Chapter 3
Uniserial and Bezout character of distributive N-group

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Uniserial and Bezout character of distributive *N*-group

3.1 Introduction

The essential definitions and outcomes of *DN*-groups are the main underlying ideas of this chapter. Stephenson [65] analyzed the notion of modules with distributive lattices of submodules by showing that if M is a D-module and $\alpha \in END(M)$, then every submodule P of M can be uniquely expressed as $P = A + A\alpha = B \cap B\alpha^{-1}$. Davidson [66] investigated the distributive homomorphism of rings and modules. This paved the path for researchers such as Victor Camilo [67] and Erdogdu [68] to investigate the concepts of distributive modules with diverse features. The relationship between D-module and annihilators of a module is derived by Camilo and different properties of the direct sum of D-modules are studied by Erdogdu. Tuganbaev [69] together with Mikhalev [1] investigated several properties of distributive rings and modules by defining distinct characteristics of uniserial and Bezout modules. They established the relationships between the uniserial and Bezout modules. The arithmetical ideals of rings, right prufer rings and noetherian rings were also studied by them. Endomorphism rings and semi-distributive rings were defined by Tuganbaev [70] to study the direct sum of distributed modules. Here, the above ideas of D-modules, uniserial and Bezout modules have been extended to the near-ring group (N-group).

3.2 Uniserial N-groups

Definition 3.2.1. *E is referred to as an uniserial N-group if any two of its N-subgroups are comparable to each other.*

Example 3.2.1. ***Example of an uniserial N-group.***

Let $N = E = \{0, s, d, t\}$ be the Klein's 4-group, which are given by the following table:-

.	0	s	d	t
0	0	0	0	0
s	0	0	s	s
d	0	s	t	d
t	0	s	d	t

+	0	s	d	t
0	0	s	d	t
s	s	0	t	d
d	d	t	0	s
t	t	d	s	0

Then $(E, +, .)$ is a near-ring as well as N-group over itself.

Also $P = \{0\}, B = \{0, s\} \leq_N E$ as $NP \subseteq P, NB \subseteq B$ such that $P \subseteq B \subseteq E$.

This shows that E is an uniserial N-group.

Now, consider the following examples-

Example 3.2.2. *[60] Let $N = E = \{0, s, d, t\}$ be the Klein's 4-group, which are given by the following table:-*

.	0	s	d	t
0	0	0	0	0
s	0	s	d	t
d	0	0	0	0
t	0	s	d	t

+	0	s	d	t
0	0	s	d	t
s	s	0	t	d
d	d	t	0	s
t	t	d	s	0

Then $(E,+,.)$ is a near-ring as well as N-group over itself.

Also $Ns = \{0,s\}, Nd = \{0,d\}, Nt = \{0,t\}$ are N-subgroups as well as ideals of E.

Example 3.2.3. *[60] Let $N = E = \{0,a,b,c,x,y\}$ is a near-ring as well as an N-group over itself under addition and multiplication as follows -*

.	0	a	b	c	x	y
0	0	0	0	0	0	0
a	0	0	0	0	a	a
b	0	0	0	0	b	b
c	0	0	o	0	c	c
x	0	0	0	0	x	x
y	0	0	0	0	y	y

+	0	a	b	c	x	y
0	0	a	b	c	x	y
a	a	0	y	x	c	b
b	b	x	0	y	a	c
c	c	y	x	0	b	a
x	x	b	c	a	y	0
y	y	c	a	b	0	x

Then $(E,+,.)$ is a near-ring as well as N-group over itself.

Also $Ax = Ay = \{0,a\}, Bx = By = \{0,b\}, Cx = Cy = \{0,c\}$ are ideals of E, where $A = \{0,a\}, B = \{0,b\}, C = \{0,c\}$ are N-subgroups of E.

The N-subgroups mentioned above have a common property that every principal N-subgroups are ideals.

So, confine our discussion on N-group restricted in the sense that every principal N-subgroup is an ideal .

Proposition 3.2.1. *E is a DN-group if and only if $N(q+g) = Nq \cap N(q+g) + Ng \cap N(q+g)$, $\forall\, q,g \in E$.*

Proof : Suppose E is DN-group.

Now, $n(q+g) - nq \in Ng$, for any $n \in N$ [since $Ng \lhd E$]

$\Rightarrow n(q+g) \in Nq + Ng$

$\Rightarrow N(q+g) \subseteq Nq + Ng$

$\Rightarrow N(q+g) = (Nq+Ng) \cap N(q+g)$.

