2

Preliminaries

2.1 Introduction

The fundamental concepts, such as definitions and findings of near-rings, N-groups, fuzzy sets and intuitionistic fuzzy sets, that are required for the entire research activity are presented in this chapter. This chapter is divided into four sections. In these sections, some pre-established definitions, propositions and lemmas that will be required for our research work have been discussed. Also, some of the important definitions and results that are needed for our investigation are also introduced here.

Several fundamental definitions of near-ring and near-ring groups, together with important results that are relevant to the study, are covered in the first section. The second section is devoted to fundamental concepts of distributive near-ring groups and arithmetic near-ring groups, both of which are essential for investigation. The third section deals with different key aspects of fuzzy sets. The fourth section presents the basic definitions and results of IF sets, which provides the basics for a new dimension of the research.

Most of the definitions related to near-ring are structured from [28]. The right near-ring N is considered commutative in chapters 5 and 6. In IF sets, $\wedge$ and $\vee$ denotes the maximum and minimum in the unit interval [0 1].

2.2 Near-rings and N-groups

The basics of near-rings, N-groups and various results are discuss in this section.

Definition 2.2.1. *[6] If the following standards are satisfied, a nonempty set N combined with the binary operations "+" and "." is referred to as the right near-ring-*

i. $(N,+)$ *is a group(not necessarily abelian).*

ii. $(N,.)$ *is a semi group.*

iii. $(p+b)c = pc+bc$, $\forall\ p,b,c \in N$.

Note that N is said to be zero-symmetric if $n0 = 0$ for all $n \in N$.

Definition 2.2.2. *[6] Let N be a near-ring. Then an additive group* $(E,+)$ *is referred to as a left N-group if* $\exists$ *a map* $N \times E \to E$ *such that* $(n,u) \to nu$ *in which the following conditions hold-*

i. $(m+n)u = mu+nu$.

ii. $(mn)u = m(nu)$, $\forall m,n \in N, u \in E$.

It is to be noted that N is itself an N-group over itself. If for $1 \in N$ *such that* $1.u = u$ $\forall\ u \in E$, *then E is called an unitary N-group.*

Throughout the work, the near-ring N is considered a zero symmetric right near-ring and E is a unitary left N-group.

Definition 2.2.3. *[28]* $e \in N$ *is called central if* $xe = ex, \forall x \in N$.

Definition 2.2.4. *[6] In the event that A is a subgroup of* $(E,+)$ *and* $NA \subseteq A$ *for any* $A \subseteq E$*, then A is referred to as an N-subgroup.*

Definition 2.2.5. *[6] If F is a normal subgroup of* $(E,+)$ *with* $na \in F$*,* $\forall\ n \in N, a \in F$*, then F is referred to be a normal N-subgroup of E.*

Definition 2.2.6. *[60] An ideal (right or left) M of N is called maximal (right or left) of N if for any ideal (right or left) L of N with* $M \subseteq L$ *then either* $M = L$ *or* $L = N$*.*

Definition 2.2.7. *If for any* $J \leq_N E$ *with* $I \subseteq J$ *implies* $J = I$ *or* $J = E$*, then* $I \leq_N E$ *is referred to be maximal.*

Definition 2.2.8. *If k or* $1-k$ *is invertible in N for any* $k \in N$ *or N has a unique maximal N-subgroup, then N is called local.*

Definition 2.2.9. *[60] N is called strongly regular if for any* $n \in N\ \exists\ m \in N$ *such that* $n = n^2m$ *or* mn^2*.*

Definition 2.2.10. *[60] Jacobson radical,* $J(N) = \cap\{I : I \in Max(N)\}$*.*

Definition 2.2.11. *[60] If* $\exists\ I, J \lhd N$ *such that* $IJ \subseteq P$ *implies* $I \subseteq P$ *or* $J \subseteq P$*, then* $P \lhd N$ *is called prime.*

Definition 2.2.12. *[61] E is referred to as generated finitely (finitely generated) if* $\exists$ *a finite set* $\{e_1, e_2, e_3, \ldots e_n\}$ *such that* $e = n_1e_1 + n_2e_2 + n_3e_3 + \cdots + n_ne_n$*,* $e_i \in E, n_i \in N$*,* $i = 1, 2, \ldots n, \forall e \in E$*.*

