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The Assam Royal Global University, Guwahati

Royal School of Applied & Pure Sciences

M.Sc. (Mathematics), 2nd Semester

Semester End Examination, June 2023

Course Title: Functional Analysis

Course Code: MAT014C203

Time: 3 Hours **Maximum Marks: 70**

Note: Attempt all questions as per instructions given.

The figures in the right-hand margin indicate marks.

Section – A

1. Attempt **all questions**. (Maximum word limit 50) **2 x 8**

a. Let T be a bounded linear operator on a normed space X. If $x_n \to x$, show that $Tx_n \to Tx$.

b. Let X and Y be normed spaces and $T: X \to Y$ a linear operator. If T is bounded and $dim T(X) < \infty$, show that T is compact

c. Expand $\langle ix + y,\ x - iz \rangle$.

d. Show that $T = 2iI$ is normal but not a self-adjoint operator, where I is an identity operator.

e. Give an example of a partially ordered set.

f. Write the statement of Zorn's lemma.

g. Define point spectrum and resolvent set of an operator.

h. Define regular value of a bounded linear operator T.

Section – B

2. Attempt **any two** of the following: **6 x 2**

a. Show that in a finite dimensional normed space X, any subset $M \subset X$ is closed and bounded if and only if M is compact.

b. Show that every finite dimensional subspace Y of a normed space X is complete.

c. i) Show that in a finite dimensional normed linear space X, every linear operator on X is bounded.

ii) Show that identity operator is not compact.

3. Attempt **any two** of the following: **7 x 2**

a. i)The space ℓ^p $(p \neq 2)$ is not an inner product space.

ii)If T is any bounded linear operator on a Hilbert space H, show that T can be uniquely expressed in the form $T = A + iB$, where A and B are self-adjoint operators.

b. If T is any bounded linear operator on a Hilbert space H, show that the following are equivalent. i) $T^*T = I$ ii) $\langle Tx, Ty \rangle = \langle x, y \rangle, \forall x, y \in H$ iii) $\|Tx\| = \|x\|; \forall\, x \in H$.

c. Let H be a Hilbert space and $f: H \to F$ be a bounded linear functional. Show that there exists an unique $z \in H$ such that $f(x) = \langle x, z \rangle$ for all $x \in H$, where z depends on f and $\|f\| = \|z\|$.

4. Attempt **any two** of the following: **7 x 2**

a. i) Let X be a normed linear space. If $f(x) = 0$ for all bounded linear functional f on X, show that $x = 0$.

ii) Show that strong convergence implies weak convergence with same limit.

b. State and prove Uniform Boundedness Theorem.

c. Prove that every bounded linear operator from a Banach space X onto a Banach space Y is an open mapping.

5. Attempt **any two** of the following: **7 x 2**

a. State and prove Banach fixed point theorem.

b. i) Show that resolvent operator is an analytic function at every point of the resolvent set $\rho(T)$.

ii) Let $X = C[a, b]$ with $\|.\|_\infty$ and $f \in X$. Let $A: X \to X$ be defined by $(Ax)(t) = f(t)x(t)$; for all $t \in [a, b]$ and $x \in X$. If f is not a constant function, show that $\sigma_p(A) = \emptyset$.

c. i) For any bounded linear operator, show that $\sigma_r(T)^* \subseteq \sigma_p(T^*)$.

ii) Let T be a bounded linear operator on a Banach space X and $\|T\| < 1$. Show that $(I - T)$ is invertible.

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