Then $Nq \cap N(q+g) + Ng \cap N(q+g)$

= $(Nq+Ng)\cap N(q+g)$[since E is DN-group]

$= N(q+g)$[since $N(q+g) \subseteq Nq+Ng$].

Conversely, suppose that $Nq\cap N(q+g) + Ng\cap N(q+g) = N(q+g)$ for all $q,g \in E$.

Let $c = q+g$.

Then $c \in (D+B)\cap L$, where $D,B,L \leq_N E$ such that $c \in L$, $q \in D$ and $g \in B$.

Clearly, $D\cap L + B\cap L \subseteq (D+B)\cap L$.

Now, $Nc = N(q+g) = Nq\cap N(q+g) + Ng\cap N(q+g)$=$Nq\cap Nc + Ng\cap Nc \subseteq D\cap L + B\cap L$.

Since $1 \in N, c \in Nc$, therefore $c \in D\cap L+B\cap L$

So,$(D+B)\cap L \subseteq D\cap L+B\cap L$.

Thus, $D\cap L + B\cap L = (D+B)\cap L$.

Hence the result.

Proposition 3.2.2. *Suppose E is a DN-group and $s,b \in E$ with $Ns\cap Nb = 0$. Then $\exists$ $c \in N$ such that $Ncb+N(1-c)s = 0$.*

Proof : Let $t = s+b$, for any $s,b \in E$.

Since E is a DN-group, by proposition 3.2.1,

$N(s+b) = Ns\cap N(s+b) + Nb\cap N(s+b)$, for all $s,b \in E$

$\Rightarrow Nt = Ns\cap Nt+Nb\cap Nt$.

Since $1 \in N$, $t = 1.t \in Nt$.

Therefore, $\exists c,d \in N$ satisfying $ct \in Ns$, $dt \in Nb$ and $c+d = 1$

$\Rightarrow c(t-s)-ct \in Ns$ [since $Ns \lhd E$]

$\Rightarrow cb-ct \in Ns$

$\Rightarrow cb \in Ns$ [since $ct \in Ns$]

$\Rightarrow cb \in Ns\cap Nb$ [since $cb \in Nb$]

$\Rightarrow Ncb \subseteq Ns\cap Nb$ [since $cb \in Ncb$].

Similarly, $ds \in Nb$.

Let $p = 1-c$ and $z = 1-c-d$, then $zt = 0$ and $z = p-d \Rightarrow ps = zs+ds$.

Again, $z(t-b)-zt \in Nb$

$\Rightarrow zs-zt \in Nb$[since $Nb \lhd E$]

$\Rightarrow zs \in Nb$ [since $zt = 0$]

$\Rightarrow ps = zs + ds \in Nb$

$\Rightarrow (1-c)s \in Nb$

$\Rightarrow (1-c)s \in Ns \cap Nb$

$\Rightarrow N(1-c)s \subseteq Ns \cap Nb.$

Thus, $Ncb + N(1-c)s \subseteq Ns \cap Nb$

$\Rightarrow Ncb + N(1-c)s = 0$ [since $Ns \cap Nb = 0$].

Proposition 3.2.3. *If E is a DN-group over a local N and $s,k \in E$. Then either $s \in Nk$ or $k \in Ns$.*

Proof : Since E is a DN-group, by **proposition 3.2.2**, $qk \in Ns$, $(1-q)s \in Nk$ for $q \in N$.

Since N is local, q or $(1-q)$ is invertible.

q is invertible implies $k \in Ns$ and $(1-q)$ is invertible implies $s \in Nk$.

Thus, either $s \in Nk$ or $k \in Ns$.

Proposition 3.2.4. *If E is a DN-group with $Ns \cap Nk = 0\ \forall\ s,k \in E$, then $\exists\ c \in N$ such that $ck = (1-c)s = 0$.*

Proof : Since E is DN-group and $s,k \in E$, by **proposition 3.2.2**, $\exists\ c \in N$ such that $ck \in Ns \cap Nk = 0$.

Therefore, $ck = 0$.

Also, $(1-c)s \in Ns \cap Nk = 0$

So, $(1-c)s = 0$.

Thus, $ck = (1-c)s = 0$.

Proposition 3.2.5. *If E' is a subfactor of a fully DN-group E and $Ns' \cap Nk' = 0'$ for $s', k' \in E'$, then $\exists\ h,t \in N$ such that $1 = h+t$ and $hs' = tk' = 0'$.*

Proof : Since E is fully DN-group, by **proposition 2.3.1**, the subfactor E' of E is also DN-group.

If $s', k' \in E'$ with $Ns' \cap Nk' = 0'$, then $\exists\ h \in N$ such that $hs' = (1-h)k' = 0'$[by **proposition 3.2.4]** .

