Chapter 4
Distributive character of multiplication N-groups

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Distributive character of multiplication N-groups

4.1 Introduction

Krull proposed the concept of a multiplication ring [71] and some important results like Nakayama Lemma and Principal Ideal Theorem. Mehdi [72] was the first to develop multiplication modules and studied direct sum of multiplication modules, faithful multiplication modules, projective multiplication modules and weak multiplication modules. However, Barnard [73], El Bast and Smith [74], Tuganbaev [75], and Atani and Ghaleh [76] all have thoroughly researched it in various ways. The relationship between the distributive module and multiplication module is established by Barnard. Jain [77] and Rajaee [78] studied generalized multiplication modules on arithmetical rings. The relationships between distributive modules and multiplication modules were also analyzed by Erdogdu [79] and Escoriza and Torrecillas [80]. Elaheh Khodadaapour and Tahereh Roodbarilor [59] studied the multiplication N-groups and cyclic N-groups in near-rings and their associations.

This chapter describes localized near-rings, localized N-groups and associated results with reference to DN-groups. The results of localized multiplication N-groups by defining multiplication N-groups with an example are established. The relationships between the multiplication N-groups and the locally cyclic N-groups and the DN-groups are established.

4.2 Localized N-groups

Definition 4.2.1. *$H \subseteq N$ is called multiplicative closed if $p \in H$ implies $p^{-1} \in H$ or $1 \in H$ and $p, y \in H$ implies $py \in H$.*

It is to be noted that for any $p \in H, pp^{-1} = p^{-1}p = 1 \in H$ and $p = (p^{-1})^{-1}$.

Let S be a multiplicative closed subset of a commutative near-ring N with identity. Define "$+$" and "$.$" in $(S^{-1}N, +, .)$ by-

$s_1^{-1}n_1 + s_2^{-1}n_2 = (s_1s_2)^{-1}(s_2n_1 + s_1n_2)$

and $(s_1^{-1}n_1).(s_2^{-1}n_2) = (s_1s_2)^{-1}(n_1n_2), \forall\, s_1, s_2 \in S, n_1, n_2 \in N.$

Then, for any $s_1^{-1}n_1, s_2^{-1}n_2, s_3^{-1}n_3 \in S^{-1}N$,

$(s_1s_2)^{-1} \in S$ and $s_2n_1 + s_1n_2 \in N$.

So, $(s_1s_2)^{-1}(s_2n_1 + s_1n_2) \in S^{-1}N$

$\Rightarrow s_1^{-1}n_1 + s_2^{-1}n_2 \in S^{-1}N.$

Now, $(s_1^{-1}n_1 + s_2^{-1}n_2) + s_3^{-1}n_3$

$= (s_1s_2)^{-1}(s_2n_1 + s_1n_2) + s_3^{-1}n_3$

$= (s_1s_2s_3)^{-1}\{s_3(s_2n_1 + s_1n_2) + (s_1s_2)n_3\}$[since $(N, .)$ is associative]

$= (s_1s_2s_3)^{-1}\{(s_2n_1 + s_1n_2)s_3 + (s_1s_2)n_3\}$[since $S \subseteq N$ and $(N, .)$ is commutative]

$= (s_1s_2s_3)^{-1}\{(s_2n_1)s_3 + (s_1n_2)s_3 + (s_1s_2)n_3\}$[by right distributive law]

$= (s_1s_2s_3)^{-1}\{s_3(s_2n_1) + s_3(s_1n_2) + (s_1s_2)n_3\}$[since $S \subseteq N$ and $(N,.)$ is commutative]

$= (s_1s_2s_3)^{-1}\{s_3(s_2n_1) + s_3(s_1n_2) + (s_1s_2)n_3\}$[since $(N,.)$ is commutative and has right distributive property]

$= (s_1s_2s_3)^{-1}\{(s_2s_3)n_1) + (s_1s_3)n_2) + (s_1s_2)n_3\}$ [since $(N,.)$ is associative and commutative] .

Similarly, it can be shown that $s_1^{-1}n_1 + (s_2^{-1}n_2 + s_3^{-1}n_3) = (s_1s_2s_3)^{-1}\{(s_2s_3)n_1) + (s_1s_3)n_2) + (s_1s_2)n_3\}$.

Therefore, $(s_1^{-1}n_1 + s_2^{-1}n_2) + s_3^{-1}n_3 = s_1^{-1}n_1 + (s_2^{-1}n_2 + s_3^{-1}n_3)$.

For any $s^{-1}x \in S^{-1}N$,

$0 + s^{-1}x$

$= s^{-1}0 + s^{-1}x$

$= s^{-1}(0 + x)$[since $s^{-1} \in N, N$ has the right distributive property and $(N,.)$ is commutative]

$= s^{-1}x$.

Similarly, $s^{-1}x + 0 = s^{-1}x$.

Thus, the identity 0 of $(N,+)$ is the identity of $(S^{-1}N,+)$.

For any $s^{-1}x \in S^{-1}N$, $-s^{-1}x$ is the inverse of $s^{-1}x$ as $s^{-1}x + (-s^{-1}x) = s^{-1}(x - x) = s^{-1}0 = 0 = (-s^{-1}x) + s^{-1}x$.

Also, $(S^{-1}N,.)$ is closed by definition.

Since N is commutative, $s_1^{-1}n_1(s_2^{-1}n_2s_3^{-1}n_3) = (s_1^{-1}n_1s_2^{-1}n_2)s_3^{-1}n_3$.

Now, for any $s, y \in S \subseteq N$,

$s^{-1}, y^{-1}, s^{-1}y^{-1}, (sy)^{-1} \in S \subseteq N$.

