Chapter 6
Intuitionistic fuzzy aspects of multiplication N-groups

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Intuitionistic fuzzy aspects of multiplication N-groups

6.1 Introduction

Lee, Park and Kim [87] researched and developed the notion of fuzzy multiplication rings. The notion of fuzzy multiplication modules was established by Lee and Park [88]. They also investigated the product of fuzzy submodules and fuzzy prime submodules. Sharma [89] also studied IF multiplication modules over a commutative ring with non-zero identity and derived the relationship between the multiplication module and the intuitionistic fuzzy multiplication module. This idea helps to carry out the work of this chapter.

Different aspects of multiplication N-groups are studied in **Chapter 4**. The relationships between DN-groups and multiplication N-groups are also established. In this chapter, these concepts have been extended to intuitionistic fuzzy aspects.

The basic concepts discussed in this chapter are found in references [5, 7]. Initiating fuzzy points, characteristic functions and multiplication N-groups of intuitionistic fuzzy sets, their several findings were elaborated. This chapter also establishes an important relationship between intuitionistic fuzzy multiplication N-group and intuitionistic fuzzy DN-group.

6.2 Prerequisite

Definition 6.2.1. *An IF point $d_{(\gamma,\lambda)}$ of a nonempty set K, where $d \in K$, is predicted as $\{d_{(\gamma,\lambda)}\} =< \phi_{d_{(\gamma,\lambda)}}, \psi_{d_{(\gamma,\lambda)}} >$, where*

$$\phi_{d_{(\gamma,\lambda)}}(h) = \begin{cases} \gamma, & \text{if } h = d \\ 0, & \text{otherwise} \end{cases} \quad \text{and } \psi_{d_{(\gamma,\lambda)}}(h) = \begin{cases} \lambda, & \text{if } h = d \\ 1, & \text{otherwise.} \end{cases}$$

Note that if for any IF set A, $\{d_{(\gamma,\lambda)}\} \subseteq A$, then it is predicted as $d_{(\gamma,\lambda)} \in A$.

Definition 6.2.2. *If $Y \subseteq X$ (non empty), then the characteristic function of Y is an IF set on X and defined by ${}_\chi Y =< \phi_{\chi Y}, \psi_{\chi Y} >$, where*

$$\phi_{\chi Y}(s) = \begin{cases} 1, & \text{if } s \in Y \\ 0, & \text{otherwise} \end{cases} \quad \text{and } \psi_{\chi Y}(s) = \begin{cases} 0, & \text{if } s \in Y \\ 1, & \text{otherwise.} \end{cases}$$

Definition 6.2.3. *Let K, B, C be IF sets on E and C be a IF set on N. Then*
$(K : B) = \{D : D$ is IF set on N such that $DB \subseteq K\}$ i.e $(K : B) =< \phi_{(K:B)}, \psi_{(K:B)} >$, where $\phi_{(K:B)}(n) = \{\phi_D(n) : D$ is an IF set on N such that $DB \subseteq K\}$ and $\psi_{(K:B)}(n) = \{\psi_D(n) : D$ is an IF set on N such that $DB \subseteq K\}$
$(K : C) = \{F : F$ is IF set on E such that $CF \subseteq K\}$ i.e $(K : C) =< \phi_{(K:C)}, \psi_{(K:C)} >$, where $\phi_{(K:C)}(n) = \{\phi_F(n) : F$ is an IF set on E such that $CF \subseteq K\}$ and $\psi_{(K:C)}(n) = \{\psi_F(n) : F$ is an IF set on E such that $CF \subseteq K\}$.
If $K \leq_{IFN} E$, then $(K :_\chi E) = \{D : D \leq_{IFN} N$ such that $D_\chi E \subseteq K\}$.

Lemma 6.2.1. *Let Z, K be IF sets on E and C be a IF set on N. Then*

(i)$(Z:K)K \subseteq Z$.

(ii) $C(Z:C) \subseteq Z$.

(iii) $CK \subseteq Z \Leftrightarrow C \subseteq (Z:K) \Leftrightarrow K \subseteq (Z:C)$.

Proof : (i) Let $Z = <\phi_Z, \psi_Z>, K = <\phi_K, \psi_K>$ be IF sets on E and $C = <\phi_C, \psi_C>$ be IF set on N.

Then $(Z:K)K = <\phi_{(Z:K)K}, \psi_{(Z:K)K}>$, where

$\phi_{(Z:K)K}(x) = \vee\{\phi_{(Z:K)}(n) \wedge \phi_K(y) : x = ny, n \in N, y \in E\}$ and $\psi_{(Z:K)K}(x) = \wedge\{\psi_{(Z:K)}(n) \vee \psi_K(y) : x = ny, n \in N, y \in E\}$.

But $\phi_{(Z:K)}(n) = \{\phi_D(n) : D$ is an IF set on N such that $DK \subseteq Z\}$.