Let us put $t = 1 - h$.

So $1 = h + t$ and $hs' = tk' = 0'$.

Theorem 3.2.1. *For any subfactor E' of E such that $Ns' \cap Nh' = 0'$, $\forall\ s', h' \in E'$ and $1 = p + q, ps' = qh' = 0'$, $\forall\ p, q \in N$, then E is a DN-group.*

Proof : Let s', h' be the natural images of s, h under the epimorphism $E \to \frac{E}{Ns \cap Nh}(= E')$ such that $Ns' \cap Nh' = 0'$ and $ps' = qh' = 0', 1 = p + q, p, q \in N$.

Now, $ps' = 0' \Rightarrow p \in l(s')$

and $qh' = 0' \Rightarrow q \in l(h')$.

Since $1 \in N$ and $1 = p + q \in l(s') + l(h')$, we have

$N \subseteq l(s') + l(h')$.

Therefore, $N = l(s') + l(h')$ [since $l(s'), l(h') \subseteq N$] .

Now, $p \in l(s')$

$\Rightarrow ps' = 0'$

$\Rightarrow p(s + Ns \cap Nh) = Ns \cap Nh$

$\Rightarrow ps + Ns \cap Nh = Ns \cap Nh$

$\Rightarrow ps \in Nh$

So,$p \in (Nh : s)$

Therefore, $l(s') \subseteq (Nh : s)$.

Also, $q \in (Nh : s)$

$\Rightarrow qs \in Nh$

$\Rightarrow qs \in Ns \cap Nh$ [since $qs \in Ns$]

$\Rightarrow qs + Ns \cap Nh = Ns \cap Nh$

$\Rightarrow q(s + Ns \cap Nh) = Ns \cap Nh$

$\Rightarrow qs' = 0'$

So, $q \in l(s')$.

Therefore, $l(s') = (Nh : s)$.

Similarly, $l(h') = (Ns : h)$.

Thus, $N = (Nh : s) + (Ns : h)$, for $s, h \in E$.

So by **theorem2.3.1**, E is a DN-group.

Proposition 3.2.6. *If E is a DN-group such that $Nf \cap Nj = 0$, for any $f, j \in E$, then $l(f) + l(j) = N$.*

Proof : For any $s, h \in N$,

$s \in l(f)$

$\Rightarrow sf = 0 \in Nf \cap Nj$

$\Rightarrow sf \in Nj$

So, $s \in (Nj : f)$

Therefore, $l(f) \subseteq (Nj : f)$.

Now, $h \in (Nj : f)$

$\Rightarrow hf \in Nf \cap Nj$[since $hf \in Nf$]

$\Rightarrow hf = 0$

So, $h \in l(f)$

Therefore,$(Nj : f) \subseteq l(f)$

Thus,$(Nj : f) = l(f)$.

Similarly, $(Nf : j) = l(j)$.

By **theorem 2.3.1**, $N = (Nj : f) + (Nf : j)$.

Thus, $N = l(f) + l(j)$.

Theorem 3.2.2. *If E is a DN-group with $Nh \cap Ns = 0$, for any non zero $h, s \in E$, then the direct sum of Nh and Ns is cyclic and $l(h+s) = l(h) \cap l(s)$.*

Proof : Since E is a DN-group such that $Nh \cap Ns = 0$, for any non zero $h, s \in E$, by **proposition 3.2.4**, $\exists\, c \in N$ such that $ch = (1-c)s = 0$.

From **proposition 3.2.1**, $N(h+s) \subseteq Nh + Ns$.

Since $N(h+s) \lhd E$, therefore

$c(h+s-s) + cs \in N(h+s)$

$\Rightarrow ch + cs \in N(h+s)$

$\Rightarrow Nh + Ns \subseteq N(h+s)$.

So, $N(h+s) = Nh + Ns$

$\Rightarrow N(h+s) = Nh \oplus Ns$[since $Nh \cap Ns = 0$] .

Thus, $Nh \oplus Ns$ is cyclic.

Let $x \in l(h+s)$

So, $x(h+s) = 0$.

Now, $x(h+s) - xh \in Ns$ [since $Ns \lhd E$]

$\Rightarrow xh \in Nh \cap Ns$ [since $xh \in Nh$]

$\Rightarrow xh = 0$[since $Nh \cap Ns = 0$]

So, $x \in l(h)$.

Similarly, $x \in l(s)$.

Therefore,$x \in l(h) \cap l(s)$.

Also, $y \in l(h) \cap l(s)$

$\Rightarrow yh = 0$ and $ys = 0$

$\Rightarrow yh = ys \in Nh \cap Ns$

$\Rightarrow y(h+s) - ys \in Nh$ [since $Nh \lhd E$]

So, $y(h+s) \in Nh$.

