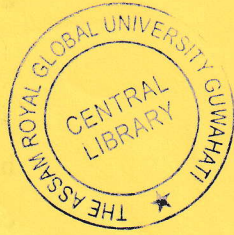


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The Assam Royal Global University, Guwahati
Royal School of Applied & Pure Sciences
M.Sc. Mathematics, 2nd Semester
Special Supplementary Examination, September 2023
Course Title: Functional Analysis
Course Code: MAT014C203



Time: 3 Hours

Maximum Marks: 70

Note: Attempt all questions as per instructions given.
The figures in the right-hand margin indicate marks.

Section – A

1. Attempt **all questions**. (Maximum word limit 50) **2 x 8**
- What do you mean by distance of a point from a set?
 - If T is a bounded linear operator on a normed space X , show that $\|T\| = \sup_{\|x\|=1} \|Tx\|$.
 - If T^* is Hilbert adjoint of the operator T . Show that $(ST)^* = T^*S^*$
 - Write the statement of Cauchy-Schwarz inequality.
 - Define maximal and minimal element in a partially ordered set.
 - Write the statement of closed graph theorem.
 - Show that the right shift operator S defined by $S(x_0, x_1, \dots) = (0, x_0, x_1, \dots)$ is bounded.
 - Define contraction of a mapping in a metric space.

Section – B

2. Attempt **any two** of the following: **6 x 2**
- Show that the normed linear space $c[0,1] = \{f: [0,1] \rightarrow \mathbb{R} | f \text{ is continuous}\}$ is not complete.
 - Show that $B(X, Y)$: set of all bounded linear operators from X to Y , is a normed linear space with norm $\|T\| = \sup_{\|x\|=1} \|Tx\|$.
 - Let X be a normed space and Y be a proper closed subspace of X . Show that for every $r \in (0,1)$ there exists $x_r \in X$ such that $\|x_r\| = 1$ and $r \leq d(x_r, Y) \leq 1$.

P.T.O.

3. Attempt **any two** of the following:

7 x 2

- a. i) The space $C[a, b]$ is not an inner product space.
ii) Let X and Y be inner product spaces and $T: X \rightarrow Y$ a bounded linear operator. Show that $T = 0$ if and only if $\langle Tx, y \rangle = 0$ for all $x \in X$ and $y \in Y$.
- b. State and prove Bessel's Inequality in an inner product space.
- c. i) If M and N are closed subspaces of a Hilbert space H such that $M \perp N$, show that subspace $M + N$ is closed.
ii) If M is a subset of a Hilbert space H , show that $M \cap M^\perp = \{0\}$.

4. Attempt **any two** of the following:

7 x 2

- a. i) Let X be a normed linear space and $x_0 \in X, x_0 \neq 0$. Show that there exists a bounded linear functional \tilde{f} on X such that $\tilde{f}(x_0) = \|x_0\|$ and $\|\tilde{f}\| = 1$.
ii) Using Uniform Boundedness Theorem, show that the normed space X of all polynomials with norm defined by

$$\|x\| = \max_j |\alpha_j| \quad (\alpha_0, \alpha_1, \dots \text{the coefficients of } x) \text{ is not complete.}$$

- b. i) Show that Uniform boundedness theorem does not hold in an incomplete norm space.
ii) Show that weak convergence does not imply strong convergence in general.
- c. i) Show that T is a closed linear transformation if and only if its graph T_G is a closed subspace.
ii) State and prove normed space version of Hahn-Banach Theorem.

5. Attempt **any two** of the following:

7x 2

- a. Consider a metric space $X = (X, d)$, where $X \neq \emptyset$. Suppose that X is complete and let $T: X \rightarrow X$ be a contraction on X . Show that T has one fixed point.
- b. i) Let X be a linear space and $A: X \rightarrow X$ be a linear operator. If $\text{Range}(A)$ is finite dimensional, show that $\sigma_p(A)$ is a finite set.
ii) Show that the spectrum $\sigma(T)$ of a bounded linear operator $T: X \rightarrow X$ on a Banach space X lies in the disk $|\lambda| \leq \|T\|$.
- c. i) Prove that the right shift operator $T: l^2 \rightarrow l^2$ defined by $T(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ has no eigenvalue
ii) Show that the spectrum $\sigma(T)$ of a bounded linear operator is compact.