Therefore, $\phi_{(Z:K)K}(x)$

$= \vee\{\phi_D(n) \wedge \phi_K(y) : D$ is an IF set on N such that $DK \subseteq Z, x = ny, n \in N, y \in E\}$

$\leq \vee\{\phi_{DK}(x) : D$ is a IF set on N such that $DK \subseteq Z\}$

$\leq \phi_Z(x), \forall x \in E$.

Similarly, $\psi_{(Z:K)K}(x) \geq \psi_Z(x)$.

Thus, $(Z:K)K \subseteq Z$.

(ii) We have, $C(Z:C) = <\phi_{C(Z:C)}, \psi_{C(Z:C)}>$,

where $\phi_{C(Z:C)}(x) = \vee\{\phi_C(n) \wedge \phi_{(Z:C)}(y) : x = ny, n \in N, y \in E\}$ and $\psi_C(x) = \wedge\{\psi_C(n) \vee \psi_{(Z:C)}(y) : x = ny, n \in N, y \in E\}$.

But $\phi_{(Z:C)}(n) = \{\phi_D(n) : D$ is an IF set on E such that $CD \subseteq Z\}$.

Therefore, $\phi_{C(Z:C)}(x)$

$= \vee\{\phi_C(n) \wedge \phi_D(y) : D$ is an IF set on E such that $CD \subseteq Z, x = ny, n \in N, y \in E\}$

$\leq \vee\{\phi_{CD}(x) : D$ is a IF set on E such that $CD \subseteq Z\}$

$\leq \phi_Z(x)$.

Similarly, $\psi_{C(Z:C)}(x) \geq \psi_Z$.

Thus, $C(Z:C) \subseteq Z$.

(iii) It is clear from the definition.

6.3 Intuitionistic Fuzzy Multiplication N-groups

Definition 6.3.1. *An N-group E is called an IF multiplication N-group if and only if for each $A \leq_{IFN} E$, $\exists\, C \lhd_{IF} N$ such that $A = C._{\chi}E$. We denote it by $A = C_{\chi}E$.*

Lemma 6.3.1. *If $Z =< \phi_Z, \psi_Z > \leq_{IFN} E$, then $(Z :_{\chi} E) \lhd_{IF} N$.*

Proof : Let $Z =< \phi_Z, \psi_Z > \leq_{IFN} E$.

Then, for any $u, h, n \in N$,

$\phi_{(Z:_{\chi}E)}(u-h) = \{\phi_D(u-h) : D \leq_{IFN} N$ such that $D_{\chi}E \subseteq Z\} \geq \{\phi_D(u) \wedge \phi_D(h) : D \leq_{IFN} N$ such that $D_{\chi}E \subseteq Z\} = \phi_{(Z:_{\chi}E)}(u) \wedge \phi_{(Z:_{\chi}E)}(h)$.

Therefore, $\phi_{(Z:_{\chi}E)}(u-h) \geq \phi_{(Z:_{\chi}E)}(u) \wedge \phi_{(Z:_{\chi}E)}(h)$.

Similarly, $\psi_{(Z:_{\chi}E)}(u-h) \leq \psi_{(Z:_{\chi}E)}(u) \wedge \psi_{(Z:_{\chi}E)}(h)$.

Now, $\phi_{(Z:_{\chi}E)}(nu) = \{\phi_D(nu) : D \leq_{IFN} N$ such that $D_{\chi}E \subseteq Z\} \geq \{\phi_D(u) : D \leq_{IFN} N$ such that $D_{\chi}E \subseteq Z\} = \phi_{(Z:_{\chi}E)}(u)$.

Therefore, $\phi_{(Z:_{\chi}E)}(nu) \geq \phi_{(Z:_{\chi}E)}(u)$.

Similarly, $\psi_{(Z:_{\chi}E)}(nu) \leq \psi_{(Z:_{\chi}E)}(u)$.

Since N is commutative, $h+u-h = u$ and so $\phi_{(Z:_{\chi}E)}(h+u-h) = \phi_{(Z:_{\chi}E)}(u)$ and $\psi_{(Z:_{\chi}E)}(h+u-h) = \psi_{(Z:_{\chi}E)}(u)$.

Again, since N is commutative, $n(u+h)-nu = nh$ and so $\phi_{(Z:_{\chi}E)}(n(u+h)-nu = nh) \geq \phi_{(Z:_{\chi}E)}(nh) \geq \phi_{(Z:_{\chi}E)}(h)$.

Similarly, $\psi_{(Z:_{\chi}E)}(n(u+h)-nu = nh) \leq \psi_{(Z:_{\chi}E)}(nh) \leq \psi_{(Z:_{\chi}E)}(h)$.

Thus the result.

Theorem 6.3.1. *E is an IF multiplication N-group if and only if for each $u \in E$ $\exists$ an IF ideal C of N such that $\{u_{(\gamma,\lambda)}\} = C_{\chi}E$.*

Proof : Let us suppose, for each $u \in E$ $\exists$ an IF ideal C of N such that $\{u_{(\gamma,\lambda)}\} = C_{\chi}E$.

Let $A =< \phi_A, \psi_A > \leq_{IFN} E$.