Again, $y(h+s) - yh \in Ns$

$\Rightarrow y(h+s) \in Ns$.

Therefore,$y(h+s) \in Nh \cap Ns = 0$

$\Rightarrow y(h+s) = 0$

So, $y \in l(h+s)$.

Thus, $l(h+s) = l(h) \cap l(s)$.

Theorem 3.2.3. *Let E be a DN-group over the local N. If E is PNG, then it is an uniserial N-group.*

Proof : Let $a, b \in E$.

Since E is DN-group over the local N, by **proposition 3.2.3**, either $a \in Nb$ or $b \in Na$.

Thus, $Na \subseteq Nb$ or $Nb \subseteq Na$.

Since E is PNG, every N-subgroup is principal.

This shows that E is an uniserial N-group over the local N.

Theorem 3.2.4. *Let E be a PNG and $\forall\, d,i \in E$, $Nd+Ni \leq_N E$ is cyclic. Then $\exists p,q,l,s \in N$ such that $(1-lp)d \in Ni, (1-sq)i \in Nd$ and if at least one of $1-lp$, lq, $1-sq$, sp is invertible, then E is an uniserial N-group.*

Proof : Since, $\forall\, d,i \in E, Nd+Ni \leq_N E$ is cyclic, $\exists p,q \in N$ such that every element of $Nd+Ni$ is generated by a single element $pd+qi$.

Since $i,d \in Nd+Ni$

$\Rightarrow \exists l,s \in N$ such that $d = l(pd+qi), i = s(pd+qi)$.

Now, $l(pd+qi) - lpd \in Ni$[since $Ni \lhd E$]

$\Rightarrow d - lpd \in Ni$

$\Rightarrow (1-lp)d \in Ni$.

If $(1-lp)$ is invertible, then $d \in Ni$.

Also, $s(pd+qi) - sqi \in Nd$

$\Rightarrow i - sqi \in Nd$

$\Rightarrow (1-sq)i \in Nd$.

If $(1-sq)$ is invertible, then $i \in Nd$.

Again, $l(pd+qi) - lqi \in Nd$

$\Rightarrow lqi \in Nd$[Since $d = l(pd+qi) \in Nd$] .

If lq is invertible, then $i \in Nd$.

and $s(pd+qi) - spd \in Ni$

$\Rightarrow spd \in Ni$[Since $i = s(pd+qi) \in Ni$] .

If sp is invertible, then $d \in Ni$.

Thus, if at least one of $1-lp$, lq, $1-sq$, sp is invertible, then $d \in Ni$ or $i \in Nd \;\Rightarrow Nd \subseteq Ni$ or $Ni \subseteq Nd$.

Thus E is an uniserial N-group.

Theorem 3.2.5. *Let E be a PNG over the local N and for any $k,d \in E$ such that $Nk+Nd$ is cyclic. Then E is an uniserial N-group.*

Proof : Since $Nk+Nd$ is cyclic N-group, by **theorem 3.2.4**, $\exists p,q,r,s \in N$ such that$(1-rp)k \in Nd, (1-sq)d \in Nk$.

Again, since N is local near-ring, $(1-rp)$ or rp is invertible.

If $(1-rp)$ is invertible, then $k \in Nd \Rightarrow Nk \subseteq Nd$.

Suppose rp is invertible.

Since $s \in N$, then s is either invertible or not invertible.

If s is invertible, then as in **theorem 3.2.4**

$spk \in Nd$

$\Rightarrow spk = gd$, for some $g \in N$

$\Rightarrow k = (rp)^{-1})rs^{-1}gd$

$\Rightarrow k \in Nd$

$\Rightarrow Nk \subseteq Nd$.

If s is not invertible, then sq is also not invertible.

Since N is local, $1-sq$ is invertible.

So, $(1-sq)d \in Nk$

$\Rightarrow (1-sq)d = gk$, for some $g \in N$

$\Rightarrow d = (1-sq)^{-1}gk \in Nk$

$\Rightarrow d \in Nk$

$\Rightarrow Nd \subseteq Nk$.

Thus the result.

Proposition 3.2.7. *Let E be a DN-group. Then $\forall\ l,i \in E$, $\exists q,p \in N$ such that $1 = q+p, ql \in Ni, pi \in Nl$ and conversely.*

Proof : Since E is DN-group and $l,i \in E$, then by **proposition3.2.2**, $\exists p,q \in N$ satisfying $t = l+i, pt \in Nl, qt \in Ni$ and $1 = p+q$.

