Chapter 5
Intuitionistic fuzzy aspects of DN-groups and weak DN-groups

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Intuitionistic Fuzzy aspects of DN-groups and weak DN-groups

5.1 Introduction

Hadi and Semeein [9] introduced and investigated fuzzy distributive modules as an expanded idea of distributive modules. They established the result between the fuzzy distributive module and the distributive of fuzzy singleton. Sharma [81] has studied the notions of intuitionistic fuzzy modules in many ways. Introducing intuitionistic fuzzy submodules of a module, Sharma proved that the intersection of two fuzzy submodules of a module is also a fuzzy submodule. Davvaz, Wieslaw and Jun [82] presented intuitionistic fuzzy submodules of modules and utilizing this notion, numerous scholars such as Isaac [83], Rahman [84], and Sharma [85] investigated it in many ways. They derived different operations on intuitionistic fuzzy sets, intuitionistic (T,S)-fuzzy submodule of a module, intuitionistic fuzzy H_v-submodules of a module and the relationship between fuzzy submodule, and intuitionistic fuzzy submodule of a module. Devi [86] developed the concept of intuitionistic fuzzy N-subgroups and ideals of N-group and discussed the association between intuitionistic fuzzy ideals and fuzzy ideals, which is being investigated by several scholars in various fields.

In **Chapter 4**, DN-groups, uniserial N-groups and Bezout N-groups are discussed thoroughly. In this chapter, the concept of DN-groups extended to intuitionistic fuzzy and intuitionistic fuzzy weak DN-groups and uniserial N-groups to intuitionistic fuzzy uniserial N-groups.

The core concepts employed in this chapter are found in [28, 6, 7]. Here, in general, $\gamma, \lambda \in [0,1]$ with $\gamma + \lambda \leq 1$.

Introducing the notion of IF DN-groups, the associated outcomes of the (γ, λ)-cut of intuitionistic fuzzy sets, intuitionistic fuzzy DN-groups and intuitionistic fuzzy weak DN-groups are examined. Finally, the correlations among the intuitionistic fuzzy concepts of uniserial N-groups, DN-groups and weak DN-groups are derived.

5.2 Intuitonistic fuzzy DN-groups

Definition 5.2.1. *E is called an IF DN-group if $(P+H)\cap C = (P\cap C)+(H\cap C)$, $\forall$ IF N-subgroups P, H, C of E.*

Definition 5.2.2. *Let $B = <\phi_B, \psi_B>$ be an IF N-subgroup, then ${}^{(\gamma,\lambda)}B$ is called an N-subgroup of ${}^{(\gamma,\lambda)}E$ if $s-y, ns \in {}^{(\gamma,\lambda)}B$, for any $s, y \in {}^{(\gamma,\lambda)}B$ and $n \in N$.*

Definition 5.2.3. *Let $B = <\phi_B, \psi_B>$ be an IF N-subgroup, then ${}^{(\gamma,\lambda)}B$ is said to be an ideal of ${}^{(\gamma,\lambda)}E$ if $p-q, m+p-m \in E$ and $n(p+m)-nm \in {}^{(\gamma,\lambda)}B$, for any $p, q \in {}^{(\gamma,\lambda)}B, m \in E$, $n \in N$.*

Lemma 5.2.1. *Let E be a commutative N-group and every principal N-subgroup is an ideal, then the sum of two subgroups of ${}^{(\gamma,\lambda)}E$ is also a subgroup.*

Proof : Let ${}^{(\gamma,\lambda)}P$ and ${}^{(\gamma,\lambda)}B$ be subgroups of ${}^{(\gamma,\lambda)}E$.

Let $p,q \in {}^{(\gamma,\lambda)}P + {}^{(\gamma,\lambda)}B$.

Then $p = s_1 + y_1, q = s_2 + y_2$, for some $s_1, s_2 \in {}^{(\gamma,\lambda)}P, y_1, y_2 \in {}^{(\gamma,\lambda)}B$.

Since E is commutative and ${}^{(\gamma,\lambda)}P, {}^{(\gamma,\lambda)}B \subseteq E$,

$p - q = (s_1 - s_2) + (y_1 - y_2) \in {}^{(\gamma,\lambda)}P + {}^{(\gamma,\lambda)}B$.

Since every principal N-subgroup is an ideal,

$n(s_1 + y_1) - ny_1 \in Ns_1$, for $n \in N$.

Also, $ny_1 \in Ny_1$.

So, $n(s_1 + y_1) - ny_1 + ny_1 \in Ns_1 + Ny_1$.

$\Rightarrow np = n(s_1 + y_1) \in Ns_1 + Ny_1 \in N^{(\gamma,\lambda)}P + N^{(\gamma,\lambda)}B \subseteq {}^{(\gamma,\lambda)}P + {}^{(\gamma,\lambda)}B$.

Thus the result.

