac{\partial}{\partial x_j}\left(T_{ij}^R\right)$$

$$\rho U_j \frac{\partial}{\partial x_j}\left(\tfrac{1}{2} U_i^2\right) = -\frac{\partial}{\partial x_i}(P U_i) + \cancel{P \frac{\partial U_i}{\partial x_i}}$$

$$+ \frac{\partial}{\partial x_j}\left[\langle \tau_{ij} \rangle U_i\right] - \langle \tau_{ij} \rangle \frac{\partial U_i}{\partial x_j}$$

$$+ \frac{\partial}{\partial x_j}\left(U_i T_{ij}^R\right) - T_{ij}^R \frac{\partial U_i}{\partial x_j}$$

$$D = \tau_{ij} \frac{\partial U_i}{\partial x_j} \qquad D^R = T_{ij}^R \frac{\partial U_i}{\partial x_j} = -\rho\langle u_i' u_j' \rangle \left(\frac{\partial U_i}{\partial x_j}\right)$$

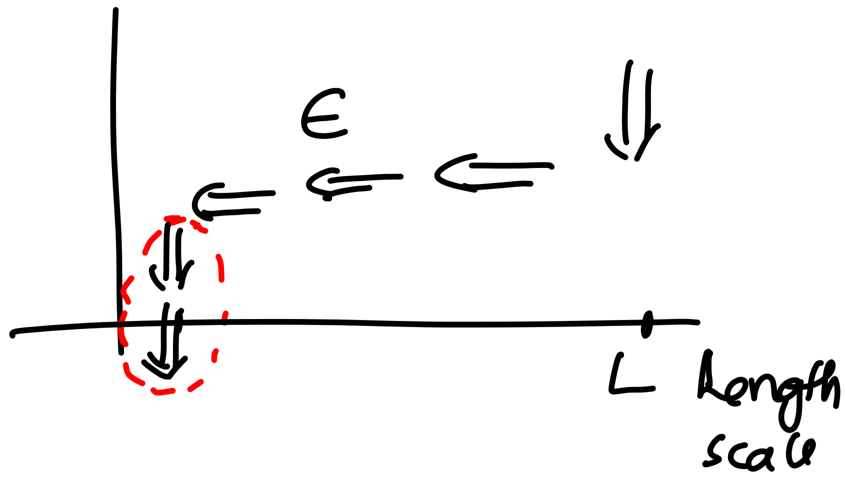
Fluctuating energy $= \frac{1}{2}\rho\langle u_i'^2\rangle$

$$\rho U_j \frac{\partial}{\partial x_j}\left(\frac{1}{2}\langle u_i'^2\rangle\right) = -\frac{\partial}{\partial x_j}\left[\langle u_j' p'\rangle + \frac{1}{2}\langle u_i'^2 u_j'\rangle - 2\mu\left\langle u_i' \frac{\partial u_i'}{\partial x_j}\right\rangle\right]$$

$$+ \tau_{ij}^R \frac{\partial U_i}{\partial x_j} - \mu\left\langle \tau_{ij}' \frac{\partial u_i'}{\partial x_j}\right\rangle$$

Kolmogorov Equilibrium Hypothesis:

$$\epsilon \simeq \rho U^2 \left(\frac{U}{L}\right) \simeq \frac{\rho U^3}{L}$$



$$\epsilon \simeq \frac{U^3}{L} = L^2 T^{-3}$$

$$\nu \simeq L^2 T^{-1}$$

Length scale $\eta = (\nu^3/\epsilon)^{1/4}$

Velocity scale $v = (\nu\epsilon)^{1/4}$

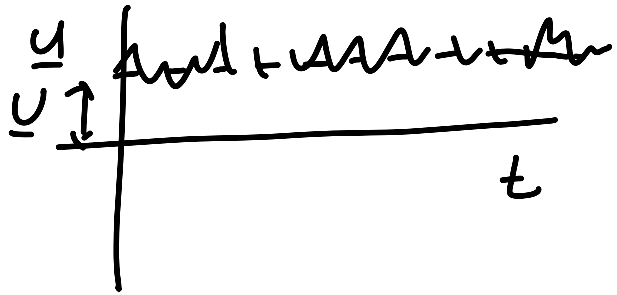
Time scale $\tau = (\nu/\epsilon)^{1/2}$

Kolmogorov scales.

