Roll No:

The Assam Royal Global University, Guwahati

Royal School of Applied & Pure Sciences

M.Sc. Mathematics, 2nd Semester

Special Supplementary Examination, September 2023

Course Title: Functional Analysis

Course Code: MAT014C203

Time: 3 Hours **Maximum Marks: 70**

Note: Attempt all questions as per instructions given.

The figures in the right-hand margin indicate marks.

Section – A

1. Attempt **all questions**. (Maximum word limit 50) **2 x 8**

a. What do you mean by distance of a point from a set?

b. If T is a bounded linear operator on a normed space X, show that $\|T\| = \sup_{\|x\|=1} \|Tx\|$.

c. If T^* is Hilbert adjoint of the operator T. Show that $(ST)^* = T^*S^*$

d. Write the statement of Cauchy-Schwarz inequality.

e. Define maximal and minimal element in a partially ordered set.

f. Write the statement of closed graph theorem.

g. Show that the right shift operator S defined by $S(x_0, x_1, \dots) = (0, x_0, x_1, \dots)$ is bounded.

h. Define contraction of a mapping in a metric space.

Section – B

2. Attempt **any two** of the following: **6 x 2**

a. Show that the normed linear space $c[0,1] = \{f: [0,1] \to R | f is\ continuous\}$ is not complete.

b. Show that $B(X, Y)$:set of all bounded linear operators from X to Y, is a normed linear space with norm $\|T\| = \sup_{\|x\|=1} \|Tx\|$.

c. Let X be a normed space and Y be a proper closed subspace of X. Show that for every $r \in (0,1)$ there exists $x_r \in X$ such that $\|x_r\| = 1$ and $r \leq d(x_r, Y) \leq 1$.

P.T.O.

3. Attempt **any two** of the following: **7 x 2**

a. i) The space $C[a,b]$ is not an inner product space.

ii) Let X and Y be inner product spaces and $T: X \to Y$ a bounded linear operator. Show that $T = O$ if and only if $\langle Tx, y \rangle = 0$ for all $x \in X$ and $y \in Y$.

b. State and prove Bessel's Inequality in an inner product space.

c. i) If M and N are closed subspaces of a Hilbert space H such that $M \perp N$, show that subspace $M + N$ is closed.

ii) If M is a subset of a Hilbert space H, show that $M \cap M^{\perp} = \{0\}$.

4. Attempt **any two** of the following: **7 x 2**

a. i) Let X be a normed linear space and $x_0 \in X, x_0 \neq 0$. Show that there exists a bounded linear functional $\tilde{f}$ on X such that $\tilde{f}(x_0) = \|x_0\|$ *and* $\|\tilde{f}\| = 1$.

ii) Using Uniform Boundedness Theorem, show that the normed space X of all polynomials with norm defined by

$$\|x\| = \max_{j} |\alpha_j| \ (\alpha_0, \alpha_1, \ldots \text{the coefficients of } x) \text{ is not complete.}$$

b. i) Show that Uniform boundedness theorem does not hold in an incomplete norm space.

ii) Show that weak convergence does not imply strong convergence in general.

c. i) Show that T is a closed linear transformation if and only if its graph T_G is a closed subspace.

ii) State and prove normed space version of Hahn-Banach Theorem.

5. Attempt **any two** of the following: **7x 2**

a. Consider a metric space $X = (X, d)$, where $X \neq \emptyset$. Suppose that X is complete and let $T: X \to X$ be a contraction on X. Show that T has one fixed point.

b. i) Let X be a linear space and $A: X \to X$ be a linear operator. If Range(A) is finite dimensional, show that $\sigma_p(A)$ is a finite set.

ii) Show that the spectrum $\sigma(T)$ of a bounded linear operator $T: X \to X$ on a Banach space X lies in the disk $|\lambda| \leq \|T\|$..

c. i) Prove that the right shift operator $T: l^2 \to l^2$ defined by $T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$ has no eigenvalue

ii) Show that the spectrum $\sigma(T)$ of a bounded linear operator is compact.

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