Choose $\gamma, \lambda \in [0,1]$ such that $\gamma + \lambda \leq 1$ with $\phi_A(u) = \gamma, \psi_A(u) = \lambda$.

Now, for any $u \in E$,

$u_{(\gamma,\lambda)}(u)$

$=< \phi_{u_{(\gamma,\lambda)}}(u), \psi_{u_{(\gamma,\lambda)}}(u) >$

$=< \gamma, \lambda >$

$=< \phi_A(u), \psi_A(u) >$

$= A(u)$.

Therefore, $\{u_{(\gamma,\lambda)}\} = A$

$\Rightarrow \{u_{(\gamma,\lambda)}\} \subseteq A$

$\Rightarrow C_\chi E \subseteq A$

$\Rightarrow C \subseteq (A :_\chi E)$ [using **lemma 6.2.1**] .

Also, $\phi_A(u)$

$= \gamma$

$= \phi_{u_{(\gamma,\lambda)}}(u)$

$= \phi_{C_\chi E}(u)$

$= \vee\{\phi_C(n) \wedge \phi_{\chi E}(u') : n \in N, u' \in E, u = nu'\}$

$\leq \vee\{\phi_{(A:_\chi E)}(n) \wedge \phi_{\chi E}(u') : n \in N, u' \in E, u = nu'\}$

$= \vee\{\phi_{(A:_\chi E)E}(nu') : n \in N, u' \in E, u = nu'\}$

$= \{\phi_{(A:_\chi E)_\chi E}(u)\}$.

Therefore, $\phi_A(u) \leq \{\phi_{(A:_\chi E)_\chi E}(u)\}$, for all $u \in E$.

Similarly, $\psi_A(u) \geq \{\psi_{(A:_\chi E)_\chi E}(u)\}$, for all $u \in E$.

Therefore, $A \subseteq (A :_\chi E)_\chi E$.

But by **lemma 6.2.1**, $(A :_\chi E)_\chi E \subseteq A$.

Therefore, $A = (A :_\chi E)_\chi E$.

Also, by **lemma 6.3.1**, $(A :_\chi E)$ is an IF ideal of N.

Thus E is an IF multiplication N-group.

Conversely, let E be an IF multiplication N-group.

Let $A =< \phi_A, \psi_A > \leq_{IFN} E$ and $u \in E$ and $\gamma, \lambda \in [0,1]$ such that $\gamma + \lambda \leq 1$ with $\phi_A(u) = \gamma, \psi_A(u) = \lambda$.

Since E is multiplication N-group, $\exists$ IF ideal C of N such that $A = C_\chi E$.

We have, $\{u_{(\gamma,\lambda)}\} = A$.

Thus, $\{u_{(\gamma,\lambda)}\} = C_\chi E$.

Proposition 6.3.1. *If* $A =< \phi_A, \psi_A >$ *is an IF set on E, then* $(A :_\chi E) = \{z_{(\gamma,\lambda)} : z \in ({}^{(\gamma,\lambda)}A : E), z \in N\}$.

Proof : Let $z \in N$ and D be IF set on N.

We can choose $\gamma, \lambda \in [0,1], \gamma + \lambda \leq 1$ with $\phi_D(z) = \gamma, \psi_D(z) = \lambda$.

Then $\phi_{z_{(\gamma,\lambda)}}(z) = \gamma = \phi_D(z), \psi_{z_{(\gamma,\lambda)}}(z) = \lambda = \psi_D(z)$.

Therefore, $\{z_{(\gamma,\lambda)}\} = D$.

Let $D_\chi E \subseteq A$

$\Rightarrow D \subseteq (A :_\chi E)$[using **lemma 6.2.1**]

$\Rightarrow \{z_{(\gamma,\lambda)}\} \subseteq (A :_\chi E)$

$\Rightarrow \{z_{(\gamma,\lambda)}\}_\chi E \subseteq A$.

Again, let $\{z_{(\gamma,\lambda)}\}_\chi E \subseteq A$

$\Rightarrow D_\chi E \subseteq A$.

Therefore, $\{z_{(\gamma,\lambda)}\}_\chi E \subseteq A \Leftrightarrow D_\chi E \subseteq A$.

So, $\{D : D$ is IF set on N such that $D_\chi E \subseteq A\} = \{z_{(\gamma,\lambda)} : z \in N, \{z_{(\gamma,\lambda)}\}_\chi E \subseteq A\}$.

Thus, $(A :_\chi E) = \{z_{(\gamma,\lambda)} : z \in N, \{z_{(\gamma,\lambda)}\}_\chi E \subseteq A\}$.