Therefore, $s^{-1}y^{-1}$

$= (s^{-1}.1)(y^{-1}.1)$[since N has the unity]

$= (sy)^{-1}(1.1)$[by hypothesis]

$= (sy)^{-1}$[since N has the unity] .

Therefore, $(s_1^{-1}n_1 + s_2^{-1}n_2).s_3^{-1}n_3$

$= (s_1s_2)^{-1}(s_2n_1 + s_1n_2)s_3^{-1}n_3$

$= (s_1s_2s_3)^{-1}(s_2n_1+s_1n_2)n_3$ [since $(N,.)$ is associative]

$= (s_1s_2s_3)^{-1}(s_2n_1n_3+s_1n_2n_3)$

$= (s_1s_2s_3)^{-1}.1.(s_2n_1n_3+s_1n_2n_3)$[since N has the unity]

$= (s_1s_2s_3)^{-1}.s_3^{-1}s_3.(s_2n_1n_3+s_1n_2n_3)$ [by definition of S]

$= (s_1s_3s_2s_3)^{-1}(s_2s_3n_1n_3+s_1s_3n_2n_3)$[since N has the right distributive property and $(N,.)$ is commutative]

$= (s_1s_3)^{-1}(n_1n_3)+(s_2s_3)^{-1}(n_2n_3)$

$= s_1^{-1}n_1.s_3^{-1}n_3+s_2^{-1}n_2.s_3^{-1}n_3.$

The above conditions shows that $S^{-1}N$ is a near-ring, called localized near-ring.

Note that if $h\in H$, then $h^{-1}0=0$, where 0 is the identity of $(N,+)$.

Let S be a multiplicative closed subset of a commutative near-ring N with identity. Then as above $(S^{-1}N,+)$ is a group.

Define a map $S^{-1}N\times S^{-1}E\to S^{-1}E$ by

$(s_1^{-1}n,s_2^{-1}e)\to(s_1s_2)^{-1}(ne)$

i. e. $s_1^{-1}n.s_2^{-1}e=(s_1s_2)^{-1}(ne)$.

Then $(s_1^{-1}n_1+s_2^{-1}n_2)(s^{-1}e)$

$= (s_1s_2)^{-1}(s_2n_1+s_1n_2)s^{-1}e$

$= (s_1s_2s)^{-1}(s_2n_1+s_1n_2)e$

$= (s_1s_2s)^{-1}(s_2n_1e+s_1n_2e)$

$= (s_1s_2)^{-1}s^{-1}(s_2n_1e+s_1n_2e)$ [as $(xy)^{-1}=x^{-1}y^{-1}$]

$= (s_1s_2)^{-1}\ (s^{-1}s_2n_1e+s^{-1}s_1n_2e)$[as $s^{-1}\in N$ and N is commutative]

$= s_1^{-1}(s^{-1}n_1e)+s_2^{-1}(s^{-1}n_2e)$[by hypothesis]

$= (s_1^{-1}s^{-1})(n_1e)+(s_2^{-1}s^{-1})(n_2e)$[since N is commutative]

$= (s_1s)^{-1}\ (n_1e)+(s_2s)^{-1})(n_2e)$ [since $(xy)^{-1}=x^{-1}y^{-1}$]

$= (s_1^{-1}n_1.s^{-1}e)+(s_2^{-1}n_2.s^{-1}e)$[by hypothesis] .

This shows that $S^{-1}E$ is an $S^{-1}N$-group called localized N-group of E or simply $S^{-1}E$ is an N-group.

Definition 4.2.2. *Let S be a multiplicative closed subset of a commutative N with unity. If $A\leq_N E$, then $S^{-1}A$ is called a $S^{-1}N$-subgroup of $S^{-1}E$ if $n(s^{-1}a)=s^{-1}(na)\in S^{-1}A$,* *for some* $s \in S, a \in A, n \in N$.

This $S^{-1}N$*-subgroup is called localized N-subgroup of E or simply* $S^{-1}A$ *is an N-subgroup of E.*

Definition 4.2.3. *Let S be a multiplicative closed subset of a commutative N with unity. For* $I \subseteq N$, $S^{-1}I$ *is an ideal of* $S^{-1}N$ *if* $S^{-1}I$ *is an additive normal subgroup of* $S^{-1}N$ *and* $s_1^{-1}x.s_2^{-1}n$, $s_1^{-1}n_1(s_2^{-1}n_2+s^{-1}x)-s_1^{-1}n_1s_2^{-1}n_2 \in S^{-1}I$, *for some* $s, s_1, s_2 \in S, n, n_1, n_2 \in N, x \in I$.

This ideal $S^{-1}I$ *is called localized ideal of N or simply* $S^{-1}I$ *is an ideal of N.*

Definition 4.2.4. *Let* $P \lhd N$ *be prime without unity.*

Then $S = N \setminus P$ *is multiplicative closed subset of N because if* $d, b \in S$, *then* $db \in N$ *and* $d, b \notin P$.

Also, P is prime, $db \notin P$ *and so* $db \in S$

and $1 \notin P, 1 \in N \Rightarrow 1 \in S$.

Then $S^{-1}N$ *is called localization of N at P and denoted by* N_P.

Therefore, $N_P = (N \setminus P)^{-1}N = S^{-1}N$.

Localization of E at a prime ideal P, $E_P = S^{-1}E = (N \setminus P)^{-1}E$.

Lemma 4.2.1. $SS = S$ *if S is a multiplicative closed subset of N.*

Proof : If $x \in SS$, then $x = s_1s_2 \in S$, for some $s_1, s_2 \in S$.