Since $qt \in Ni$, $qt = xi$, for some $x \in N$.

Since $Ni \lhd E$, $i \in Ni, t \in E$, $q(i-t)+qt \in Ni$

$\Rightarrow q(i-t) \in Ni$[Since $qt \in Ni$]

$\Rightarrow ql \in Ni$ [Since $t = l+i$] .

Similarly, $pi \in Nl$.

Conversely, let $F,Q,K \leq_N E$ such that $c \in (F+Q) \cap K$.

Then $c = l+i$, for some $l \in F$, $i \in Q$ and $c \in K$.

Since $Nl \lhd E$, $p(l+i) - pi \in Nl$

$\Rightarrow p(l+i) \in Nl$[Since $pi \in Nl$]

$\Rightarrow pc \in Nl$ [Since $c = l+i$]

$\Rightarrow pc \in Nl \cap Nc$[since $pc \in Nc$]

$\Rightarrow pc \subseteq F \cap K$ [Since $Nl \subseteq F$ and $Nc \subseteq K$] .

Similarly, $qc \in Q \cap K$.

Now, $1 = p+q$

$\Rightarrow c = pc + qc \in (F \cap K) + (Q \cap K)$.

Again, let $x \in F \cap K + Q \cap K$

$\Rightarrow y+z = x$, $y \in F$, K and $z \in Q$, K

$\Rightarrow y+z = x \in F+Q$ and $y+z = x \in K$[Since $K \leq_N E$]

$\Rightarrow x \in (F+Q) \cap K$.

Thus, $(F \cap K) + (Q \cap K) = (F+Q) \cap K$.

Hence E is a DN-group.

Theorem 3.2.6. *Let E be a PNG and* $(Nf : j) + (Nj : f) = N$, $\forall\ f, j \in E$. *Then* $\exists q, p, r, s \in N$ *such that* $1 = p+q, pj \in Nf, qf \in Nj$. *If at least one of p and q is invertible, then E is an uniserial N-group.*

Proof : Since $(Nf : j) + (Nj : f) = N$, $\forall\ f, j \in E$, by **theorem 2.3.1**, E is DN-group.

So, by **proposition 3.2.7**, $\exists p, q, r, s \in N$ such that $1 = p+q, qf \in Nj, pj \in Nf$.

If p is invertible, then $j \in Nf$ and if q is invertible, then $f \in Nj$.

So, either $Nf \subseteq Nj$ or $Nj \subseteq Nf$.

Thus E is an uniserial N-group.

3.3 Bezout N-groups

Definition 3.3.1. *E is called a Bezout N-group if any of its finitely generated N-subgroups is cyclic.*

Example 3.3.1. *Example of a Bezout N-group.*

Let $N = E = \{0,s,k,t\}$. Then, the following table defines $(N,+,.)$ as a near-ring-

.	0	s	k	t
0	0	0	0	0
s	0	0	s	s
k	0	s	k	k
t	0	s	t	t

+	0	s	k	t
0	0	s	k	t
s	s	0	t	k
k	k	t	0	s
t	t	k	s	0

Also $(E,+,.)$ is an N-group over itself.

Then $T = \{0\}, L = \{0,s\}$, $E \leq_N E$ and $T = N0, L = Ns$, $E = Nk = Nt$.

This shows that T, L and E are cyclic N-subgroups.

Hence E is a Bezout N-group.

Theorem 3.3.1. *E is a Bezout N-group if and only if $\forall\ h,k \in E$,$\exists\ s,q \in N$ such that h and k are generated by (sh+qk).*

Proof : Let $W = Nh + Nk$, $h,k \in E$ and E be a Bezout N-group.

Then W is cyclic and there exits $s,q \in N$ such that every element of W is generated by (sh + qk).

Since $1 \in N$, therefore $h,k \in W$.

So, h and k are generated by (sh + qk).

Conversely, suppose W be any 2-generated N-subgroup.

It is enough to establish that W is cyclic.

Now, for any $x \in W$,

$x = s'h' + q'k'$, for some $s', q' \in N$ and $h', k' \in W$[by definition of finitely generated] .

Now, $h', k' \in W$.

$\Rightarrow h', k'$ are generated by $(sh' + qk')$.

So, by assumption, $h' = t(sh' + qk')$ and $k' = d(sh' + qk')$, for some $t, d \in N$.

Therefore,$x = s't(sh' + qk') + q't(sh' + qk')$

$\Rightarrow x = (s't + q't)(sh' + qk')$[by left distributive property]

$\Rightarrow x$ is generated by $(sh' + qk')$.

This shows that W is cyclic generated by ($sh' + qk'$).