Now, for each $u \in E$,

$$\phi_{z_{(\gamma,\lambda)\chi}E}(u) = \begin{cases} \vee\{\phi_{z_{(\gamma,\lambda)}}(z) \wedge \phi_{\chi E}(u')\}, & u = zu', u' \in E \\ 0, & otherwise. \end{cases}$$

Since $\phi_{z_{(\gamma,\lambda)}}(z) = \gamma$ and $\phi_{\chi E}(u') = 1$, therefore

$$\phi_{z_{(\gamma,\lambda)\chi}E}(u) == \begin{cases} \vee\{\gamma \wedge 1\}, & u = zu', u' \in E \\ 0, & otherwise \end{cases} = \begin{cases} \gamma, & u = zu', u' \in E \\ 0, & otherwise. \end{cases}$$

Similarly, $\psi_{z_{(\gamma,\lambda)\chi}E}(u) = \begin{cases} \lambda, & u = zu', u' \in E \\ 1, & otherwise. \end{cases}$

Now, $\{z_{(\gamma,\lambda)}\}_\chi E \subseteq A$

$\Rightarrow \phi_{z_{(\gamma,\lambda)\chi}E}(u) \leq \phi_A(u)$ and $\psi_{z_{(\gamma,\lambda)\chi}E}(u) \geq \psi_A(u)$, for $u \in E$

$\Rightarrow \phi_A(zu') \geq \gamma$ and $\psi_A(zu') \leq \lambda$, for $z \in N, u' \in E$.

Therefore, $(A :_\chi E)$

$= \{z_{(\gamma,\lambda)} : z \in N, \phi_A(zu') \geq \gamma$ and $\psi_A(zu') \leq \lambda, u' \in E\}$

$= \{z_{(\gamma,\lambda)} : z \in N, zu' \in {}^{(\gamma,\lambda)}A, u' \in E\}$

$= \{z_{(\gamma,\lambda)} : z \in N, zE \subseteq {}^{(\gamma,\lambda)}A\}$

$= \{z_{(\gamma,\lambda)} : z \in N, z \in ({}^{(\gamma,\lambda)}A : E)\}$.

Lemma 6.3.2. *If $z \in E$, then $z_{(\gamma,\lambda)} \in_\chi E$.*

Proof : For $y \in E$, $\{z_{(\gamma,\lambda)}\} = < \phi_{z_{(\gamma,\lambda)}}, \psi_{z_{(\gamma,\lambda)}} >$,

where $\phi_{z_{(\gamma,\lambda)}}(y) = \begin{cases} \gamma, & \text{if } y = z \\ 0, & otherwise \end{cases}$ and $\psi_{z_{(\gamma,\lambda)}}(y) = \begin{cases} \lambda, & \text{if } y = z \\ 1, & otherwise. \end{cases}$

Therefore, $z_{(\gamma,\lambda)}(y) = \begin{cases} < \gamma, \lambda >, & \text{if } y = z \\ < 0,1 >, & otherwise \end{cases}$ and $\chi E(y) = \begin{cases} < 1,0 >, & \text{if } y \in E \\ < 0,1 >, & otherwise. \end{cases}$

Since $0 \leq \gamma, \lambda \leq 1$, $\{z_{(\gamma,\lambda)}\} \subseteq_\chi E$ and so $z_{(\gamma,\lambda)} \in_\chi E$.

Lemma 6.3.3. *If $A \leq_N E$, then $(A : E) \lhd N$.*

Proof : Since $(A : E) = \{u \in N\colon uE \subseteq A\}$, $(A : E) \subseteq N$.

Now, $u_1, u_2 \in (A : E)$ and $u \in N$

$\Rightarrow u_1E \subseteq A, u_2E \subseteq A$.

Now, for any $e \in E$,

$(u_1 - u_2)e = u_1e - u_2e$.

Since $A \leq_N E$, $u_1 - u_2 \in A$.

Therefore, $(u_1 - u_2)e \in A$

$\Rightarrow (u_1 - u_2)E \subseteq A$

$\Rightarrow (u_1 - u_2) \in (A : E)$.

Since N is commutative $(uu_1)e = (u_1u)e = u_1(ue) \in u_1E \subseteq A$[since $ue \in E$].

Therefore, $(uu_1)E \subseteq A$

$\Rightarrow uu_1 \in (A : E)$.

This proves the result.

Proposition 6.3.2. *E is a multiplication N-group if and only if every $Z \leq_N E$ is structured like $Z = (Z : E)E$.*

Proof : Let $n \in (Z : E)$.

Then $nE \subseteq Z$ and $n \in N$

$\Rightarrow (Z:E)E \subseteq Z$.

Since E is a multiplication N-subgroup, $Z = IE$, for some $I \lhd N$.

Now, $IE = Z$

$\Rightarrow IE \subseteq Z$

$\Rightarrow I \subseteq (Z:E)$.

Again, $(Z:E) \subseteq N$ and $Z \subseteq IE$

$\Rightarrow Z \subseteq (Z:E)E$.

Therefore, $Z = (Z:E)E$.

Conversely, let $Z = (Z:E)E$.

Since by **lemma 6.3.3**, $(Z:E) \lhd N$, therefore Z is multiplication N-subgroup.

Proposition 6.3.3. *If E is a multiplication N-group, then for every $K = <\phi_K, \psi_K>$ $\leq_{IFN} E$, ${}^{(\gamma,\lambda)}K = ({}^{(\gamma,\lambda)}K : E)E$.*

Proof : Since $K = <\phi_K, \psi_K> \leq_{IFN} E$, by **proposition 2.5.1**, ${}^{(\gamma,\lambda)}K \leq_N E$.

