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The Assam Royal Global University, Guwahati

Royal school of Applied & Pure Sciences

M.Sc. Mathematics 2nd Semester

Special Supplementary Examination, September 2023

Course Title: Complex Analysis

Course Code: MAT014C204

Time: 3 Hours **Maximum Marks: 70**

Note: Attempt all questions as per instructions given.

The figures in the right-hand margin indicate marks.

Section – A

1. Attempt **all** questions: **2x8=16**

(a) Find the value of the integral

$$\int_0^{1+i}(x-y+ix^2)\,dz$$

along the straight line from $z=0$ to $z=1+i$.

(b) Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}dz$ if C is

(i) the circle $|z|=3$

(ii) the circle $|z|=1$.

(c) Determine the number of roots of the equation $z^8-4z^5+z^2-1=0$ that lie inside the circle $|z|=1$.

(d) Find Laurent series about the indicated singularity of the function $\frac{z}{(z+1)(z+2)}$; $z=-2$.

(e) Find the residue of the function $\frac{z^2-2z}{(z+1)^2}$ at its pole in the finite plane.

(f) Define Mittag-Leffler's expansion theorem.

(g) What do you mean by bilinear or fractional transformation?

(h) Find the fixed points of the transformation $w=\frac{3iz+1}{z+i}$.

Section-B

2. Attempt **any two** of the following: **6x2=12**

(a) If for all z in the entire complex plane (i) $f(z)$ is analytic and (ii) $f(z)$ is bounded i.e. $|f(z)|\le M$ for some constant M, then show that $f(z)$ must be a constant.

(b) If $f(z)$ is analytic inside and on a circle C with centre at 'a' and radius 'r',then prove that $f(a)$ is the mean of the values of $f(z)$ on C i.e.

$$f(a)=\frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\,d\theta$$

(c) If $f(z)$ is continuous in a simply connected region and if $\int_C f(z)\,dz=0$ around every closed contour in R, then show that $f(z)$ is analytic in R.

3. Attempt **any two** of the following: **7x2=14**

(a) Let $f(z)$ be analytic within and on a simple closed curve C except for a finite number of poles inside C , then show that

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = N - P$$

where N and P are respectively the number of zeros and number of poles of $f(z)$ inside C.

(b) If $a > e$, use Rouche's theorem to prove that the equation $e^z = az^n$ has n roots inside the circle $|z| = 1$.

(c) If $f(z)$ is analytic inside a circle C with centre at 'a', then prove that for all z inside C,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n \qquad \text{where} \quad a_n = \frac{f^n(a)}{n!}$$

4. Attempt **any two** of the following: **7x2=14**

(a) Apply calculus of residues to evaluate

$$\int_0^{\pi} \frac{a}{a^2+sin^2\theta} d\theta \quad \text{where } a > 0$$

(b) By the method of contour integration, prove that

$$\int_0^{\infty} \frac{\cos mx}{x^2+a^2} dx = \frac{\pi e^{-ma}}{2a} \qquad \text{where} \quad m \geq 0\,,\ a > 0$$

And hence deduce that $\int_0^{\infty} \frac{\cos x}{x^2+4} dx = \frac{\pi}{4} e^{-a}$.

(c) By integrating $\frac{e^{iz}}{\sqrt{z}}$ along a suitable path, show that

$$\int_0^{\infty} \frac{\cos x}{\sqrt{x}} dx = \int_0^{\infty} \frac{\sin x}{\sqrt{x}} dx = \sqrt{\frac{\pi}{2}}$$

5. Attempt **any two** of the following: **7x2=14**

(a) Discuss the transformation

$$w = \frac{i(1-z)}{1+z}$$

and show that it transforms the circle $|z| = 1$ into the real axis of the w-plane and the interior of the circle $|z| < 1$ into the upper half of the w-plane.

(b) Define cross-ratio. Show that the cross-ratio remains invariant under a bilinear transformation.

(c) In the transformation $z = \frac{i-w}{i+w}$, show that half of w-plane given by $v \geq 0$ corresponds to the circle $|z| \leq 1$ in z plane.

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