So, $SS \subseteq S$.

Again, if $s \in S$, then $s = 1.s$ [since N has identity] .

Since $1 \in S$ and S is Multiplicative closed, therefore $s = 1.s \in SS$.

Thus, $S \subseteq SS$ and hence $SS = S$.

Lemma 4.2.2. *Let S be a multiplicative closed subset of N and* $I \lhd N$. *Then* $S^{-1}I \lhd S^{-1}N$.

Proof : Since $I \lhd N$, for any $x, y \in I$ and $n, n_1, n_2 \in N$, $xn \in I, n+x-n \in I$ and $n_1(n_2+x)-n_1n_2 \in I$.

Now, for any $s_1, s_2 \in S$,

$s_1^{-1}x - s_2^{-1}y = (s_1s_2)^{-1}(s_2x - s_1y) \in S^{-1}I$

and $s_1^{-1}n + s_2^{-1}x - s_1^{-1}n = (s_1s_2)^{-1}(s_2n + s_1x) + s_1^{-1}(-n) = (s_1s_2s_1)^{-1}[s_1(s_2n + s_1x) - s_1s_2n]$.

Now, $s_1 \in S \Rightarrow s_1 \in N \Rightarrow s_1x \in I$.

Also, $s_2n \in N$ and so $s_1(s_2n + s_1x) - s_1s_2n \in I$.

Again, $s_1s_2s_1 \in S$ and therefore $(s_1s_2s_1)^{-1}[s_1(s_2n + s_1x) - s_1s_2n] \in S^{-1}I$.

Now, $s_2^{-1}x.s_1^{-1}n = (s_2s_1)^{-1}(xn) \in S^{-1}I$

and $s_1^{-1}n_1(s_2^{-1}n_2 + s^{-1}x) - s_1^{-1}n_1s_2^{-1}n_2$

$= s_1^{-1}n_1[(s_2s)^{-1}(sn_2 + s_2x)] - (s_1s_2)^{-1}(n_1n_2)$

$= (s_1s_2s)^{-1}\ n_1(sn_2 + s_2x) - (s_1s_2)^{-1}(n_1n_2)$

$= (s_1s_2s)^{-1}[n_1(sn_2 + s_2x) - s(n_1n_2)] \in S^{-1}I$ [since $s_2x \in I$] .

This shows that $S^{-1}I$ is an ideal of $S^{-1}N$.

Since by **lemma 2.2.1**, maximal ideal in N with unity is prime ideal. So, $P \in Max(N)$ implies $S = N \setminus P$ is closed subset as shown earlier. Now, utilizing this idea, demonstrate some findings.

Lemma 4.2.3. *Let $X \leq_N E$. Then $X_P \leq_N E_P$, $\forall\ P \in Max(N)$.*

Proof : We have, $X_P = S^{-1}X, E_P = S^{-1}E$.

Since $X \leq_N E$, X is subgroup of E and so $NX \subseteq X$.

Now, $a, b \in X_P$ implies $a = s_1^{-1}x_1, b = s_2^{-1}x_2$, for some $s_1, s_2 \in S, x_1, x_2 \in X$.

So, $a - b = (s_1s_2)^{-1}(s_2x_1 - s_1x_2)$.

Since $s_2x_1, s_1x_2 \in X$ and X is subgroup of E, $s_2x_1 - s_1x_2 \in X$.

Also, $s_1.s_2 \in S$ [since S is multiplicative closed] .

Therefore, $a - b \in S^{-1}X$.

Also, let $y \in N_P.X_P$.

Then $y = nx$, for some $n \in N_P, x \in X_P$.

Therefore, $n = s_1^{-1}n_1$ and $x = s_2^{-1}x_1$, for some $n_1 \in N, x_1 \in X$

$\Rightarrow nx = (s_1s_2)^{-1}(n_1x_1) \in S^{-1}X$.

Thus, $y = nx \in X_P$.

Hence the result.

Lemma 4.2.4. *If $E_P = 0$, $\forall\, P \in Max(N)$, then $\exists\, s \in S = N \setminus P$ such that $se = 0\ \forall\, e \in E$.*

Proof : $E_P = 0 \Rightarrow S^{-1}E = 0$.

So, for any $e \in E\ \exists\, s \in S$ such $s^{-1} \in S$ and $(s^{-1})^{-1}e = 0$[Since S is closed] .

Also $s = (s^{-1})^{-1}$.

Thus the result.

Theorem 4.2.1. *$E = 0$ if and only if $E_P = 0$, $\forall\, P \in Max(N)$.*

Proof : If $E = 0$, $\exists\, s \in (N \setminus P)^{-1}$ such that $se = 0$. So, $E_P = 0$.

Let $e \in E$.

Then, for any $n_1, n_2 \in Ann(e), n \in N$, $(n_1 - n_2)e = n_1e - n_2e = 0$ and $nn_1e = 0$.

So, $n_1 - n_2, nn_1 \in Ann(e)$.

Therefore, $Ann(e) \lhd N$.

Let $Ann(e) \lhd N$ be proper.

Then $\exists\, P \in Max(N)$ such that $Ann(e) \subseteq P$ [since every proper ideal in N is contained in a maximal ideal] .

Since $E_P = 0$, $\exists\, s \in S = N \setminus P$ such that $se = 0$[by **lemma 4.2.4**] .

Therefore, $s \in Ann(e) \subseteq P$

$\Rightarrow s \in P$-which contradicts $s \in N \setminus P$.

Thus, $Ann(e) = N$

$\Rightarrow ne = 0, \forall\, n \in N$.