Definition 2.2.13. *[59] If D is a normal subgroup of* $(E,+)$ *such that* $n(a+e) - ne \in D$*,* $\forall\ n \in N, a \in D, e \in E$*, then D is referred to as an ideal of E.*

Definition 2.2.14. *[61] Let* $A \lhd E$*. Then the set* $\frac{E}{A} = \{a + A : a \in E\}$ *forms an N-group under the operations* $(k+A) + (s+A) = (k+s) + A$ *and* $m(s+A) = ms + A, \forall s, k \in E, m \in N$*, called quotient N-group.*

Definition 2.2.15. *[59] E is called cyclic if for every* $e \in E, e = nl$ *for some* $n \in N$*,* $l \in E$*.*

Definition 2.2.16. *[60] When $x \in E$, Nx is referred to as the principal N-subgroup of E.*

Definition 2.2.17. *[60] E is referred to as a principal N-group (PNG) if each $A \leq_N E$ is principal.*

Definition 2.2.18. *E is referred to as an ideal N-group if each $A \leq_N E$ is an ideal.*

Definition 2.2.19. *[60] If $K \lhd E$, then the N-subgroup of E/K is referred to as a sub-factor of E.*

Definition 2.2.20. *[60] For $e \in E$, $Ann(e) = l(e) = \{n \in N : ne = 0\}$.*

Lemma 2.2.1. *[28][2.72 corollary] Maximal ideals of a near-ring with unity is a prime ideal.*

Lemma 2.2.2. *N-subgroups of an ideal DN-group E are also ideal DN-groups.*

Proof : Let $T \leq_N E$.

If $T_1 \leq_N T$, then $T_1 \leq_N E$.

Since E is an ideal N-group, T_1 is ideal.

Also, since every N-subgroup of a DN-group is also DN-group, T is a DN-group.

Thus T is an ideal DN-group.

Hence the results follows.

2.3 *DN*-groups

Some basic preliminary concepts of DN-groups are discussed in this section.

Definition 2.3.1. *[60] If $(P+T)\cap C = P\cap C + T\cap C \ \forall\ P, T, C \leq_N E$, then E is called DN-group (distributive N-group).*

Definition 2.3.2. *[60] A fully DN-group is a DN-group if each factor group is DN-group.*

Definition 2.3.3. *[60] For $f, g \in E, (Nf : g) = \{x \in N : xg \in Nf\}$.*

Proposition 2.3.1. *Let E be a fully DN-group, then N-subgroups of the quotient N-group are also DN-group.*

Proof : Since E is a fully DN-group, each factor group of E is also DN-group. Also, by **lemma 2.2.2**, N-subgroups of a DN-group are also DN-group.
Thus the result.

The results obtained are restricted in the sense that every principal N-subgroup is an ideal.

Theorem 2.3.1. *[2] If for each $f,i \in E, (Nf:i)+(Ni:f)=N$, then E is a DN-group.*

Theorem 2.3.2. *If for each g,i in a DN-group E, then $Ng+Ni=N(g+i)+Ng\cap Ni$.*

Proof : For any $n \in N$, $ng+ni=n(g+i-i)+ni \in N(g+i)$, for $N(g+i) \lhd E$.
Thus, $ng \in N(g+i)+Ni$.
Therefore, $Ng \subseteq Ng+Ni \subseteq N(g+i)+Ni$
So, $Ng=[N(g+i)+Ni]\cap Ng$
$=N(g+i)\cap Ng+Ng\cap+Ni.$
Similarly, $Ni=N(g+i)\cap Ni+Ng\cap+Ni.$
So, $Ng+Ni=[N(g+i)\cap Ng+Ng\cap+Ni]+[N(g+i)\cap Ni+Ng\cap+Ni]$
$=(Ng+Ni)\cap N(g+i)+Ng\cap Ni$
$=N(g+i)+Ng\cap Ni.$
Thus, $Ng+Ni=N(g+i)+Ng\cap Ni.$

2.4 Fuzzy sets

In this section, fuzzy sets and related fuzzy definitions are defined.

Definition 2.4.1. *[7] A fuzzy subset of a set X, which is not empty, is a function ϕ from X to* $[0,1]$.