Since E is multiplication, ${}^{(\gamma,\lambda)}K = ({}^{(\gamma,\lambda)}K : E)E$.

Lemma 6.3.4. *Given a non-empty set K, if $z_{(\gamma,\lambda)} \in K$, then $z \in {}^{(\gamma,\lambda)}K$.*

Proof : $z_{(\gamma,\lambda)} \in K$

$\Rightarrow \{z_{(\gamma,\lambda)}\} \subseteq K$.

So, $\phi_{z_{(\gamma,\lambda)}} \leq \phi_K, \psi_{z_{(\gamma,\lambda)}} \geq \psi_K$.

Therefore, $\phi_K(z) \geq \phi_{z_{(\gamma,\lambda)}}(z) = \gamma$ and $\psi_K(z) \leq \psi_{z_{(\gamma,\lambda)}}(z) = \lambda$.

Thus, $z \in {}^{(\gamma,\lambda)}K$.

Lemma 6.3.5. *If $u \in E, s \in N$, then $(su)_{(\gamma,\lambda)} = s_{(\gamma,\lambda)}u_{(\gamma,\lambda)}$.*

Proof : For any $l \in E$,

$\{(su)_{(\gamma,\lambda)}\}(l)$

$= <\phi_{(su)_{(\gamma,\lambda)}}(l), \psi_{(su)_{(\gamma,\lambda)}}(l)>$

$$= \begin{cases} <\gamma,\lambda>, & if\, l = su \\ <0,1>, & otherwise \end{cases}$$

and $\{s_{(\gamma,\lambda)}u_{(\gamma,\lambda)}\}(l)$

$=< \phi_{s_{(\gamma,\lambda)}u_{(\gamma,\lambda)}}(l), \psi_{s_{(\gamma,\lambda)}u_{(\gamma,\lambda)}}(l) >$.

Now, $\phi_{s_{(\gamma,\lambda)}u_{(\gamma,\lambda)}}(l) = \vee\{\phi_{s_{(\gamma,\lambda)}(s')} \wedge \psi_{u_{(\gamma,\lambda)}(u')}, l = s'u', s' \in N, u' \in E\}$.

If $s = s', u = u'$, then $\phi_{s_{(\gamma,\lambda)}u_{(\gamma,\lambda)}}(l) = \gamma$.

Similarly, if $l = su$ then $\psi_{s_{(\gamma,\lambda)}u_{(\gamma,\lambda)}}(l) = \lambda$.

Again if $s \neq s', l \neq l'$ then $\phi_{s_{(\gamma,\lambda)}u_{(\gamma,\lambda)}}(l) = 0$ and $\psi_{s_{(\gamma,\lambda)}u_{(\gamma,\lambda)}}(l) = 1$.

Therefore, $\{s_{(\gamma,\lambda)}u_{(\gamma,\lambda)}\}(l) = \begin{cases} < \gamma, \lambda >, & if l = su \\ < 0, 1 >, & otherwise. \end{cases}$

Therefore, $(su)_{(\gamma,\lambda)} = s_{(\gamma,\lambda)}u_{(\gamma,\lambda)}$.

Lemma 6.3.6. *If $B \leq_N E$, then $_\chi B \leq_{IFN} E$.*

Proof : Let $u, z \in E$ and $n \in N$.

We have, $_\chi B =< \phi_{\chi B}, \psi_{\chi B} >$.

If $u, z \in B$, then $u - z \in B$[since B is subgroup of $(E, +)$] .

So, $\phi_{\chi B}(u) = 1, \phi_{\chi B}(z) = 1, \phi_{\chi B}(u - z) = 1$.

Therefore, $\phi_{\chi B}(u - z) = 1 \wedge 1 = \phi_{\chi B}(u) \wedge \phi_{\chi B}(z)$.

If $u, z \notin B$, then either $u - z \in B$ or $u - z \notin B$.

If $u - z \in B$, then $\phi_{\chi B}(u) = 0, \phi_{\chi B}(z) = 0, \phi_{\chi B}(u - z) = 1$ and so $\phi_{\chi B}(u - z) > \phi_{\chi B}(u) \wedge \phi_{\chi B}(z)$.

If $u - z \notin B$, then $\phi_{\chi B}(u) = 0, \phi_{\chi B}(z) = 0, \phi_{\chi B}(u - z) = 0$.

So,$\phi_{\chi B}(u - z) = 0 \wedge 0 = \phi_{\chi B}(u) \wedge \phi_{\chi B}(z)$.

If $u \in B$ but $z \notin B$, then $u - z \notin B$ and so $\phi_{\chi B}(u) = 1, \phi_{\chi B}(z) = 0, \phi_{\chi B}(u - z) = 0$.

So, $\phi_{\chi B}(u - z) = 1 \wedge 0 = \phi_{\chi B}(u) \wedge \phi_{\chi B}(z)$.