Since $1 \in N, 1e = 0 \Rightarrow e = 0 \Rightarrow E = 0$.

Corollary 4.2.1. *$A_P = B_P \Rightarrow A = B$, $\forall\, A, B \leq_N E$, $P \in Max(N)$.*

Proof : Let $a \in A$.

Then $s_1^{-1}a \in S^{-1}A$, for some $s_1 \in S = N \setminus P$

$\Rightarrow s_1^{-1}a \in S^{-1}B$[since $A_P = B_P$]

$\Rightarrow s_1^{-1}a = s_2^{-1}b$, for some $s_2 \in S, b \in B$

$\Rightarrow s_1^{-1}a - s_2^{-1}b = 0$

$\Rightarrow (s_1s_2)^{-1}(s_2a - s_1b) = 0$

$\Rightarrow (s_2a - s_1b)_P = 0$

$\Rightarrow (s_2 a - s_1 b) = 0$[by using **theorem 4.2.1**]

$\Rightarrow s_2 a = s_1 b$

$\Rightarrow s_2^{-1} s_2 a = s_2^{-1} s_1 b \in NB$

$\Rightarrow a \in NB \subseteq B.$

Thus, $A \subseteq B$.

Similarly, $B \subseteq A$.

Hence, $A = B$.

Lemma 4.2.5. *If $I \lhd N$, then $I_P \lhd N_P$, $\forall\, P \in Max(N)$*

Proof : As in **lemma 4.2.2** it can be proved the result.

Theorem 4.2.2. *An ideal N-group E is a DN-group if and only if E_P is also a DN-group, $\forall\, P \in Max(N)$.*

Proof : Since E is a DN-group, $(D \cap T) + (K \cap T) = (D + K) \cap T$, $\forall\, D, K, T \leq_N E$.

Now, to show $(D_P + K_P) \cap T_P = (D_P \cap T_P) + (K_P \cap T_P)$, $\forall\, D_P, K_P, T_P \leq_N E\ E_P$.

It is enough to show that, $D_P + K_P = (D + K)_P$ and $D_P \cap K_P = (D \cap K)_P$.

Let $x \in D_P + K_P$

$\Rightarrow x = a + b$, where $a \in D_P, b \in K_P$.

So, $a = s_1^{-1} a_1, b = s_2^{-1} b_1$ for some $a_1 \in D, b_1 \in K$ and $s_1, s_2 \in S$.

Therefore, $x = s_1^{-1} a_1 + s_2^{-1} b_1 = (s_1 s_2)^{-1}(s_2 a_1 + s_1 b_2)$.

Since $D, K \leq_N E$, $s_2 a_1 \in D, s_1 b_2 \in K$.

Also, since S is multiplicative closed, $s_1, s_2 \in S \Rightarrow s_1.s_2 \in S$.

Therefore, $x \in S^{-1}(D + K)$.

$\Rightarrow D_P + K_P \subseteq (D + K)_P$.

Let $y \in (D + K)_P = S^{-1}(D + K)$.

Then, $y = s^{-1}(x + b)$, for some $s \in S, x \in D, b \in K$.

Since $S^{-1}x$ is an ideal, $s^{-1}(x + b) - s^{-1} b \in S^{-1} x$

$\Rightarrow y = s^{-1}(x + b) \in S^{-1}x + S^{-1}b \subseteq S^{-1}D + S^{-1}K = D_P + K_P$.

Therefore, $(D + K)_P \subseteq D_P + K_P$.

Thus, $D_P + K_P = (D + K)_P$.

Again, let $x \in D_P \cap K_P = S^{-1}D \cap S^{-1}K$.

Then, $x = s_1^{-1}a = s_2^{-1}b$, for some $s_1, s_2 \in S, a \in D, b \in K$.

Since $s_1^{-1}, s_2^{-1} \in S = N \setminus P$ and $D, K \leq_N E$, $s_1^{-1}a \in D, s_2^{-1}b \in K$.

Therefore, $x \in D \cap K$.

Since $D \cap K \leq_N E$ and $S = N \setminus P \subseteq N$, $S(D \cap K) \subseteq D \cap K$ and so $D \cap K \subseteq S^{-1}(D \cap K)$.

Therefore, $x \in S^{-1}(D \cap K) = (D \cap K)_P$.

$\Rightarrow D_P \cap K_p \subseteq (D \cap K)_P$.

So, $D_P \cap K_P = (D \cap K)_P$.

Thus, $(D_P + K_P) \cap T_P = (D+K)_P \cap T_P = [(D+K) \cap T]_P = [(D \cap T) + (K \cap T)]_P$

$= (D \cap T)_P + (K \cap T)_P = (D_P \cap T_P) + (K_P \cap T_P)$.

But by **lemma 4.2.3**, $D_P, K_P, T_P \leq_N E_P$.

Hence E_P is a DN-group.

Let E_P be DN-group, then for any $D, K, T \leq_N E$,

$(D_P + K_P) \cap T_P = (D_P \cap T_P) + (K_P \cap T_P)$

$\Rightarrow ((D+K) \cap T)_P = ((D \cap T) + (K \cap T))_P$

$\Rightarrow (D+K) \cap T = (D \cap T) + (K \cap T)$[by **corollary 4.2.1**] .

This shows that E is a DN-group.

Proposition 4.2.1. *An ideal N-group E is generated finitely if and only if E_P is generated finitely, $\forall\ P \in Max(N)$.*

Proof : Let $e_p \in E_P = S^{-1}E = (N \setminus P)^{-1}E$.