Definition 2.4.2. *[62] A fuzzy subset ϕ of E is called a fuzzy N-subgroup if the following conditions hold:*

i. $\phi(x-y) \geq \phi(x) \wedge \psi(x)$.

ii.$\phi(-x) = \phi(x)$.

iii. $\phi(nx) \geq \phi(x)$, *for all* $n \in N, x, y \in E$.

Definition 2.4.3. *[63] A fuzzy subset* ϕ *of N is referred to as a fuzzy sub near-ring if-*

i. $\phi(s+m) \geq \phi(s) \wedge \phi(m)$.

ii. $\phi(-s) = \phi(s)$.

iii. $\phi(sm) \geq \phi(s) \wedge \phi(m)$, $\forall\ s, m \in N$.

Definition 2.4.4. *[63] A fuzzy subset* ϕ *of E is called a fuzzy left ideal of E if the following conditions hold-*

i. $\phi(p-m) \geq \phi(p) \wedge \phi(m)$.

ii. $\phi(np) \geq \phi(p)$.

iii. $\phi(m+p-m) \geq \phi(p)$.

iv. $\phi(n(p+m)-np) \geq \phi(m)$, $\forall\ p, m \in E, n \in N$..

2.5 Intuitionistic Fuzzy sets

This section discusses IF sets, operations on IF sets and some of the important results that are needed for the work.

Definition 2.5.1. *[7] The object* $A =< \phi_A, \psi_A >= \{< s, \phi_A(s), \psi_A(s) > |s \in S\}$ *is referred to as an intuitionistic fuzzy (IF) set on a non empty set S, where* ϕ_A *and* ψ_A *are fuzzy subsets of S such that* $0 \leq \phi_A(s) + \psi_A(s) \leq 1$.

If $M =< \phi_M, \psi_M >$ *and* $B =< \phi_B, \psi_B >$ *are IF sets on S, then*

i. $\phi_M \leq \phi_B$ *if* $\phi_M(x) \leq \phi_B(x)$ *and* $\psi_M \leq \psi_B$ *if* $\psi_M(x) \leq \psi_B(x) \forall x \in S$.

ii. $\phi_M = \phi_B$ *if* $\phi_M(x) = \phi_B(x)$ *and* $\psi_M = \psi_B$ *if* $\psi_M(x) = \psi_B(x) \forall x \in S$.

iii. $\phi_M \vee \phi_B = \phi_M(x) \vee \phi_B(x)$ *and* $\phi_M \wedge \phi_B = \phi_M(x) \wedge \phi_B(x)$ *such that* $x \in S$.

iv. $\psi_M \vee \psi_B = \psi_M(x) \vee \psi_B(x)$ *and* $\psi_M \wedge \psi_B = \psi_M(x) \wedge \psi_B(x)$ *such that* $x \in S$.

v. $\phi_M.\phi_B = \phi_M(x).\phi_B(x)$ *and* $\psi_M.\psi_B = \psi_M(x).\psi_B(x)$ *such that* $x \in S$.

vi. $\phi_M + \phi_B = \phi_M(x) + \phi_B(x)$ *and* $\psi_M - \psi_B = \psi_M(x) - \psi_B(x)$ *such that* $x \in S$.

[7] **Some operations on IF sets are discuss below**:

Let $M =< \phi_M, \psi_M >$ and $B =< \phi_B, \psi_B >$ be IF sets on S. Then

i. $M \subseteq B \Leftrightarrow \phi_M \leq \phi_B, \psi_M \geq \psi_B$.

ii. $M = B \Leftrightarrow \phi_M = \phi_B, \psi_M = \psi_B$.

iii. $M^c = \bar{M} =< \psi_M, \phi_M >$.

iv. $M \cup B =< \phi_M \vee \phi_B, \psi_M \wedge \psi_B >$.

v. $M \cap B =< \phi_M \wedge \phi_B, \psi_M \vee \psi_B >$.

vi. $M \oplus B =< \phi_M + \phi_B - \phi_M \phi_B, \psi_M \psi_B >$.

Note that if M and H are IF sets on S, then $M^c, M \cup H, M \cap C, M \oplus H, M + H$, are all IF sets on S.

