1

General Introduction

1.1 Prelude

A near-ring is a structure of algebra that can carry out two binary operations, addition and multiplication, which obey all ring structure postulates except addition commutativity and one of the distributive laws. The collection of all maps of a group into itself is a natural example of a near-ring, with addition specified point-wise and multiplication represented as map composition. Dickson [10] made the first contribution to near-rings through axiomatic study. Since 1930, the near-ring theory has been thoroughly researched and it is today a well-established theory. The study and research on near-rings are now being expanded in a very methodical manner.

The near-ring theory is well established not just by its significant theory but also by its various applications. Planar near-rings, for example, are recognized in computer science theory to provide good balanced incomplete block designs, which in turn produce very efficient statistical experiment designs. It has a wide range of applications, including combinatorial problems, interpolation theory, polynomials, matrices and graph theory (see [11, 12, 13, 14]).

Fittings [15], Zassenhaus [16], Taussky [17] and Wielandt [18] had done substantial work in near-ring theory from 1930 forward. Later, Blackett [19], Frohlich [20], Laxton [21], Beidleman [22], Malone [23], Oswald [24], Maxson [25], Meldrum [26]and others performed significant studies on various areas of near-ring theory. Clay and Pliz have made an elegant contribution to near-ring theory with their books [27, 28]. Apart from the conference materials, there are many significant books available on near-rings.

To deal with uncertainty, Zadeh expanded Cantor's classical set theory to a new concept called fuzzy subset (in short, FS) in 1965. A fuzzy subset of a non-empty set X is a function from X to [0,1]. Since then, scholars have been increasingly using fuzzy subsets to extend many mathematical notions.

Atanassov [5] created a new concept called intuitionistic fuzzy (in short, IF) set in 1986 to deal with vagueness by offering a non-membership grade to the elements, which was confined to a membership grade solely in the case of fuzzy subsets. Since then, scholars have been drawing the notion to apply it to the study of diverse algebraic structures.

1.2 Literature Survey

Blackett [29] studied various properties of simple and semi-simple near-rings, such as "Every right ideal R of a semi-simple near-ring A contains an idempotent e such that $eR = R$.". Radicals of near-rings were investigated by Beidleman [30] and different results like " Let M be an R-module that is direct sum of minimal submodules. Then M is not a radical module such that $J(M) = 0$" are derived. Distributive near-rings are thoroughly described by Hearthly [31] and the relation between distributive near-ring and ideals of near-ring is established. Oswald [32] has studied various important results of near-ring in which each N-subgroup is principal. Mason[33] worked on strongly regular near-rings and derived the associations with regular near-rings. Near-rings and their embedded conditions with the group near-ring have been investigated by Meldrum[34]. Introducing essential ideals of N-groups, Reddy and Satyanarayana[35] proved that the intersection of the finite number of essential ideals is again essential and every ideal containing an essential ideal is essential. Choudhury, Saikia and Das [36, 2] worked on various results on quasi direct sum of N-groups and distributive N-groups. Defining valuation near-rings, Khodadadpour and Roodbarylor [3] established their related results. The concept of local near-rings is introduced by them and different associated results on it are analyzed. The above mentioned notions and results of modules have a great significance to pursue research on DN-groups.

"Multiplication ideals, multiplication rings and the ring $R[x]$" were surveyed by Anderson [37]. The condition for which a multiplication ideal in a ring is principal is established by Anderson. Mehdi and Singh [38] introduced the multiplication module and explored various important results on multiplication modules. Smith [39] established some remarkable results on multiplication modules such as an R-module M is a multiplication module if and only if $N = (N : M)M$ for all submodules N of M. Again, Ameri [40] analyzed significant results on the prime submodules of multiplication modules like "If B prime submodule of a module M, then $ann(M/B)$ is a prime ideal of a ring R". Atani and Khojasteh [41, 42] investigated various exceptional results on multiplication modules such as if R is a commutative ring and M is a multiplication R-module, then M is finitely generated. Here, the notions about modules have been extended to N-groups to investigate multiplication N-groups and their related results.

Liu [43] proposed the notion of fuzzy ring and ideal. Several researchers, including Marashdeh and Salleh [44] have attempted to construct intuitionistic fuzzy rings based on the concept of Liu . Banerjee and Basnet [45] explored IF subrings and ideals and several other scholars followed it. Jianming and Xueling [46] introduced and studied the notions of IF ideals of near-rings. Sharma [47] also reviewed the algebraic nature of it subsequently.

Intuitionistic fuzzy subgroups were explored by Biswas [48]. Hur, Jang and Kang [49] explained the intuitionistic fuzzy ideals of rings and their associated results. Also, various properties of intuitionistic fuzzy groups were studied by Sharma [50]. Ejegwa, Akubo and Joshua [51] got motivated towards the application of an intuitionistic fuzzy set in career determination via the normalized Euclidean distance method. The conditions of intuitionistic fuzzy ideals in a ring to be a prime were developed by Bakhadach, Melliani, Oukesson, and Chadli [52]. Residual questions and annihilators of intuitionistic fuzzy sets of rings and modules were broadly discussed by Sharma and Kaur [8]. By motivation to extend this concept in near ring theory, Saikia and Saikia [7] explained in detail the intuitionistic fuzzy N-subgroup and intuitionistic fuzzy ideals in near-rings. Also, Saikia and Barthakur [53] redefined N-subgroups of a near-ring. Recently, Geetha and Ramadoss [54] studied fuzzy algebra and fuzzy near-rings. Saikia and Barthakur [6] designed (T,S)-intuitionistic fuzzy N-subgroups of an N-group and established its corresponding results. The different substructures of intuitionistic fuzzy concepts mentioned above play an important role in studying intuitionistic fuzzy aspects of DN-groups and weak DN-groups.