Turbulent flows:

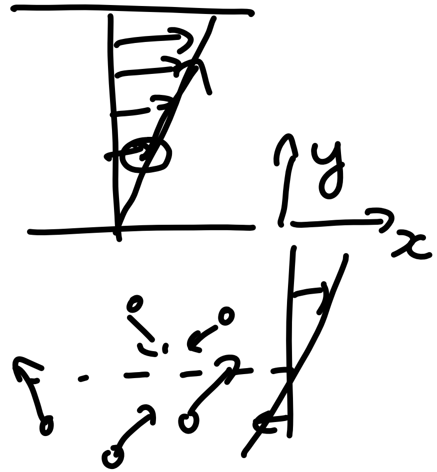
$$\nabla \cdot \underline{u} = 0$$

$$\rho\left(\cancel{\frac{\partial \underline{u}}{\partial t}} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \nabla\cdot\underline{\underline{\tau}}$$



$$\underline{U} = \frac{1}{T}\int_0^T dt\, \underline{u}$$

$$\underline{u}' = \underline{u} - \underline{U} \Rightarrow \langle \underline{u}' \rangle = \frac{1}{T}\int_0^T dt\,(\underline{u} - \underline{U}) = 0$$



$$\underline{u} = \underline{U} + \underline{u}' \qquad p = P + p'$$

$$\rho\, \underline{U}\cdot\nabla\underline{U} = -\nabla P + \nabla\cdot\underline{\underline{\tau}} + \nabla\cdot\underline{\underline{T}}^R$$

$$\rho\, U_j \frac{\partial U_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}(\tau_{ij}) + \frac{\partial}{\partial x_j}\left(T_{ij}^R\right)$$

$$T_{ij}^R = -\rho\langle u_i' u_j' \rangle \qquad T_{xy} = -\rho\langle u_x' u_y' \rangle$$

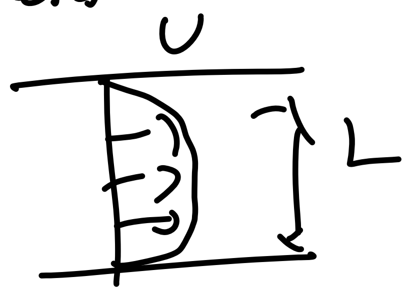
$$k = \tfrac{1}{2}\rho U_i^2 \quad ; \quad k' = \tfrac{1}{2}\rho \langle u_i'^2 \rangle$$

$$\rho U_j \frac{\partial}{\partial x_j}\left(\tfrac{1}{2} U_i^2\right) = -\frac{\partial}{\partial x_i}\left(P U_i\right) + \cancel{P \frac{\partial U_i}{\partial x_i}} + \frac{\partial}{\partial x_j}\left(\langle \tau_{ij} \rangle U_i\right) - \langle \tau_{ij} \rangle \frac{\partial U_i}{\partial x_j} + \frac{\partial}{\partial x_j}\left(T_{ij}^R U_i\right) - T_{ij}^R \left(\frac{\partial U_i}{\partial x_j}\right)$$

$$\rho U_j \frac{\partial}{\partial x_j}\left(\tfrac{1}{2}\langle u_i'^2 \rangle\right) = -\frac{\partial}{\partial x_j}\left[\langle u_j' p' \rangle + \tfrac{1}{2}\langle u_i'^2 u_j' \rangle - 2\mu \left\langle u_i' \frac{\partial u_i'}{\partial x_j} \right\rangle\right] + T_{ij}^R \frac{\partial U_i}{\partial x_j} - 2\mu \left\langle \frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j} \right\rangle$$

Smallest turbulence scales:

'Kolmogorov universal equilibrium hypothesis'



$$\rho U_j \frac{\partial}{\partial x_j}\left(\tfrac{1}{2} U_i^2\right) \equiv \frac{\rho U^3}{L}$$

$$P = \left(\frac{U^3}{L}\right) \equiv L^2 T^{-3} \equiv \epsilon$$

$$\nu = L^2 T^{-1}$$

$$\eta = \left(\nu^3/\epsilon\right)^{1/4} \qquad v = (\nu\epsilon)^{1/4} \qquad \tau = (\nu/\epsilon)^{1/2}$$