Also, note that $\phi_{M \cap B}(x) = \phi_M(x) \wedge \phi_B(x)$ and $\psi_{M \cap B}(x) = \psi_M(x) \vee \phi_B(x)$.

Definition 2.5.2. *Let $M =< \phi_M, \psi_M >$ and $B =< \phi_B, \psi_B >$ be IF sets of E. Then $M + B =< \phi_{M+B}, \psi_{M+B} >$, where $\phi_{M+B}(x) = \vee\{\phi_M(p) \wedge \phi_B(m) : p, m \in E, x = p + m\}$ and $\psi_{M+B}(x) = \wedge\{\psi_M(p) \vee \psi_B(m) : p, m \in E, x = p + m\}$.*

$M.B = MB =< \phi_{MB}, \psi_{MB} >$, where $\phi_{MB}(x) = \vee\{\phi_M(p) \wedge \phi_B(m) : p, m \in E, x = pm\}$ and $\psi_{MB}(x) = \wedge\{\psi_M(p) \vee \psi_B(m) : p, m \in E, x = pm\}$.

Definition 2.5.3. *[7] An IF set $A =< \phi_A, \psi_A >$ in N is called an IF near-ring in N if*

i. $\phi_A(p + m) \geq \phi_A(p) \wedge \phi_A(m)$.

ii. $\phi_A(-p) \geq \phi_A(p)$.

iii. $\phi_A(pm) \geq \phi_A(p) \wedge \phi_A(m)$.

iv. $\psi_A(p + m) \leq \psi_A(p) \vee \psi_A(m)$.

v. $\psi_A(-p) \leq \psi_A(p)$.

vi. $\psi_A(pm) \leq \psi_A(p) \vee \psi_A(m)$, $\forall\ p, m \in N$.

Definition 2.5.4. *If $A \leq_N E$, then $(A : E) = \{n \in N$: $nE \subseteq A\}$.*

Lemma 2.5.1. *If $Z \leq_N E$, then $(Z : E)E \subseteq Z$.*

Proof : Let $n \in (Z : E)$. Then $nE \subseteq Z$ and so $(Z : E)E \subseteq Z$.

Definition 2.5.5. *[47] An IF set $A =< \phi_A, \psi_A >$ in N is called an IF N-subgroup of N $(A \leq_{IFN} N)$ if* *i.* $\phi_A(p-m) \geq \phi_A(p) \wedge \phi_A(m)$.

ii. $\phi_A(np) \geq \phi_A(p)$.

iii. $\psi_A(p-m) \leq \psi_A(p) \vee \psi_A(m)$.

iv. $\psi_A(np) \leq \psi_A(p)$, $\forall\, p,m,n \in N$.

Definition 2.5.6. *An IF set* $A =< \phi_A, \psi_A >$ *in* E *is called IF an N-subgroup of* E $(A \leq_{IFN} E)$ *if*

i. $\phi_A(p-m) \geq \phi_A(p) \wedge \phi_A(m)$.

ii. $\phi_A(np) \geq \phi_A(p)$.

iii. $\psi_A(p-m) \leq \psi_A(p) \vee \psi_A(m)$.

iv. $\psi_A(np) \leq \psi_A(p)$, $\forall\, p,m \in E, n \in N$.

Definition 2.5.7. *An IF normal N-subgroup of* E *is an IF set* $A =< \phi_A, \psi_A >$ *in* E $(A \trianglelefteq_{IFN} E)$ *such that*

i. $\phi_A(p-m) \geq \phi_A(p) \wedge \phi_A(m)$.

ii. $\phi_A(np) \geq \phi_A(p)$.

iii. $\phi_A(m+p-m) \geq \phi_A(p)$.

iv. $\psi_A(p-m) \leq \psi_A(p) \vee \psi_A(m)$.

v. $\psi_A(np) \leq \psi_A(p)$.

vi. $\psi_A(m+p-m) \leq \psi_A(p), \forall p,m \in E, n \in N$.