Again, if $u \notin B$ but $z \in B$, then $u - z \notin B$ and so $\phi_{\chi B}(u) = 0, \phi_{\chi B}(z) = 1, \phi_{\chi B}(u - z) = 0$.

Therefore, $\phi_{\chi B}(u - z) = 0 \wedge 1 = \phi_{\chi B}(u) \wedge \phi_{\chi B}(z)$.

ans so, $\phi_{\chi B}(u - z) \geq \phi_{\chi B}(u) \wedge \phi_{\chi B}(z)$, for $u, z \in E$.

Similarly, $\psi_{\chi B}(u - z) \leq \psi_{\chi B}(u) \vee \psi_{\chi B}(z)$, for $u, z \in E$.

Now, if $u \in B$, then $nu \in B$ and so $\phi_{\chi B}(u) = 1, \phi_{\chi B}(nu) = 1$.

Therefore, $\phi_{\chi B}(nu) = \phi_{\chi B}(u)$, if $u \in B$.

Also, if $u \notin B$, then either $nu \in B$ or $nu \notin B$.

So, if $u \notin B$ and $nu \in B$, then

$\phi_{\chi B}(u) = 0, \phi_{\chi B}(nu) = 1.$

Therefore, $\phi_{\chi B}(nu) > \phi_{\chi B}(u)$

and if $u \notin B$ and $nu \notin B$, then

$\phi_{\chi B}(u) = 0, \phi_{\chi B}(nu) = 0.$

Therefore, $\phi_{\chi B}(nu) = \phi_{\chi B}(u)$.

Thus, $\phi_{\chi B}(nu) \geq \phi_{\chi B}(u)$, for $u \in E$.

Similarly, $\psi_{\chi B}(nu) \leq \psi_{\chi B}(u)$, for $u \in E$.

Hence the result.

Theorem 6.3.2. *E be an IF multiplication N-group if and only if for every $A \leq_{IFN} E$, $A = (A :_\chi E)_\chi E$.*

Proof : By **lemma 6.2.1**, $(A :_\chi E)_\chi E \subseteq A$.

So, it is sufficient to show that $A \subseteq (A :_\chi E)_\chi E$.

Since E is an IF multiplication N-group, $\exists$ an IF ideal C of N such that $A = C_\chi E$.

Now, $A = C_\chi E$

$\Rightarrow C_\chi E \subseteq A$

$\Rightarrow C \subseteq (A :_\chi E)$

$\Rightarrow C_\chi E \subseteq (A :_\chi E)_\chi E$

$\Rightarrow A \subseteq (A :_\chi E)_\chi E.$

Therefore, $A = (A :_\chi E)_\chi E$.

Conversely, suppose $A = (A :_\chi E)_\chi E$.

Since by **lemma 6.3.1**, $(A :_\chi E)$ is an IF ideal of N, by definition A is an IF multiplication N-group.

Theorem 6.3.3. *E is a multiplication N-group if and only if E is an IF multiplication N-group.*

Proof : Let E be a multiplication N-group and $A =< \phi_A, \psi_a > \leq_{IFN} E$.

By **lemma 6.2.1**, $(A :_\chi E)_\chi E \subseteq A$.

Since by **lemma 6.3.1**, $(A :_\chi E)$ is an IF ideal of N, it is sufficient to show that $A \subseteq (A :_\chi E)_\chi E$.

For $u \in E$, it can be choose $\gamma, \lambda \in [0,1], \gamma+\lambda \leq 1$ with $\phi_A(u) = \gamma, \psi_A(u) = \lambda$.

Then $u \in {}^{(\gamma,\lambda)}A$.

Since E is a multiplication N-group, by **proposition 6.3.3**,

${}^{(\gamma,\lambda)}A = ({}^{(\gamma,\lambda)}A : E)E$.

Therefore, $u = nu'$, for some $n \in ({}^{(\gamma,\lambda)}A : E), u' \in E$.

By **proposition 6.3.1**, $n \in ({}^{(\gamma,\lambda)}A : E) \Rightarrow n_{(\gamma,\lambda)} \in (A :_\chi E)$.

Since $u' \in E$, by **lemma 6.3.2**, $u'_{(\gamma,\lambda)} \in_\chi E$.

So, by **lemma 6.3.5**, $u_{(\gamma,\lambda)} = (nu')_{(\gamma,\lambda)} = n_{(\gamma,\lambda)}u'_{(\gamma,\lambda)}$

$\Rightarrow u_{(\gamma,\lambda)} \in (A :_\chi E)_\chi E$

$\Rightarrow u \in {}^{(\gamma,\lambda)}\{(A :_\chi E)_\chi E\}$[by **lemma 6.3.4**]

$\Rightarrow \phi_{(A:_\chi E)_\chi E}(u) \geq \gamma = \phi_A(u), \psi_{(A:_\chi E)_\chi E}(u) \leq \lambda = \psi_A(u)$.

Therefore, $A \subseteq (A :_\chi E)_\chi E$.

So, $A = (A :_\chi E)_\chi E$.

Thus E is an IF multiplication N-group.

Conversely, let E be an IF multiplication N-group.

Let $B \leq_N E$.