Proposition 5.2.1. *If $P =< \phi_P, \psi_P >$ and $B =< \phi_B, \psi_B >$ are IF N-subgroups of E, then*

i. ${}^{(\gamma,\lambda)}(P+B) = {}^{(\gamma,\lambda)}P + {}^{(\gamma,\lambda)}B$ *with* $\gamma + \lambda = 1$.

ii. ${}^{(\gamma,\lambda)}(P \cap B) = {}^{(\gamma,\lambda)}P \cap {}^{(\gamma,\lambda)}B$.

Proof : (i). Let $s \in {}^{(\gamma,\lambda)}(P+B)$.

Then $\phi_{P+B}(s) \geq \gamma$ and $\psi_{P+B}(s) \leq \lambda$.

Now, $\phi_{P+B}(s) \geq \gamma$

$\Rightarrow \vee\{\phi_P(r) \wedge \phi_B(t) : r,t \in E, s = r+t\} \geq \gamma$

$\Rightarrow \phi_P(r_1) \wedge \phi_B(t_1) \geq \gamma$, for some $r,t \in E$ such that $s = r_1 + t_1$.

Since $\phi_P(r_1), \phi_B(t_1) \geq \phi_P(r_1) \wedge \phi_B(t_1) \geq \gamma$

$\Rightarrow \phi_P(r_1) \geq \gamma$ and $\phi_B(t_1) \geq \gamma$, for some $r_1, t_1 \in E$ such that $s = r_1 + t_1$

$\Rightarrow \psi_P(r_1) \leq 1 - \phi_P(r_1) = 1 - \gamma = \lambda$ and $\psi_B(t_1) \leq 1 - \phi_B(t_1) = 1 - \gamma = \lambda$ [Since $\gamma + \lambda = 1$]

$\Rightarrow r_1 \in {}^{(\gamma,\lambda)}P$ and $t_1 \in {}^{(\gamma,\lambda)}B$

$\Rightarrow s = r_1 + t_1 \in {}^{(\gamma,\lambda)}P + {}^{(\gamma,\lambda)}B$.

Converselr, let $s \in {}^{(\gamma,\lambda)}P + {}^{(\gamma,\lambda)}B$.

Then $s = r + t$, for some $r \in {}^{(\gamma,\lambda)}P, t \in {}^{(\gamma,\lambda)}B$

$\Rightarrow \phi_P(r) \geq \gamma, \psi_P(r) \leq \lambda, \phi_B(r) \geq \gamma, \psi_B(r) \leq \lambda$

$\Rightarrow \phi_P(r) \wedge \phi_B(r) \geq \gamma$ and $\psi_P(r) \vee \psi_B(r) \leq \lambda$

$\Rightarrow \vee\{\phi_P(r) \wedge \phi_B(r) \geq \gamma : s = r + t\}$ and $\wedge\{\psi_P(r) \vee \psi_B(r) \leq \lambda : s = r + t\}$

$\Rightarrow s \in {}^{(\gamma,\lambda)}(P + B)$.

Thus the result.

(ii). Let $s \in {}^{(\gamma,\lambda)}(P \cap B)$

$\Rightarrow \phi_{P\cap B}(s) \geq \gamma$ and $\psi_{P\cap B}(s) \leq \lambda$.

But $P \cap B \subseteq P, B$.

So $\phi_P(s), \phi_B(s) \geq \phi_{P\cap B}(s) \geq \gamma$ and $\psi_P(s), \psi_B(s) \leq \psi_{P\cap B}(s) \leq \lambda$.

Thus, $s \in {}^{(\gamma,\lambda)}P$ and $s \in {}^{(\gamma,\lambda)}B$

$\Rightarrow s \in {}^{(\gamma,\lambda)}P \cap {}^{(\gamma,\lambda)}B$.

Conversely, let $s \in {}^{(\gamma,\lambda)}P \cap {}^{(\gamma,\lambda)}B$

$\Rightarrow s \in {}^{(\gamma,\lambda)}P$ and $s \in {}^{(\gamma,\lambda)}B$

$\Rightarrow \phi_P(s) \geq \gamma, \psi_P(s) \leq \lambda$ and $\phi_B(s) \geq \gamma, \psi_B(s) \leq \lambda$

$\Rightarrow \phi_P(s) \wedge \phi_B(s) \geq \gamma$ and $\psi_P(s) \vee \psi_B(s) \leq \lambda$

$\Rightarrow \phi_{P\cap B}(s) \geq \gamma$ and $\psi_{P\cap B}(s) \leq \lambda$

$\Rightarrow s \in {}^{(\gamma,\lambda)}(P \cap B)$.

Thus the result.

Definition 5.2.4. *If $P = <\phi_P, \psi_P>$ and $B = <\phi_B, \psi_B>$ are IF N-subgroups of E, then ${}^{(\gamma,\lambda)}P = {}^{(\gamma,\lambda)}B$ if and only if $\phi_P(s) = \phi_B(s) \geq \gamma$ and $\psi_P(s) = \psi_B(s) \leq \lambda$, $\forall\ s \in E$.*

Lemma 5.2.2. *If $P = <\phi_P, \psi_P>$ and $M = <\phi_M, \psi_M>$ are IF N-subgroups of E, then $P = M$ if and only if ${}^{(\gamma,\lambda)}P = {}^{(\gamma,\lambda)}M$.*

Proof : It is clear from the definition.