Definition 2.5.8. *[47] An IF set* $A =< \phi_A, \psi_A >$ *in* N *is recognized as an IF ideal of* N $(A \triangleleft_{IF} N)$ *if*

i. $\phi_A(p-m) \geq \phi_A(p) \wedge \phi_A(m)$.

ii. $\phi_A(np) \geq \phi_A(p)$.

iii. $\phi_A(m+p-m) \geq \phi_A(p)$.

iv. $\phi_A(n(p+m)-np) \geq \phi_A(m)$.

v. $\psi_A(p-m) \leq \psi_A(p) \vee \psi_A(m)$.

vi. $\psi_A(np) \leq \psi_A(p)$.

vii. $\psi_A(m+p-m) \leq \psi_A(p)$.

viii. $\psi_A(n(p+m)-np) \leq \psi_A(m)$, $\forall\, p,m,n \in N$.

Definition 2.5.9. *[7] An IF set* $A = <\phi_A, \psi_A>$ *in* E *is called an IF ideal of* E $(A \lhd_{IF} E)$ *if*

i. $\phi_A(p-m) \geq \phi_A(p) \wedge \phi_A(m)$.

ii. $\phi_A(np) \geq \phi_A(p)$.

iii. $\phi_A(m+p-m) \geq \phi_A(p)$.

iv. $\phi_A(n(p+m)-np) \geq \phi_A(m)$.

v. $\psi_A(p-m) \leq \psi_A(p) \vee \psi_A(m)$.

vi. $\psi_A(np) \leq \psi_A(p)$.

vii. $\psi_A(m+p-m) \leq \psi_A(p)$.

viii. $\psi_A(n(p+m)-np) \leq \psi_A(m)$, $\forall\ p,m \in E, n \in N$.

It should be emphasized that an IF ideal of E is an IF N-subgroup.

Definition 2.5.10. *[47] Let* $P = <\phi_P, \psi_P>$ *be an IF sets in* E*. Then* (γ,λ)*-cut of* P *is referred by-*

$^{(\gamma,\lambda)}P = \{m \in E : \phi_P(m) \geq \gamma, \psi_P(m) \leq \lambda\}$.

$^{(\gamma,\lambda)}E$={ $^{(\gamma,\lambda)}P : P = <\phi_P, \psi_P>$ is an IF N-subgroup of E}.

$^{(\gamma,\lambda)}P$ is called unitary if $1.x = x$, $\forall\ x \in {}^{(\gamma,\lambda)}P$.

Since $^{(\gamma,\lambda)}P \subseteq E$, so if E is unitary then $^{(\gamma,\lambda)}P$ is also unitary and hence $^{(\gamma,\lambda)}E$ is also unitary.

$Ann(^{(\gamma,\lambda)}P) = \{n \in N : nx = 0\ \forall\ x \in {}^{(\gamma,\lambda)}P\}$

Note that $^{(\gamma,\lambda)}P, {}^{(\gamma,\lambda)}E \subseteq E$.

Proposition 2.5.1. *If* $L = <\phi_L, \psi_L> \leq_{IFN} E$*, then* $^{(\gamma,\lambda)}L \leq_N E$.

Proof : By definition, $^{(\gamma,\lambda)}L \subseteq E$.

For any $n \in N$, $s, y \in {}^{(\gamma,\lambda)}L$,

$\phi_L(s), \phi_L(y) \geq \gamma$ and $\psi_L(s), \psi_L(y) \leq \lambda$.

Therefore, $\phi_L(ns) \geq \phi_L(s) \geq \gamma$ [since L is an IF N-subgroup] and $\psi_L(ns) \leq \psi_L(s) \leq \lambda$.

Also, $ns \in E$.

Therefore, $ns \in {}^{(\gamma,\lambda)}L$.

Also, $s-y \in E$ such that $\phi_L(s-y) \geq \phi_L(s) \wedge \phi_L(y) \geq \gamma \wedge \gamma = \gamma$ and $\psi_L(s-y) \leq \psi_L(s) \vee$

$\psi_L(y) \leq \lambda \vee \lambda = \lambda$.

So, $s - y \in {}^{(\gamma,\lambda)}L$.

This shows that ${}^{(\gamma,\lambda)}L$ is an N-subgroup of E.

2.6 Conclusion

In this chapter, definitions of near-ring, near-ring groups and all related definitions are presented. Some of the lemmas and propositions that will be required for the research have been reviewed and illustrated here. Also, a few simple theorems have been proved, which will be used in the subsequent chapters.