Then $(B : E)E \subseteq B$ by **lemma 2.5.1**.

To show $B \subseteq (B : E)E$.

Now, define an IF set P on E by,

$$\phi_P(x) = \begin{cases} 1, & \text{if } x \in B \\ 0, & otherwise \end{cases}$$

and $$\psi_P(x) = \begin{cases} 0, & \text{if } x \in B \\ 1, & otherwise. \end{cases}$$

Then $P =_\chi B$ and ${}^{(\gamma,\lambda)}P = B$ with $\gamma, \lambda \in (0,1], \gamma+\lambda \leq 1$.

By **lemma 6.3.6**, $P =_\chi B \leq_{IFN} E$.

Since E is an IF multiplication N-group, by **theorem 6.3.2**, $P = (P :_\chi E)_\chi E$.

Let $b \in B$.

Then $\phi_P(b) = \phi_{(P:_\chi E)_\chi E}(b) = 1$ and $\psi_P(b) = \psi_{(P:_\chi E)_\chi E}(b) = 0$ [by assumption of P] .

But $\phi_{(P:_\chi E)_\chi E}(b)$

$= \vee\{\phi_{(P:_\chi E)}(n') \wedge \phi_{\chi E}(u') : b = n'u'$, for some $n' \in N, u' \in E\}$

$= \vee\{\phi_{(P:_\chi E)}(n') : b = n'u'$, for some $n' \in N, u' \in E\}$[since $\phi_{\chi E}(u') = 1$]

$= \vee\{\phi_{n_{\gamma,\lambda}}(n') : nE \subseteq {}^{(\gamma,\lambda)}P = B, b = n'u'$, for some $n' \in N, u' \in E$ with $\gamma, \lambda \in (0,1], \gamma + \lambda \leq 1\}$ [by **proposition 6.3.1**]

$= \vee\{\phi_{n_{\gamma,\lambda}}(n') : n \in (B:E), b = n'u'$, for some $n' \in N, u' \in E$ with $\gamma, \lambda \in (0,1], \gamma + \lambda \leq 1\}$.

Similarly, $\psi_{(P:_\chi E)_\chi E}(b) = \wedge\{\psi_{n_{\gamma,\lambda}}(n') : n \in (B:E), b = n'u'$, for some $n' \in N, u' \in E$ with $\gamma, \lambda \in (0,1], \gamma + \lambda \leq 1\}$.

Let us consider $S = \{n : n \in (B:E), b \in nE\}$.

If S is empty, then for each $b \in tE$, $t \notin (B:E)$ when $t \in N$.

Then $\phi_{(P:_\chi E)_\chi E}(b) = \vee\{\phi_{n_{\gamma,\lambda}}(t) : n \in (B:E), b \in tE, t \in N$, with $\gamma, \lambda \in (0,1], \gamma + \lambda \leq 1\}$.

Since $n \in (B:E), t \notin (B:E), n \neq t$ and so $\phi_{n_{\gamma,\lambda}}(t) = 0$.

Therefore, $\phi_{(P:_\chi E)_\chi E}(b) = 0$.

Similarly, $\psi_{(P:_\chi E)_\chi E}(b) = 1$.

These are contradictions.

So, it can be conclude that S is non-empty.

Thus, $\exists\, n \in N$ such that $b \in nE$ and $n \in (B:E)$.

Therefore, $b \in nE \Rightarrow b \in (B:E)E$.

But $b \in B$.

Therefore, $B \subseteq (B:E)E$.

Thus, $B = (B:E)E$.

Hence E is a multiplication N-group.

Definition 6.3.2. *Let $A = <\phi_A, \psi_A> \leq_{IFN} E$, then ${}^{(\gamma,\lambda)}A \leq_N {}^{(\gamma,\lambda)}E$ if $m - y, nm \in {}^{(\gamma,\lambda)}A$, for any $m, y \in {}^{(\gamma,\lambda)}A$ and $n \in N$.*

Theorem 6.3.4. *An IF multiplication N-group is an IF DN-group.*

Proof : Let $F, K, C \leq_{IFN} E$.

Since E is an IF multiplication N-group,

$F = (F :_\chi E)_\chi E, K = (K :_\chi E)_\chi E, C = (C :_\chi E)_\chi E$.

Let $u \in E$.

Now, $\phi_F(u)$

$= \phi_{(F:_\chi E)_\chi E}(u)$

$= \vee\{\phi_{(F:_\chi E)}(n) \wedge \phi_{\chi E}(e) : u = ne, n \in N, e \in E\}$

$= \vee\{\phi_{(F:_\chi E)}(n) : u \in nE, n \in N\}$ [since $\phi_{\chi E}(e) = 1$].

Similarly, $\psi_F(u) = \wedge\{\psi_{(F:_\chi E)}(n) : u \in nE, n \in N\}$.

But by **proposition 6.3.1**, $(F :_\chi E) = \{n_{(\gamma,\lambda)} : \gamma, \lambda \in [0,1], \gamma + \lambda \leq 1, nE \subseteq^{(\gamma,\lambda)} F\}$.