Kolmogorov scales

$$\frac{\eta}{L} = \frac{1}{L}\left(\frac{\nu^3}{\epsilon}\right)^{1/4} = \left(\frac{\nu^3}{U^3 L^3}\right)^{1/4} = Re^{-3/4}$$

$$\frac{v}{U} = \frac{1}{U}(\nu\epsilon)^{1/4} = Re^{-1/4}$$

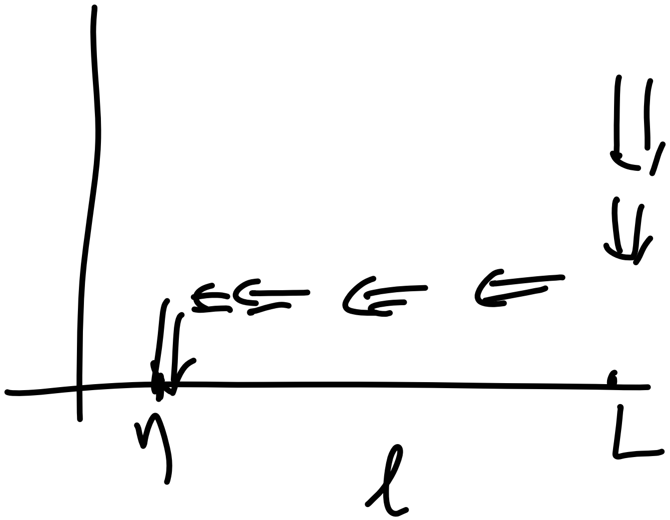
$$s \cong \left(\frac{v}{\eta}\right) = \left(\frac{\epsilon}{\nu}\right)^{1/2} \quad \Bigg| \quad \frac{(v/\eta)}{(U/L)} = \left(\frac{\epsilon}{\nu}\right)^{1/2}\left(\frac{L}{U}\right) = Re^{+1/2}$$

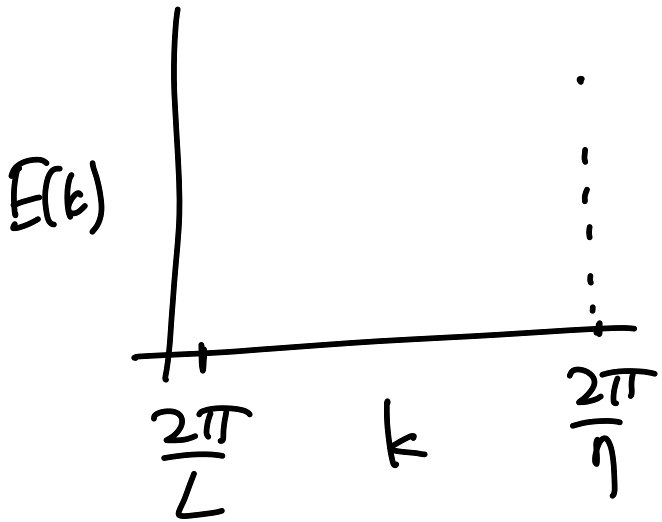
Dissipation rate $\propto \mu S^2$

$$\frac{\text{Dissipation rate (Kolmogorov)}}{\text{Dissipation Rate (Mean flow)}} = Re^{+1}$$

$$Re = \left(\frac{UL}{\nu}\right) \qquad Re_k = \left(\frac{v\eta}{\nu}\right) = \frac{(\nu^3/\epsilon)^{1/4}(\nu\epsilon)^{1/4}}{\nu}$$

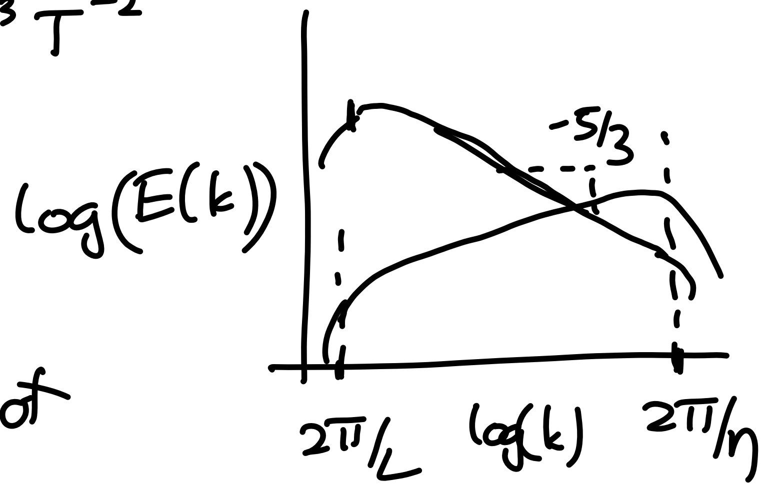
$$\equiv 1$$





Energy spectrum:

$$E(k) \cong L^3 T^{-2}$$



$$E \equiv L^2 T^{-2}$$

$$E = \int dk\, E(k)$$

$k = \frac{2\pi}{\lambda}$ where λ = wavelength of fluctuation

$$E(k) = L^3 T^{-2} \equiv \eta^{5/4} \epsilon^{4/4} f(k\eta)$$

$$E(k) = U^2 L f(kL)$$

Inertial sub-range $E(k)$ depends only on ϵ & k

$$E(k) \propto \epsilon^{2/3} k^{-5/3}$$

$$D \propto \mu S^2 \propto \mu (U/\ell)^2 = \int dk\, D(k)$$

$$D(k) \equiv E(k)/L^2 \propto L T^{-2} \equiv \epsilon^{2/3} k^{+1/3}$$

K-ε model:

$$\tau_{ij} = (\mu + \mu_t)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\underline{\mu_t = \rho C_\mu \frac{k^2}{\epsilon}} \equiv \underline{\underline{\rho C_\mu l u'}} = \underline{\underline{\rho C_\mu l^2 \left|\frac{\partial U}{\partial y}\right|}}$$

$$\frac{\partial}{\partial t}(\rho k) + \frac{\partial}{\partial x_j}(\rho U_j k) = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] + P_k - \rho\epsilon$$

$$\frac{\partial}{\partial t}(\rho\epsilon) + \frac{\partial}{\partial x_j}(\rho U_j \epsilon) = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_\epsilon}\right)\frac{\partial \epsilon}{\partial x_j}\right] + C_{1\epsilon} P_k (\epsilon/k) - C_{2\epsilon}\rho\frac{\epsilon^2}{k}$$

$$P_k = (\mu + \mu_T)\frac{\partial U_i}{\partial x_j}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

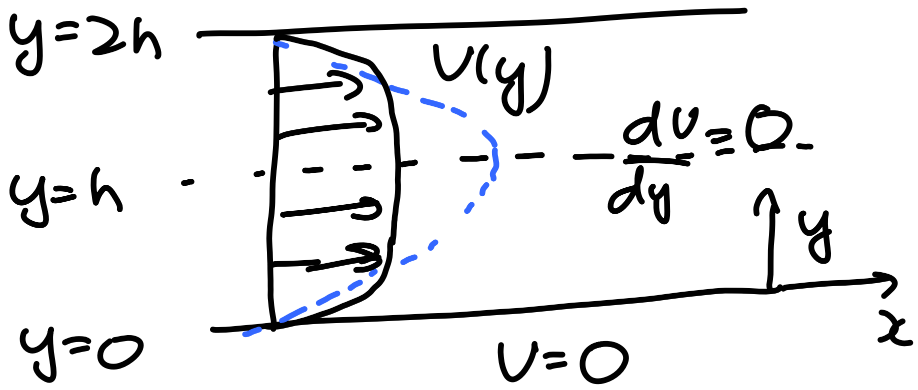
$$= 2(\mu + \mu_T) S_{ij}^2$$

$C_\mu = 0.09$; $C_{1\epsilon} = 1.44$, $C_{2\epsilon} = 1.92$, $\sigma_k = 1.0$

$\sigma_\epsilon = 1.3$

Turbulent flow in a channel:

$$U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j^2} - \frac{\partial}{\partial x_j}\left(\langle u_i' u_j' \rangle\right)$$



$$\cancel{U_x \frac{\partial U_x}{\partial x}} + \cancel{U_y \frac{\partial U_x}{\partial y}} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left[\cancel{\frac{\partial^2 U_x}{\partial x^2}} + \frac{\partial^2 U_x}{\partial y^2}\right] - \frac{\partial}{\partial x}\left(\cancel{\langle u_x'^2 \rangle}\right) - \frac{\partial}{\partial y}\left(\langle u_x' u_y' \rangle\right)$$

