

The Assam Royal Global University, Guwahati

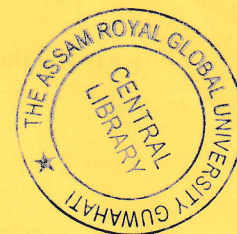
Royal School of Applied and Pure Sciences

MSc (Mathematics), First Semester

Special Supplementary Examination, September 2023

Course Title: Real Analysis

Course Code: MAT014C103



Time: 3 Hours

Maximum Marks: 70

Note: Attempt all questions as per instructions given.

*The figures in the right hand side indicate marks.*

**Section – A**

Q. 1. Attempt all questions (Maximum word limit – 50) 2 x 8 = 16

(a) Define the concept of *refinement* of a partition of an interval  $[a, b]$  with a numerical example.

(b) If  $f$  is continuous and non-negative on  $[a, b]$ , show that  $\int_a^b f dx \geq 0$ .

(c) State the necessary and sufficient condition of integrability of a function  $f(x)$  with respect to  $\alpha(x)$  on  $[a, b]$ .

(d) Test the convergence of  $\int_0^1 \frac{1}{\sqrt{1-x}} dx$ .

(e) State Abel's Test of uniform convergence of a series.

(f) Define what is meant by Uniform Convergence of a series.

(g) For  $f(x, y) = 2x^2 - xy + 2y^2$  find  $f_x$  and  $f_y$ .

(h) Find the radius of convergence of the series

$$1 + 2x + 3x^2 + \dots \dots .$$

**Section – B**

Q. 2. Attempt any two questions 6 x 2 = 12

(a) State and prove the Second Mean Value Theorem of Integral Calculus.

(b) If both  $f$  and  $g$  are differentiable on  $[a, b]$  and if both  $f'$  and  $g'$  are integrable on  $[a, b]$ , show that

$$\int_a^b f(x) g'(x) dx = [f(x) \cdot g(x)]_a^b - \int_a^b g(x) f'(x) dx.$$

(c) If  $f_1$  and  $f_2$  are bounded and integrable on  $[a, b]$ , show that  $(f_1 - f_2)$  is integrable on  $[a, b]$  and

$$\int_a^b f_1 dx - \int_a^b f_2 dx = \int_a^b (f_1 - f_2) dx.$$

Q. 3. Attempt any two questions

7 x 2 = 14

(a) If  $f_1 \in R(\alpha)$  and  $f_2 \in R(\alpha)$  on  $[a, b]$ , show that  $(f_1 + f_2) \in R(\alpha)$  and

$$\int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha = \int_a^b (f_1 + f_2) d\alpha.$$

(b) If  $P^*$  is a refinement of a partition  $P$ , then for bounded functions  $f$  and  $\alpha$  on  $[a, b]$  for monotone increasing  $\alpha$ , prove that

$$U(P^*, f, \alpha) \leq U(P, f, \alpha)$$

where  $U$  stands for the Upper Riemann-Stieltje Sum.

(c) Prove that every absolutely convergent improper integral is convergent.

Q. 4. Attempt any two questions.

7 x 2 = 14

(a) Prove that a sequence of functions  $\{f_n\}$  defined on  $[a, b]$  converges uniformly on  $[a, b]$  if and only if for every  $\varepsilon > 0$  and for all  $x \in [a, b]$ , there exists an integer  $N$  such that

$$|f_{n+p}(x) - f_n(x)| < \varepsilon,$$

for all  $n \geq N, p \geq 1$ .

(b) Deduce Dirichlet's Test for uniform convergence of a series of monotonic functions. Hence show that the series

$$\sum (-1)^n \frac{x^2 + n}{n^2}$$

converges uniformly in every bounded interval.



(c) Prove that a uniformly convergent series of continuous functions can be integrated term by term.

Q. 5. Attempt any two questions.

7 x 2 = 14

(a) Prove that a power series is absolutely convergent within its interval of convergence.

(b) Prove that for a continuous function  $f(x, y)$ , where  $x = \varphi(t)$ ,  $y = \psi(t)$ ,

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

(c) Establish Lagrange's Mean Value Theorem for a bivariate function  $f(x, y)$ .

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