Then $e_P = s^{-1}e$, for some $s \in S, e \in E$.

Since E generated finitely, $e = n_1e_1 + n_2e_2 + \cdots + n_ne_n$, where $n_i \in N, e_i \in E, i = 1, 2, ..n$.

Since $S^{-1}n_1e_1$ is an ideal, $s^{-1}(n_1e_1 + n_2e_2) - s^{-1}n_2e_2 \in S^{-1}n_1e_1$

$\Rightarrow s^{-1}(n_1e_1 + n_2e_2) = s_1^{-1}n_1e_1 + s^{-1}n_2e_2$, for some $s_1 \in S$

$\Rightarrow s^{-1}(n_1e_1 + n_2e_2) = n_1(s_1^{-1}e_1) + n_2(s_2^{-1}e_2)$[since N is commutative] , where $s = s_2$.

In the same way, it can be extended to a finite number n of steps, i. e

$e_P = s^{-1}e = s^{-1}(n_1e_1 + n_2e_2 + n_3e_3 + \cdots + n_ne_n) = n_1(s_1^{-1}e_1) + n_2(s_2^{-1}e_2) + n_3(s_3^{-1}e_3) + \ldots n_n(s_n^{-1}e_n)$, where $s_i \in S, e_i \in E$ and $n_i \in N$, for $i = 1, 2, 3, \ldots n$.

This shows that E_P generated finitely.

Lemma 4.2.6. *If E_P is a cyclic N-group, then $\frac{N_P}{I_P} \cong E_P$, for some $P \in Max(N)$.*

Proof : Let E_P be cyclic N-group generated by e_p.

Now, let us define a function $\phi : N_P \to E_P$ by

$\phi(n_p) = (ne)_p$, where $n \in N, e \in E$.

i. e.$\phi(s^{-1}n) = s^{-1}(ne)$,where $s \in S = N \setminus P$.

Clearly, ϕ is well defined and onto.

For any $m_p, n_p \in N_P$,

$\phi(m_P + n_p)$

$= \phi(s_1^{-1}m + s_2^{-1}n)$, where $s_1, s_2 \in S$

$= \phi((s_1s_2)^{-1})(s_2m + s_1n))$

$= (s_1s_2)^{-1}((s_2m + s_1n)e)$

$= (s_1s_2)^{-1}(s_2me + s_1ne)$

$= s_1^{-1}(me) + s_2^{-1}(ne)$

$= (me)_p + (ne)_p$

$= \phi(m_p) + \phi(n_p)$.

Also, for any $n_p \in N_p, x_p \in E_p$,

$\phi(n_p x_p)$

$= \phi(s_1^{-1}n.s_2^{-1}x)$, where $s_1, s_2 \in S$

$= \phi((s_1s_2)^{-1}(nx))$

$= (s_1s_2)^{-1}(nxe)$

$= s_1^{-1}(n)s_2^{-1}(xe) = n_p\phi(x_p)$.

Therefore, $\frac{N_P}{ker\phi} \cong E_P$.

Since $ker\phi$ is an ideal, taking $ker\phi = I_p$,

$\frac{N_P}{I_P} \cong E_P$, where I_p is an ideal of N_P.

4.3 Multiplication N-groups

Definition 4.3.1. *N is referred to as arithmetical if N is considered as an N-group is a DN-group or* $N_P = (N \setminus P)^{-1}N$ *is uniserial,* $\forall\, P \in Max(N)$.

Example 4.3.1. *If* $E = N = \{0, s, b, m\}$ *is the Klein's* 4*-group given by the following table:-*

.	0	s	b	m
0	0	0	0	0
s	0	0	s	s
b	0	s	b	b
m	0	s	m	m

+	0	s	b	m
0	0	s	b	m
s	s	0	m	b
b	b	m	0	s
m	m	b	s	0

Then $(E, +, .)$ *is a near-ring and N-group over itself.*
$P = \{0\}, L = \{0, s\}$, $E \leq_N E$ *as* $NP = P, NL = L$ *and* $NN = N$ *such that* $P \subset L \subset N$.
We have, $P+L = L+P = L, P+E = E+P = E, L+E = E+L = E, P+P = P$.
and $(P+L)\cap E = L = (P\cap E)+(L\cap E), (P+E)\cap L = L = (P\cap L)+(E\cap L), (L+E)\cap P = P = (L\cap P)+(E\cap P), (L+P)\cap E = L = (L\cap E)+(P\cap E), ((E+P)\cap L = L = (E\cap L)+(P\cap L), (E+L)\cap P = P = (E\cap P)+(L\cap P)$.
Thus E is a DN-group and hence E is arithmetical.

Definition 4.3.2. *If an N-subgroup A of E has the form IE for some* $I \lhd N$*, it is referred to as multiplication.*

Definition 4.3.3. *E is referred to as a multiplication N-group if A is multiplication* $\forall A \leq_N E$.

Example 4.3.2. ***Example of a multiplication*** N***-group.***

Let $N = (E, +, .) = \{0, s, b, k\}$ *be the Klein's* 4*-groups under the operations given below-*

.	0	s	b	k
0	0	0	0	0
s	0	0	s	s
b	0	s	k	b
k	0	s	b	k

+	0	s	b	k
0	0	s	b	k
s	s	0	k	b
b	b	k	0	s
k	k	b	s	0

Then $(N,+,.)$ is a near-ring as well as N-group over itself.

We have, $D=\{0\}, K=\{0,s\}, E\leq_N E$.

Also, D,K, $N\lhd N$ such that $D=DE, K=KE$ and $E=NE$.

Thus E is a multiplication N-group.