The relationships between distributive and multiplication modules (rings) were established by Tuganbaev [55]. L-fuzzy multiplication modules, which are most useful in our work, were discussed by Atani and Saraei [56] and Nimbhorkar and Khubchandani [57]. The concepts of intuitionistic fuzzy modules by Xu [58] have been extended to intuitionistic fuzzy N-groups in our work. The above mentioned relationship works are also our motivation to deal with their intuitionistic fuzzy concepts.

1.3 Research objectives

The objectives of the research work are as follows:

1. To investigate uniserial and Bezout N-groups with examples and their relationships with distributive N-groups.

2. To study multiplication N-groups, localized conditions of N-groups, N-subgroups, ideals and multiplication N-groups and their relationships.

3. To extend the notion of distributive near-ring groups (DN-groups) to intuitionistic fuzzy DN-groups, intuitionistic fuzzy uniserial N-groups, intuitionistic fuzzy weak DN-groups and their related results.

4. To extend the concept of multiplication N-groups to intuitionistic fuzzy multiplication N-groups and the association between multiplication N-groups with intuitionistic fuzzy multiplication N-groups.

1.4 Organisation of the thesis

In **Chapter 1**, a prelude and an extensive literature survey are presented. It also discusses the objectives of the research. The organisation of the thesis has also been illustrated there.

Chapter 2 deals with the methods in terms of definition and pre-established results of near-ring theory, which is a pre-requisite for the research. The first section of the chapter describes some fundamental definitions and results on the near-rings and near-ring groups and their substructures.

In the second section, notions such as DN-groups, totally DN-groups and arithmetical near-rings are discussed. The third section elaborates fuzzy sets, fuzzy N-subgroups, and fuzzy left ideals. The last section of this chapter addresses intuitionistic fuzzy (IF) sets, operations on IF sets and related definitions and relations.

All the established definitions and results will be guidelines for extending the work in subsequent chapters.

In **Chapter 3**, an attempt has been made to extend the notions of the uniserial and Bezout modules to near-ring groups. This chapter is divided into two sections.

The first section discusses uniserial N-groups with examples and their association with DN-groups. It establishes some important results of DN-groups like for any sub-factor E' of E such that $Ns' \cap Nh' = 0', \forall s', h' \in E'$ and $1 = p+q, ps' = qh' = 0', \forall p, q \in N, E$ is DN-group. It also proved that if E is a PNG DN-group over the local N , then it is a uniserial N-group. The results of this section are needed in the subsequent sections and chapters.

The second part of this chapter is devoted to Bezout N-groups with suitable example and some important results. The result "E is a Bezout N-group if and only if $\forall$ $h, k \in E$, $\exists$ $s, q \in N$ such that h and k are generated by $(sh+qk)$“ is one of the important results in this section. The relationships between uniserial N-groups and Bezout N-groups are established by showing that if E is a Bezout PNG over the local N, then E is a Uniserial N-group. Moreover, defining semi-simple N-groups and small N-subgroups, some associated results are also derived. Finally, the relationships between DN-groups and Bezout N-groups are established, showing that if E is a DN-group over a strongly regular N, then E is a Bezout N-group.

The first part of **Chapter 4** describes the concepts of localized near-rings, N-groups, N-subgroups, ideal and localization of an N-group at a prime ideal. It is shown that for an N-group $E, E = 0$ if and only if $E_P = 0, \forall P \in Max(N)$, which is the association between N-group and localized N-group. The distributive conditions of N-groups and localized N-groups are also an interest of research in this section. It is shown that an ideal N-group E is a DN-group if and only if E_P is also a DN-group, $\forall P \in Max(N)$.

The next section deals with the multiplication N-groups together with examples. It is ensured that every cyclic localized N-group is a localized multiplication N-group. "A localized multiplication N-group is also a multiplication N-group" is another important result of this section. Finally, it is discussed that if E is the multiplication ideal N-group generated finitely and N is an arithmetical local near-ring, then E is a DN-group. The link between uniserial DN-groups and multiplication N-groups is also established in this chapter.

Chapter 5 consists of IF *DN*-groups and IF weak *DN*-groups. The first part outlines IF *DN*-groups and their essential outcomes. The basic definitions of *N*-subgroups related to (γ, λ)- cut are coined and their characteristics are studied. The subsequent section discusses the weak *DN*-group, IF weak *DN*-group and their associated findings. "If $(\gamma, \lambda)E$ is a weak *DN*-group with $\gamma + \lambda = 1$, then E is an IF weak *DN*-group" is the result linking between *DN*-group and IF weak *DN*-group .

Chapter 6 is about intuitionistic fuzzy aspects of multiplication *N*-groups. In this chapter, introducing IF fuzzy point and characteristic function of IF set, IF multiplication *N*-groups and its essential results are demonstrated. "E be an IF multiplication *N*-group if and only if for every $A \leq_{IFN} E$, $A = (A :_{\chi} E)_{\chi} E$." is one of the core results in this chapter. It is also established that a multiplication *N*-group is also an IF multiplication *N*-group and vice versa. Finally, it is ensured that an IF multiplication *N*-group is an IF *DN*-group.

Chapter 7 gives the conclusion and future scope of the work. Here, the summary of the outcomes of the research work has been discussed. Also, a direction for the extension of the work has been provided in the future scope of the work.