Since the result of W can be extended to any finitely generated N-subgroups i.e finitely generated N-subgroup is cyclic and so E is a Bezout N-group.

Theorem 3.3.2. *Let E be a Bezout PNG over a local N, then E is an uniserial N-group.*

Proof : Since E is a Bezout N-group, for any $k,t \in E$, $Nk+Nt$ is cyclic N-subgroup. Since N is local, E is an uniserial N-group[by **theorem 3.2.5**] .

Definition 3.3.2. *E is referred to as a simple N-group if $E = Nm$, for any non-zero $m \in M$, where $M \leq_N E$.*

Definition 3.3.3. *An N-group is said to be semi simple if it is the sum of simple N-subgroups or direct sum of simple N-subgroups.*

Proposition 3.3.1. *If E_1 and E_2 are cyclic N-subgroups of a DN-group E with $E_1 \cap E_2 = 0$, then the direct sum of E_1 and E_2 is also a cyclic N-subgroup of E.*

Proof : Suppose E_1 and E_2 are generated by $g \in E_1$ and $q \in E_2$ respectively.
Then $E_1 = Ng$ and $E_2 = Nq$.
Since E is a DN-group,
$N(g+q)+Ng \cap Nq = Ng+Nq$[by **theorem 2.3.2**] .
Since E_1 and E_2 are disjoint, $E_1 \cap E_2 = 0$ implies $Ng+Nq = N(g+q)$.
Now, for any $y \in E_1+E_2$,
$y = e_1+e_2$, where $e_1 = n_1 g$ and $e_2 = n_2 q$, for some $e_1 \in E_1$, $e_2 \in E_2$ and $n_1,n_2 \in N$.
Therefore,$y = n_1 g+n_2 q \in Ng+Nq = N(g+q)$
$\Rightarrow y = h(g+q)$, for some $h \in N$.
This shows that direct sum of E_1 and E_2 is cyclic N-subgroup.

Corollary 3.3.1. *As a consequence of* ***proposition*** *3.3.1, the finite direct sum of disjoint cyclic N-subgroups of a DN-group is also cyclic N-subgroups.*

Definition 3.3.4. *$S \leq_N E$ is referred to be* ***Small*** *in E if for any ideal D of E, $S+D = E$ implies $D = E$.*

Proposition 3.3.2. *If $W \leq_N E$ is small, then E is cyclic if and only if $\frac{E}{W}$ is cyclic.*

Proof : Let E be a cyclic N-group generated by $x \in E$.

Now, for any $k+W \in \frac{E}{W}$, $k \in E$

$\Rightarrow k = hx$, for some $h \in N$

$\Rightarrow k+W = hx+W = h(x+W)$.

This shows that $\frac{E}{W}$ is cyclic.

conversely, let $\frac{E}{W}$ is cyclic generated by $(x+W)$.

Let $y \in E$

$\Rightarrow y+W \in \frac{E}{W}$

$\Rightarrow y+W = h(x+W)$, for some $h \in N$

$\Rightarrow y+W = hx+W$

$\Rightarrow y = hx+m$, for some $m \in W$.

Therefore,$E = Nx+W$

$\Rightarrow E = Nx$ [since W is small in E] .

This shows that E is cyclic N-group.

Proposition 3.3.3. *If E is a DN-group and $M \trianglelefteq_N E$ is small such that $\frac{E}{M}$ is the finite direct sum of disjoint cyclic N-subgroups, then E is cyclic.*

Proof : By **corollary** 3.3.1, $\frac{E}{M}$ is cyclic.

Also, by **proposition** 3.3.2, E is cyclic.

Definition 3.3.5. *Jacobson radical $J(N)$ is an ideal of N satisfying $a \in J(N)$ implies that $1-a$ is invertible.*

Lemma 3.3.1. *$J(N)E = E$ implies $E = 0$ if E is finitely generated.*

Proof : Let E be generated by $e_1 \in E$.

Then, for any $e \in E$,

$e = n_1e_1$, for some $n_1 \in E$.

Now, $e_1 \in E$ and $J(N)E = E$

$\Rightarrow e_1 = ae_1$, for some $a \in J(N)$

$\Rightarrow (1-a)e_1 = 0$

$\Rightarrow (1-a)^{-1}(1-a)e_1 = 0$

$\Rightarrow e_1 = 0$[since E is unitary]

$\Rightarrow e = 0$

$\Rightarrow$E= 0.

Let E be generated by $\{e_1, e_2\}$.

Then, for any $e \in E$,

$e = n_1e_1 + n_2e_2$, for some $n_1, n_2 \in N$.