Theorem 4.3.1. *If $K\lhd N$ such that $K\subseteq J(N)$ and E is a multiplication N-group, then $KE=0$ implies $E=0$.*

Proof : Let $x\in E$.

Since E is a multiplication N-group, therefore by definition of multiplication N-group, $Nx=JE$, for some $J\lhd N$[since Nx is a principal N-subgroup] .

Now, $KE=0$

$\Rightarrow JKE=0$

$\Rightarrow KJE=0$[since N is commutative]

$\Rightarrow KNx=0$.

Since $x\in Nx, ax=0 \forall a\in K$

$\Rightarrow a^{-1}ax=0$[since $a\in K\subseteq J(N)$]

$\Rightarrow x=0$[since E is unitary]

$\Rightarrow E=0$.

Definition 4.3.4. $(A_P:E_P)=\{n_P\in N_P: n_PE_P\subseteq A_P\}$*, for any* $A_P\leq_N E_P$.

Definition 4.3.5. *An $I_P\leq_N N_P$ is called an ideal of N_P if $x_P-y_P\in I_P, n_P+x_P-n_P\in I_P, n_P(n'_P+y_P)-n_Pn'_P\in I_P, \forall x_P,y_P\in I_P, n_P,n'_P\in N_P$.*

Definition 4.3.6. *E_P is referred to as a multiplication N-group if for every $A_P\leq_N E_P$, $A_P=I_PE_P$, for some $I_P\lhd N_P$.*

Theorem 4.3.2. *Every cyclic localized N-group is a localized multiplication N-group.*

Proof : Let E_P be cyclic generated by e_P, for some $e\in E$.

Let $A_P\leq_N E_P$.

Now, $(A_P:E_P)=\{n_P\in N_P: n_PE_P\subseteq A_P\}$.

So, $(A_P:E_P)E_P\subseteq A_P$.

Let $a_P \in A_P \subseteq E_P$

$\Rightarrow a_P = n_P e_P$, for some $n \in N$.

Now, for any $m_P \in E_P$,

$n_P m_P$

$= n_P n'_P e_P$, for some $n' \in N$

$= n'_P(n_P e_P)$[since N is commutative]

$= n'_P a_P \in N_P A_P \subseteq A_P$[since $A_P \leq_N E_P$] .

Therefore, $n_P E_P \subseteq A_P$

$\Rightarrow n_P \in (A_P : E_P)$

$\Rightarrow a_P \in (A_P : E_P)E_P$

$\Rightarrow A_P \subseteq (A_P : E_P)E_P$.

Therefore, $A_P = (A_P : E_P)E_P$.

Let $x_P, y_P \in (A_P : E_P)$ and $n_P, n'_P \in N_P$.

Since $A_P \leq_N E_P$, for any $e \in E$, $(x_P - y_P)e_P = x_P e_P - y_P e_P \in A_P$.

Therefore, $(x_P - y_P)E_P \subseteq A_P$

$\Rightarrow x_P - y_P \in (A_P : E_P)$.

Since N is commutative $n_P + x_P - n_P = x_P \in (A_P : E_P)$ and $n_P(n'_P + y_P) - n_P n'_P = n_P y_P$.

But, for any $e \in E$,

$(n_P y_P)e_p = (y_P n_P)e_P = y_P(n_P e_P) \in y_P E_P \subseteq A_P$.

Therefore, $n_P y_P \in (A_P : E_P)$.

Thus $(A_P : E_P)$ is an ideal of N_P and hence E_P is a multiplication N-group.

Theorem 4.3.3. *Every localized multiplication N-group over local N is cyclic.*

Proof : Let E_P be multiplication N-group over local N.

Therefore, $E_P = I_P E_P$, for some $I_P \lhd N_P$

$\Rightarrow E_P = I_P E_P \subseteq N_P E_P \subseteq E_P$.

Therefore, $E_P = N_P E_P$.

So, for any $e \in E$,

$N_P e_P \subseteq N_P E_P = E_P$

$\Rightarrow N_P e_P \subseteq E_P$.

Let $e_P \in E_P$ and $a \in N$.

Since N is local, a or $1-a$ is invertible in it.

If a is invertible, then

$a_P e_P \in N_P E_P$

$\Rightarrow a_P e_P = n_p e_P$, for some $n \in N$

$\Rightarrow (s_1^{-1}a)(s_2^{-1}e) = (s_3^{-1}n)(s_4^{-1}e)$, for some $s_1, s_2, s_3, s_4 \in S, n \in N$

$\Rightarrow (s_1 s_2)^{-1}(ae) = (s_3 s_4)^{-1}(ne)$

$\Rightarrow a^{-1}(s_1 s_2)^{-1}(ae) = a^{-1}(s_3 s_4)^{-1}(ne)$

$\Rightarrow (s_1 s_2)^{-1}(a^{-1}(ae)) = (s_3 s_4)^{-1}(a^{-1}(ne))$[since N is commutative]

$\Rightarrow (s_1 s_2)^{-1}(e) = (s_3 s_4)^{-1}((a^{-1}n)e)$

$\Rightarrow (s_1 s_2)^{-1}(e) = (s_3^{-1}(a^{-1}n))(s_4^{-1}e)$

$\Rightarrow e_p \in N_P e_P$

$\Rightarrow E_P \subseteq N_P e_P$.

Thus, $E_P = N_P e_P$ and hence E_P is cyclic.

Theorem 4.3.4. *Every localized multiplication N-group is also multiplication N-group.*

Proof :

Let $M' \leq_N S^{-1}E = E_P$.

Then $\exists\, M \leq_N E$ such that $M' = S^{-1}M$.