Therefore, $\phi_F(u)$

$= \vee\{\phi_{n_{(\gamma,\lambda)}}(n) : u \in nE \subseteq^{(\gamma,\lambda)} F, n \in N\}$

$= \gamma$, where $u \in^{(\gamma,\lambda)} F$.

Similarly, $\psi_F(u) = \lambda$, where $u \in^{(\gamma,\lambda)} F$.

Now, define

$$\phi_F(u) = \begin{cases} \gamma, & u \in X \\ 0, & u \notin X \end{cases}, \quad \psi_F(u) = \begin{cases} \lambda, & u \in X \\ 1, & u \notin X \end{cases},$$

$$\phi_K(u) = \begin{cases} \gamma, & u \in Y \\ 0, & u \notin Y \end{cases}, \quad \psi_K(u) = \begin{cases} \lambda, & u \in Y \\ 1, & u \notin Y \end{cases},$$

$$\phi_C(u) = \begin{cases} \gamma, & u \in Z \\ 0, & u \notin Z \end{cases}, \quad \psi_C(u) = \begin{cases} \lambda, & u \in Z \\ 1, & u \notin Z \end{cases}$$

with $\gamma, \lambda \in (0,1]$.

Then, for $u \in X$, $\phi_F(u) = \gamma, \psi_F(u) = \lambda$ and so $u \in {}^{(\gamma,\lambda)}F$.

Also, if $u \in {}^{(\gamma,\lambda)}F$, then either $u \in X$ or $u \notin X$.

If $u \notin X$, then $\phi_F(u) = 0 \geq \gamma$ and $\psi_F(u) = 1 \leq \lambda$-which is a contradiction to the fact that $\gamma, \lambda \in (0,1]$.

Thus, ${}^{(\gamma,\lambda)}F = X$.

Similarly, ${}^{(\gamma,\lambda)}K = Y, {}^{(\gamma,\lambda)}C = Z$ with $\gamma, \lambda \in (0,1]$ and so X, Y, Z are subsets of E.

Now, for any $u \in X \cap Y$, $(F+K)(u) = < \phi_{F+K}(u), \psi_{F+K}(u) >$,

where $\phi_{F+K}(u) = \vee\{\phi_F(y) \wedge \phi_K(z) : y, z \in X \cap Y$ and $u = y + z \in X \cap Y\}$ and $\psi_{F+K}(u) = \wedge\{\psi_F(y) \vee \psi_K(z) : y, z \in X \cap Y$ and $u = y + z \in X \cap Y\}$.

Therefore, $\phi_{F+K}(u) = \gamma$ and $\psi_{F+K}(u) = \lambda$, where $u \in X \cap Y$[since $\phi_F(u) = \gamma$ and

$\psi_F(u) = \lambda$ for all $u \in X$ and $\phi_K(u) = \gamma$ and $\psi_K(u) = \lambda$ for all $u \in Y$] .

Thus, $(F+K)(u) =< \gamma, \lambda >$, where $u \in X \cap Y$.

Also, $(F \cap K)(u) =< \phi_F(u) \wedge \phi_K(u), \psi_F(u) \vee \psi_K(u) >=< \gamma, \lambda >$, if $u \in X \cap Y$.

If $u \in Z$, then $u \in X \cap Y \cap Z$ and $((F+K) \cap C)(u) =< \gamma, \lambda > \cap < \gamma, \lambda >=< \gamma, \lambda >$.

If $u \notin Z$, then $u \notin X \cap Y \cap Z$ and $((F+K) \cap C)(u) =< \gamma, \lambda > \cap < 0, 1 >=< 0, 1 >$.

Again, $((F \cap C) + (K \cap C))(u) =< \gamma, \lambda > + < \gamma, \lambda >=< \gamma, \lambda >$, where $u \in X \cap Y \cap Z$[since $(F+F)(u) = F(u)$ for all $u \in X \subseteq E$] .

If $u \notin Z$ and $u \in X \cap Y$, then $u \notin X \cap Y \cap Z$ and $((F \cap C) + (K \cap C))(u) =< \phi_F(u) \wedge \phi_C(u), \psi_F(u) \vee \psi_C(u) > + < \phi_K(u) \wedge \phi_C(u), \psi_K(u) \vee \psi_C(u) >=< 0, 1 > + < 0, 1 >=< 0, 1 >$.

So, it can be conclude that $((F+K) \cap C)(u) = ((F \cap C) + (K \cap C))(u)$, for all $u \in E$.

Thus, $(F+K) \cap C = (F \cap C) + (K \cap C)$ and hence E is an IF DN-group.

6.4 Conclusion

The aim of this chapter is to extend the notions of multiplication near-ring groups to intuitionistic fuzzy multiplication near-ring groups. In section 6.2, some prerequisites of intuitionistic fuzzy multiplication N-groups are discussed. The **theorem** 6.3.3 draws the relationship between IF multiplication N-group and multiplication N-group. Also, the **theorem** 6.3.4 illustrates the linkage between IF multiplication N-group and IF DN-group.