The Assam Royal Global University, Guwahati

Royal School of Applied and Pure Sciences

MSc (Mathematics), First Semester

Special Supplementary Examination, September 2023

Course Title: Real Analysis

Course Code: MAT014C103

Time: 3 Hours **Maximum Marks: 70**

Note: Attempt all questions as per instructions given.

The figures in the right hand side indicate marks.

Section – A

Q. 1. Attempt all questions (Maximum word limit – 50) $2 \times 8 = 16$

(a) Define the concept of *refinement* of a partition of an interval $[a, b]$ with a numerical example.

(b) If f is continuous and non-negative on $[a, b]$, show that $\int_a^b f dx \geq 0$.

(c) State the necessary and sufficient condition of integrability of a function $f(x)$ with respect to $\alpha(x)$ on $[a, b]$.

(d) Test the convergence of $\int_0^1 \frac{1}{\sqrt{(1-x)}} dx$.

(e) State Abel's Test of uniform convergence of a series.

(f) Define what is meant by Uniform Convergence of a series.

(g) For $f(x, y) = 2x^2 - xy + 2y^2$ find f_x and f_y.

(h) Find the radius of convergence of the series

$$1 + 2x + 3x^2 + \dots \dots .$$

Section – B

Q. 2. Attempt any two questions $6 \times 2 = 12$

(a) State and prove the Second Mean Value Theorem of Integral Calculus.

(b) If both f and g are differentiable on $[a,b]$ and if both f' and g' are integrable on $[a,b]$, show that

$$\int_a^b f(x)\, g'(x)dx = [f(x).g(x)]_a^b - \int_a^b g(x)\, f'(x)dx.$$

(c) If f_1 and f_2 are bounded and integrable on $[a,b]$, show that $(f_1 - f_2)$ is integrable on $[a,b]$ and

$$\int_a^b f_1 dx - \int_a^b f_2 dx = \int_a^b (f_1 - f_2)\, dx.$$

Q. 3. Attempt any two questions 7 x 2 = 14

(a) If $f_1 \in R(\alpha)$ and $f_2 \in R(\alpha)$ on $[a,b]$, show that $(f_1 + f_2) \in R(\alpha)$ and

$$\int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha = \int_a^b (f_1 + f_2)\, d\alpha.$$

(b) If P^* is a refinement of a partition P, then for bounded functions f and α on $[a,b]$ for monotone increasing α, prove that

$$U(P^*, f, \alpha) \leq U(P, f, \alpha)$$

where U stands for the Upper Riemann-Stieltje Sum.

(c) Prove that every absolutely convergent improper integral is convergent.

Q. 4. Attempt any two questions. 7 x 2 = 14

(a) Prove that a sequence of functions $\{f_n\}$ defined on $[a,b]$ converges uniformly on $[a,b]$ if and only if for every $\varepsilon > 0$ and for all $x \in [a,b]$, there exists an integer N such that

$$| f_{n+p}(x) - f_n(x) | < \varepsilon,$$

for all $n \geq N, p \geq 1$.

(b) Deduce Dirichlet's Test for uniform convergence of a series of monotonic functions. Hence show that the series

$$\Sigma (-1)^n \frac{x^2 + n}{n^2}$$

converges uniformly in every bounded interval.

(c) Prove that a uniformly convergent series of continuous functions can be integrated term by term.

Q. 5. Attempt any two questions. 7 x 2 = 14

(a) Prove that a power series is absolutely convergent within its interval of convergence.

(b) Prove that for a continuous function $f(x, y)$, where $x = \varphi(t), y = \psi(t)$,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

(c) Establish Lagrange's Mean Value Theorem for a bivariate function $f(x, y)$.

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