Since E is a multiplication N-group, $M = IE$, for some $I \lhd N$.

Then $M' = S^{-1}(IE) = (SS)^{-1}(IE)$[using **lemma 4.2.1**] .

So, $M' = (S^{-1}I)(S^{-1}E)$.

Also, by **lemma 4.2.2**, $S^{-1}I$ is an ideal of $S^{-1}N$.

Thus the result.

Corollary 4.3.1. *Since every multiplication N-group over local N is cyclic and the localized N-group of a multiplication N-group is also a multiplication N-group, every localized multiplication N-group over local N is cyclic.*

Theorem 4.3.5. *If E is generated finitely, then E is multiplication if and only if E_P is multiplication, $\forall\ P \in Max(N)$.*

Proof : Let E be multiplication.

So, by **theorem 4.3.4**, E_P is a multiplication N-group.

Conversely, let E_P be multiplication N-group.

Let $X \leq_N E$.

Then $X_P = I_P.E_P$, for some ideal I_P of N_P.

Therefore, $X_P = S^{-1}I.S^{-1}E = (SS)^{-1}(IE) = S^{-1}(IE) = (IE)_P$ [since $SS = S$] .

So, by **corollary 4.2.1**, $X = IE$.

Hence E is a multiplication N-group.

Theorem 4.3.6. *If $Ann(E) \subseteq P_i$ only, $P_i \in Max(N)$ such that each principal N-subgroup is an ideal and E_{P_i} is cyclic, then E_P is a multiplication N-group, for $i = 1, 2 \ldots n$.*

Proof : Since E_{P_i} is cyclic, $E_{P_i} = (Ne_i)_{P_i}$, where $e_i \in E$, $i = 1, 2, 3, \ldots n$.

Let us choose $b_i \in (\bigcap_{i \neq j} P_i) \setminus P_i$, i $\neq j$, $i = 1, 2 \ldots n$.

Let $X \leq_N E$ be cyclic and generated by $x = \sum_{i=1}^{n} b_i e_i$.

Now, $E_{P_1} = (Ne_1)_{P_1}$

$\Rightarrow (N \setminus P_1)^{-1}E = (N \setminus P_1)^{-1}(Ne_1)$

$\Rightarrow (N \setminus P_1)E = Ne_1$ [since $(N \setminus P_1)N \subseteq N$]

$\Rightarrow s_i e_i = n_i e_1$, for some $s_i \in N \setminus P_1, n_i \in N$.

Let $s = s_1 s_2 s_3 \ldots s_n$ and $s'_i = s_1 s_2 s_3 \ldots s_{i-1} s_{i+1} \ldots s_n$ such that $s = s_i s'_i$.

Therefore, $sx = s(b_1 e_1 + b_2 e_2 + \ldots b_n e_n)$.

Now, $s(b_1 e_1 + b_2 e_2 + \ldots b_n e_n) - s(b_2 e_2 + \ldots b_n e_n) \in S b_1 e_1$

$\Rightarrow sx - s(b_2 e_2 + \ldots b_n e_n) = s' b_1 e_1$,for some $s' \in S$

$\Rightarrow (ss)^{-1}[sx - s(b_2 e_2 + \ldots b_n e_n)] = (ss)^{-1}(s' b_1 e_1)$

$\Rightarrow (ss)^{-1}[sx - s(b_2 e_2 + \ldots b_n e_n)] = (ss)^{-1}(s' b_1 e_1)$

$\Rightarrow s^{-1}(x) - s^{-1}(b_2 e_2 + \ldots b_n e_n)] = (ss)^{-1}(s' b_1 e_1)$.

Since $s' b_1 e_1 \in E, ss \in S, x \in X, b_2 e_2 + \ldots b_n e_n \in J(N)E$, therefore $E_P = X_P - (J(N)E)_P$.

Since $Ann(E) \subseteq P_i$ only, $Ann(E) \nsubseteq P$, $\forall\ P \in Max(N)$.

So, $\exists\, s \in Ann(E)$, but $s \notin P$

$\Rightarrow sE = 0$, for $s \in N$, but $s \notin P$

$\Rightarrow E_P = 0$[since $s \in N \setminus P = S$].

So, $X_P = (J(N)E)_P$

$\Rightarrow X_P = S^{-1}(J(N)E)$

$\Rightarrow X_P = (SS)^{-1}(J(N)E)$[since SS=S]

$\Rightarrow X_P = (S^{-1}J(N)).(S^{-1}E)$

$\Rightarrow X_P = J(N)_P.E_p$.

By **lemma 4.2.5**, $J(N)_P \lhd N_P$[since $J(N) \lhd N$] .

This shows that E_P is a multiplication N-group.

Definition 4.3.7. *E is called locally cyclic if E_P is cyclic, $\forall\, P \in Max(N)$.*

Theorem 4.3.7. *If E is generated finitely on local N, then E is multiplication if and only if it is locally cyclic N-group.*

Proof : If E is a multiplication which generated finitely on local N, then by **theorem 4.3.5**, E_P is multiplication N-group, $\forall\, P \in Max(N)$.

Since N is local, E_P is cyclic N-group $\forall P \in Max(N)$[by **theorem4.3.3**] .

So, by definition E is locally cyclic N-group.

Conversely, suppose E is locally cyclic N-group.

Then E_P is cyclic, $\forall\, P \in Max(N)$.

So, E_P is multiplication N-group, $\forall\, P \in Max(N)$[by **theorem 4.3.2**] and so E is multiplication [by **theorem 4.3.5**] .