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu \frac{\partial^2 U}{\partial y^2} - \frac{\partial}{\partial y}\left(\langle u_x' u_y' \rangle\right) = 0$$

$$\cancel{U_x \frac{\partial U_y}{\partial x}} + \cancel{U_y \frac{\partial U_y}{\partial y}} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\cancel{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}}\right)U_y - \cancel{\frac{\partial}{\partial x}\left(\langle u_x' u_y' \rangle\right)} - \frac{\partial}{\partial y}\langle u_y'^2 \rangle$$

$$-\frac{1}{\rho}\frac{\partial P}{\partial y} - \frac{\partial}{\partial y}\left[\langle u_y'^2\rangle\right] = 0$$

$$-\frac{1}{\rho}\boxed{\frac{\partial P}{\partial x}} + \nu \frac{\partial^2 U_x}{\partial y^2} - \frac{\partial}{\partial y}\left(\langle u_x' u_y'\rangle\right) = 0$$

$$\frac{P}{\rho} + \frac{1}{2}\langle u_y'^2\rangle = \frac{P_0}{\rho}$$

P_0 = wall pressure

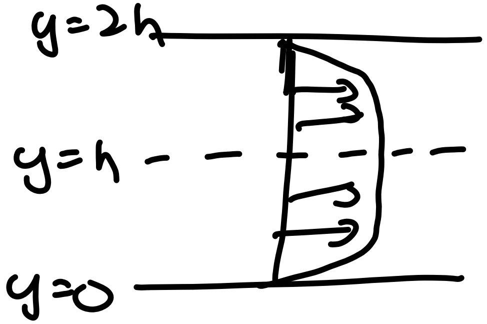
$$\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left(\cancel{\frac{1}{2}\langle u_y'^2\rangle}\right) = \frac{1}{\rho}\frac{\partial P_0}{\partial x}$$

$$\frac{\partial P}{\partial x} = \boxed{\frac{\partial P_0}{\partial x}}$$

$$-\frac{1}{\rho}\frac{\partial P_0}{\partial x} + \nu \frac{\partial^2 U_x}{\partial y^2} - \frac{\partial}{\partial y}\left(\langle u_x' u_y'\rangle\right) = 0$$

$$\boxed{-\frac{y}{\rho}\frac{\partial P_0}{\partial x} + \nu \frac{\partial U_x}{\partial y} - \langle u_x' u_y'\rangle - \boxed{u_*^2} = 0}$$

Take value of this equation at $y=0$

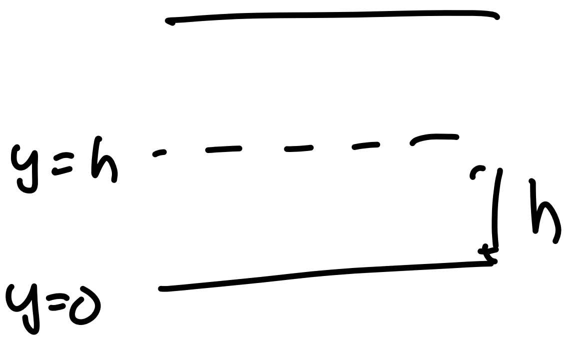


$$+\nu \frac{\partial U_x}{\partial y} - u_*^2 = 0$$

$$\rho u_*^2 = \mu \frac{\partial U_x}{\partial y} = \tau_{xy}$$

u_* = Friction velocity

Take value of this equation at $y=h$



$$-\frac{h}{\rho}\frac{\partial P_0}{\partial x} + \cancel{\nu\frac{\partial U_x}{\partial y}} - \cancel{\langle u_x' u_y'\rangle} - u_*^2 = 0$$

$$u_*^2 = -\frac{h}{\rho}\frac{\partial P_0}{\partial x} \Longrightarrow \frac{\partial P_0}{\partial x} = -\frac{\rho}{h}u_*^2$$

$$-\langle u_x' u_y'\rangle + \nu\frac{\partial U_x}{\partial y} = u_*^2\left(1 - y/h\right)$$