Now, if $e_1 \in E$ and $J(N)E = E$, as above it can be shown that $e_1 = 0$.

Again, if $e_2 \in E$ and $J(N)E = E$, as above it can be shown that $e_2 = 0$.

Thus $e = 0$ and hence $E = 0$.

Also, if E is generated by $\{e_1, e_2, \ldots e_n\}$, then in the same way $E = 0$.

Proposition 3.3.4. *If E is finitely generated, then $J(N)E$ is small in E.*

Proof : First, try to show that for any normal N-subgroups W, $\frac{E}{W}$ is finitely generated unitary factor N-group of E.

Let E be generated by a finite set $\{e_1, e_2, e_3 \ldots e_n\}$ of E .

Then for any $e + W \in \frac{E}{W}$,

$e = n_1e_1 + n_2e_2 + \ldots n_ne_n$, for $n_i \in N$, $i = 1, 2, 3, \ldots .n$

$\Rightarrow e + W = n_1(e_1 + W) + n_2(e_2 + W) + \ldots n_n(e_n + W)$.

This shows that $\frac{E}{W}$ is finitely generated factor N-group of E.

Now, $x + W \in \frac{E}{W}$

$\Rightarrow x \in E$

$\Rightarrow 1.x = x$[using unitary in E]

$\Rightarrow 1.(x + W) = x + W$.

Therefore, $\frac{E}{W}$ is finitely generated unitary factor N-group of E.

Secondly, to show $J(N)E + W = E$ implies $J(N)(\frac{E}{W}) = (\frac{E}{W})$.

Now, $e \in J(N)E + W$

$\Rightarrow e = ae_1 + m$, where $a \in J(N), e_1 \in E, m \in W$

$\Rightarrow e + W = ae_1 + W$

$\Rightarrow e+W = a(e_1+W) \in J(N)(\frac{E}{W})$

Therefore, $\frac{E}{W} \subseteq J(N)(\frac{E}{W})$.

Again, for $b \in J(N)$ and $e \in E$, $b(e+W) \in J(N)(\frac{E}{W})$.

Now, $be+m \in J(N)E+W$.

$\Rightarrow be+m \in E$

$\Rightarrow be+m+W \in \frac{E}{W}$

$\Rightarrow be+W \in \frac{E}{W}$

$\Rightarrow b(e+W) \in \frac{E}{W}$.

Therefore, $J(N)(\frac{E}{W}) \subseteq \frac{E}{W}$

$\Rightarrow J(N)(\frac{E}{W}) = \frac{E}{W}$.

By lemma 3.3.1,

$\frac{E}{W} = \bar{0} = W$

$\Rightarrow E = W$.

Thus the result.

Theorem 3.3.3. *E is a Bezout N-group if and only if for any finitely generated ideal L of E, the factor N-group* $\frac{L}{J(N)L}$ *is cyclic.*

Proof : Since E is a Bezout N-group and L is finitely generated, L is cyclic.

By **proposition** 3.3.4, $J(N)L$ is small in L.

So, by **proposition** 3.3.2, $\frac{L}{J(N)L}$ is cyclic.

Conversely, since L is finitely generated and $\frac{L}{J(N)L}$ is cyclic,

So by **proposition 3.3.2**, L is cyclic.

Thus E is a Bezout N-group.

Theorem 3.3.4. *Let E be a DN-group such that for any finitely generated* $L \leq_N E$*, the factor N-group* $\frac{L}{J(N)L}$ *is direct sum of finite cyclic N-subgroups. Then E is a Bezout N-group.*

Proof : Since $L \leq_N E$ is finitely generated, $J(N)L \leq_N L$ is small[by **proposition 3.3.4**] .

Also L is DN-group as a N-subgroup of a DN-group.

Also, by hypothesis, $J(N)L \trianglelefteq_N L$.

Thus $J(N)L$ is small normal N-subgroup of an DN-group L such that the factor N-group $\frac{L}{J(N)L}$ is direct sum of finite cyclic N-subgroups.

So, by **proposition 3.3.3** L is cyclic.

This shows that E is a Bezout N-group.

Definition 3.3.6. *If $a = axa$ is true for all $a, x \in N$, then N is said to be regular.*

Proposition 3.3.5. *A strongly regular near-ring is regular.*

Proof : Let N be a strongly regular near-ring. If $t \in N$, then $t = yt^2$, for some $\in N$

$\Rightarrow t^2 - tyt = 0$

$\Rightarrow (t - tyt)t = 0$

$\Rightarrow \{t(t - tyt)\}^2 = t(t - tyt)t(t - tyt) = 0.$

If $t(t - tyt) \neq 0$, then $t(t - tyt) = y\{t(t - tyt)\}^2 = 0$, for some $y \in N$[since N is strongly regular] -which is a contradiction.