Definition 4.3.8. *Every N-subgroup of E is referred to as a principal ideal N-group if it is both principal and ideal.*

Theorem 4.3.8. *If E is principal DN-group over local N and every localized DN-group over local N is uniserial, then E is multiplication N-group.*

Proof : Since E is a principal DN-group, every N-subgroup is principal and so generated finitely. Let $M \leq_N E$ generated finitely.

Since by **lemma 2.2.2**, N-subgroups of a DN-group are also ideal DN-groups, M is a DN-group.

Therefore, by **theorem** 4.2.2, M_P is a DN-group.

Since N is a local, M_P is an uniserial N-group[by hypothesis] .

Since M generated finitely, M_P is also generated finitely[by **proposition 4.2.1**] .

So, for $m_P \in M_p$,

$m_P = n_{1p}e_{1p} + n_{2p}e_{2p} + n_{3p}e_{3p} + \cdots + n_{np}e_{np}$, where $n_{iP} \in N_P, e_{iP} \in M_P$

$\Rightarrow m_P \in N_p e_{1P} + N_p e_{2P} + N_p e_{3P} + \cdots + N_p e_{nP}$.

Since M_P is uniserial, so any two of its N-subgroups are comparable, it may assume

$N_p e_{1P} \subseteq N_p e_{2P} \subseteq N_p e_{3P} \subseteq \cdots \subseteq N_p e_{nP}$.

Therefore, $m_P \in N_P e_{nP}$

$\Rightarrow M_P \subseteq N_P e_{nP}$.

Since M_P is N-subgroup and $e_{nP} \in M_P, N_P e_{nP} \subseteq M_P$.

Therefore, $M_P = N_P e_{nP}$

$\Rightarrow M_P$ is cyclic

$\Rightarrow M$ is locally cyclic.

So, by **theorem 4.3.7**, M is a multiplication N-group and hence E is multiplication.

Definition 4.3.9. *A local near-ring is referred to as convey if it is strongly regular.*

Theorem 4.3.9. *If E is an ideal DN-group that generated finitely over a convey N with inverse property and every ideal DN-group over a strongly regular near-ring is Bezout, then E is a multiplication N-group.*

Proof : Let $M \leq_N E$. Since E generated finitely, M generated finitely.

Since N-subgroups of an ideal DN-group are also ideal DN-group, M is an ideal DN-group.

So, by **theorem 4.2.2**, M_P is also DN-group.

Since M generated finitely, M_P is also generated finitely [by **proposition 4.2.1**] .

Since N is convey, M_P is an ideal DN-group over a strongly regular near-ring.

By hypothesis, M_P is a Bezout N-group.

So, every generated finitely N-subgroup is cyclic.

Since M_P generated finitely, M_P is cyclic.

So, by definition M is locally cyclic.

Thus by **theorem 4.3.7**, M is a multiplication N-group.

Hence E is multiplication.

Proposition 4.3.1. *If N is an arithmetical, then $\forall\ Q \in Max(N)$, $\frac{N_Q}{I_Q}$ is an uniserial N-group.*

Proof : Let N be an arithmetical.

Then by definition, N_Q is uniserial, $\forall\ Q \in Max(N)$.

Now, to show for any sub factors (ideals of $\frac{N_Q}{I_Q}$) $\bar{X_Q} = \frac{N_Q}{I_{1p}}$ and $\bar{Y_Q} = \frac{N_Q}{I_{2p}}$,$\bar{X_Q} \subseteq \bar{Y_Q}$ or $\bar{Y_Q} \subseteq \bar{X_Q}$. Since I_{1p}, I_{2p} are ideals of the uniserial N-group N_Q, $I_{1p} \subseteq I_{2p}$ or $I_{2p} \subseteq I_{1p}$.

Let $\bar{a} \in \bar{X_Q}$.

Then $a \in I_{1p}$

$\Rightarrow a \in I_{2p}$

$\Rightarrow \bar{a} \in \bar{Y_Q}$.

Therefore, $\bar{X_Q} \subseteq \bar{Y_Q}$.

Thus, if $I_{1p} \subseteq I_{2p}$, then $\bar{X_Q} \subseteq \bar{Y_Q}$.

Similarly, if $I_{2p} \subseteq I_{1p}$, then $\bar{Y_Q} \subseteq \bar{X_Q}$.

This shows the result.

Theorem 4.3.10. *If $E$is a multiplication ideal N-group that generated finitely and N is an arithmetical local near-ring, then E is a DN-group.*

Proof : E_P is also multiplication N-group as E is a multiplication N-group[by **theorem 4.3.4**] .

Also, by **theorem 4.3.3**, E_P is cyclic.

So, by **lemma 4.2.6**, $\frac{N_P}{I_P} \cong E_P\ \forall P \in Max(N)$.

Since N is a arithmetical local, $\frac{N_P}{I_P}$ is an uniserial N-group[by **proposition 4.3.1**] .

So, E_P is an uniserial N-group

$\Rightarrow E_P$ is a DN-group[since uniserial N-group is DN-group]

$\Rightarrow E$ is a DN-group[by **theorem 4.2.2**] .

4.4 Conclusion

To extend the notions of near -ring groups to localized near-ring groups and multiplication modules to multiplication near-ring groups, are the prime objectives of this chapter. Defining related definitions of localized N-groups and multiplication N-groups, some lemmas and theorems are derived. The theorem 4.2.2 describes the relationship between DN-groups and localized DN-groups. Theorems 4.3.4 to 4.3.10 demonstrate the connection among DN-groups, multiplication N-groups and localized multiplication N-groups. The results obtained here will be used in the subsequent chapters.