$$y^* = (y/h)$$

$$-\frac{\langle u_x' u_y'\rangle}{u^{*2}} + \frac{\nu}{u_*^2\, h}\frac{\partial U_x}{\partial y^*} = (1-y^*)$$

$$-\frac{\langle u_x' u_y' \rangle}{u_*^2} + \left(\frac{\nu}{u_* h}\right)\frac{\partial}{\partial y^*}\left(\frac{U_x}{u_*}\right) = 1 - y^*$$

$$Re_*^{-1}$$

$$-\frac{\langle u_x' u_y' \rangle}{u_*^2} + Re_*^{-1}\frac{\partial}{\partial y^*}\left(\frac{U_x}{u_*}\right) = 1 - y^*$$

$$y^+ = \left(\frac{y u^*}{\nu}\right) \Longrightarrow y = \frac{\nu y^+}{u_*}$$

$$-\frac{\langle u_x' u_y' \rangle}{u_*^2} + \frac{\nu}{u^{*2}}\frac{\partial U_x}{\partial y} = \left(1 - (y/h)\right)$$

$$-\frac{\langle u_x' u_y' \rangle}{u_*^2} + \frac{\partial (U_x/u_*)}{\partial y_+} = \left(1 - \frac{\nu}{u_* h} y^+\right)$$

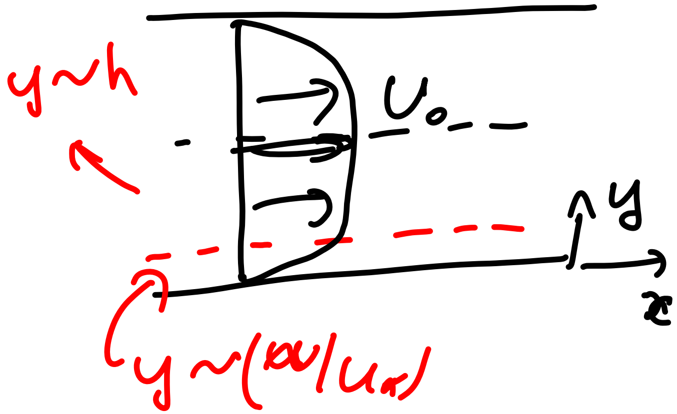
$$-\frac{\langle u_x' u_y' \rangle}{u_*^2} + \frac{\partial (U_x/u_*)}{\partial y^+} = \left(1 - \cancel{Re_*^{-1}}\, y^+\right)$$

$$\frac{dU_x}{dy} = \left(\frac{u_*}{h}\right)\frac{dF(y^*)}{dy^*}$$

$$\boxed{\frac{U_x - U_0}{u^*} = F(y^*)}$$

Core of channel.

$y \sim h$



$$\frac{dU_x}{dy} = \frac{u_*}{(\nu/u_*)}\frac{df(y^+)}{dy^+} = \frac{u_*^2}{\nu}\frac{df}{dy^+}$$

Wall layer $y \sim (\nu/u_*)$

$$\boxed{\frac{U_x}{u_*} = f(y^+)}$$

Intermediate region $\nu/u_* \ll y \ll h$

$$\frac{y^*}{y^+} = Re_*^{-1}$$

$y^* = y/h \ll 1$

$y^+ = \left(\frac{y u_*}{\nu}\right) \gg 1$

$$\frac{dU_x}{dy} = \frac{u_*}{h}\frac{dF}{dy^*} = \frac{u_*^2}{\nu}\frac{df}{dy^+}$$

$$y^* \frac{dF}{dy^*} = y^+ \frac{df}{dy^+} = \frac{1}{k}$$

$$K = 0.4 \Rightarrow \frac{1}{k} = 2.5$$

$$A = 5 \qquad B = -1$$

$$F = \frac{1}{k}\log(y^*) + B = \boxed{\frac{U_x - U_0}{u_*}}$$

$$f(y^+) = \frac{1}{k}\log(y^+) + A = \boxed{\frac{U_x}{u_*}}$$

$$\frac{U_0}{u_*} = \frac{1}{k}\log(Re_*) + A - B$$

$$\frac{U_0}{u_*} = 2.5\log(Re_*) + 0.6$$

$$F = 2.5\log(y^*) - 1 \;;\; f = 2.5\log(y_+) + 5$$