Thus, $t(t - tyt) = 0$

$\Rightarrow tyt(t - tyt) = 0$

$\Rightarrow (t - tyt)^2 = (t - tyt)(t - tyt) = t(t - tyt) - tyt(t - tyt) = 0$

$\Rightarrow (t - tyt) = 0$[as above]

$\Rightarrow t = tyt.$

Hence N is regular.

Theorem 3.3.5. *If E is a DN-group over a strongly regular N, then E is a Bezout N-group.*

Proof : Let $L = Ns + Ng$, for any $s, g \in L$.

It is sufficient to show that L is cyclic.

Since E is DN-group over a strongly regular N, by **proposition** 3.2.7, $\exists\ k, t \in N$ such that $1 = k + t, ks \in Ng, tg \in Ns$.

N is regular, by **proposition** 3.3.5 and so $k = kxk$, $\forall x \in N$.

Let us put $xk = z$, then $z \in N$ and z is idempotent as $z^2 = zz = xkxk = xk = z$.

Also,

$k = kxk$

$\Rightarrow k = kz \in Nz$

$\Rightarrow Nk \subseteq Nz$.

Also, $z \in Nz$ and $z = xk \in Nk$.

Therefore, $Nz \subseteq Nk$.

$\Rightarrow Nz = Nk$.

Now, for any $x \in N$,

$xz^2 = xz$

$\Rightarrow z = xz$[since N is strongly regular]

$\Rightarrow xz = x^2z = x$[since N is strongly regular] .

Again, $z = xz$

$\Rightarrow zx = xzx = x$ [Since N is regular] .

Therefore, $zx = xz$, $\forall\ x \in N$

$\Rightarrow z$ is central.

Again, $(1-z)^2 = 1-z-z+z^2 = 1-z$.

So, $(1-z)$ is idempotent.

So $x(1-z)^2 = x(1-z)$

$\Rightarrow (1-z) = x(1-z)$ [since $(1-z) \in N$ and N is strongly regular]

$\Rightarrow x(1-z) = x^2(1-z) = x$ [since N is strongly regular] .

Again, $(1-z) = x(1-z)$

$\Rightarrow (1-z)x = x(1-z)x = x$ [since N is regular] .

Therefore, $(1-z)x = x(1-z)\ \forall\ x \in N$.

Thus, $(1-z)$ is also central.

Since $1, z \in N$ and $Ns, Ng \in E$, so by using distributive property of N-group,

$L = zNs + zNg + (1-z)Ns + (1-z)Ng = Nzs + Nzg + N(1-z)s + N(1-z)g$[since $z, 1-z$ are central] .

Now, $z = zkz = zkxk = zk[since\text{N}isregular]$

$\Rightarrow zs = zks \in zNg = Nzg$

$\Rightarrow Nzs \subseteq Nzg$.

Again, since $1-z$ is central,

$(1-z)k = k(1-z) = k(1-z)^2$[since $1-z$ is idempotent]

$\Rightarrow (1-z)k = 1-z$[since $1-z$ is strongly regular]

$\Rightarrow (1-z)s = (1-z)ks \in (1-z)Ng = N(1-z)g$

$\Rightarrow N(1-z)s \subseteq N(1-z)g$.

Therefore, $L = Nzg + N(1-z)g$.

Again, $x \in Nzg \cap N(1-z)g$

$\Rightarrow x = n'zg = n''(1-z)g$, for some $n', n'' \in N$

$\Rightarrow zn'zg = zn''(1-z)g$

$\Rightarrow n'zzg = n''z(1-z)g = n''(1-z)zg$

$\Rightarrow n'zg = n''(z-z^2)g$

$\Rightarrow n'zg = 0$

$\Rightarrow x = 0$.

Again, L is DN-group being a N-subgroup of a DN-group and $zg, (1-z)g \in L$,

We get, $L = N(zg + (1-z)g)$[by **theorem 2.3.2]** .

This shows that L is cyclic.

Hence it can be conclude that E is a Bezout N-group.

3.4 Conclusion

To study near ring group concepts of uniserial and Bezout modules is the motivation of this chapter. Introducing uniserial and Bezout N-groups, their relationships are developed and proved. The **theorems** 3.2.1 to 3.2.6 describe the correlation between DN-groups and uniserial N-groups. **Theorems** 3.3.1 to 3.3.5 illustrate the linkage among DN-groups, uniserial N-groups and Bezout N-groups. The results obtained here will be used in the subsequent chapters.