Proposition 5.2.2. *If $L = <\phi_L, \psi_L>$ is an IF N-subgroup of E, then ${}^{(\gamma,\lambda)}L$ is also an N-subgroup of ${}^{(\gamma,\lambda)}E$.*

Proof : By definition, ${}^{(\gamma,\lambda)}L$ is a subset of ${}^{(\gamma,\lambda)}E$.

For any $n \in N$, $s, y \in {}^{(\gamma,\lambda)}L$,

$\phi_L(s), \phi_L(y) \geq \gamma, \psi_L(s), \psi_L(y) \leq \lambda$.

Therefore, $\phi_L(ns) \geq \phi_L(s) \geq \gamma$ and $\psi_L(ns) \leq \psi_L(s) \leq \lambda$[since L is an IF N-subgroup].

Also, $ns \in E$.

Therefore, $ns \in {}^{(\gamma,\lambda)}L$.

Again, $s - y \in E$ such that $\phi_L(s-y) \geq \phi_L(s) \wedge \phi_L(y) \geq \gamma \wedge \gamma = \gamma$ and $\psi_L(s-y) \leq \psi_L(s) \vee \psi_L(y) \leq \lambda \vee \lambda = \lambda$.

So, $s - y \in {}^{(\gamma,\lambda)}L$.

This shows that ${}^{(\gamma,\lambda)}L$ is an N-subgroup of ${}^{(\gamma,\lambda)}E$.

Theorem 5.2.1. *E is an IF DN-group if and only if ${}^{(\gamma,\lambda)}E$ is a DN-group with $\gamma, \lambda \in (0,1], \gamma + \lambda = 1$.*

Proof : Let E be an IF DN-group and X, I, L be N-subgroups of ${}^{(\gamma,\lambda)}E$ such that

$$\phi_P(h) = \begin{cases} 1, & h \in X \\ 0, & h \notin X \end{cases}, \ \psi_P(h) = \begin{cases} 0, & h \in X \\ 1, & h \notin X \end{cases},$$

$$\phi_M(h) = \begin{cases} 1, & h \in I \\ 0, & h \notin I \end{cases}, \ \psi_M(h) = \begin{cases} 0, & h \in I \\ 1, & h \notin I \end{cases},$$

$$\phi_T(h) = \begin{cases} 1, & h \in L \\ 0, & h \notin L \end{cases}, \ \psi_T(h) = \begin{cases} 0, & h \in L \\ 1, & h \notin L \end{cases}$$

where $\gamma, \lambda \in (0,1], \gamma + \lambda = 1$.

Now, if $s \in {}^{(\gamma,\lambda)}P$, then either $s \in X$ or $s \notin X$.

If $s \notin X$, then $\phi_P(s) = 0 \geq \gamma$ and $\psi_P(s) = 1 \leq \lambda$[since $\phi_P(s) \geq \gamma, \psi_P(s) \leq \lambda$]- which is a contradiction as $\gamma, \lambda \in (0,1]$.

Thus, ${}^{(\gamma,\lambda)}P = X$.

Similarly, ${}^{(\gamma,\lambda)}M = I, {}^{(\gamma,\lambda)}T = L$.

Claim P, M, T are IF N-subgroups of E.

Since X is a subgroup of $(E,+)$, for any $s, y \in X, n \in N$,

$s - y \in X$ and $ns \in X$

$\Rightarrow \phi_P(s-y) = 1 = \phi_P(s) \wedge \phi_P(y)$ and $\phi_P(ns) = 1 = \phi_P(s)$ [since $s, y \in X, \phi_P(s) =$

$1, \phi_P(y) = 1]$.

Similarly, $\psi_P(s-y) = \psi_P(s) \vee \psi_P(y)$ and $\psi_P(ns) = \psi_P(s)$.

This shows that P ia an IF N-subgroup.

Similarly, M and T are IF N-subgroups of E.

Since E is an IF DN-subgroup, therefore

$(P+M) \cap T = (P \cap T) + (M \cap T)$

$\Rightarrow {}^{(\gamma,\lambda)}[(P+M) \cap T] = {}^{(\gamma,\lambda)}[(P \cap T) + (M \cap T)]$ [using **lemma 5.2.2**]

$\Rightarrow {}^{(\gamma,\lambda)}(P+M) \cap {}^{(\gamma,\lambda)}T = {}^{(\gamma,\lambda)}(P \cap T) + {}^{(\gamma,\lambda)}(M \cap T)$[using **proposition 5.2.1**]

$\Rightarrow ({}^{(\gamma,\lambda)}P + {}^{(\gamma,\lambda)}M) \cap {}^{(\gamma,\lambda)}T = ({}^{(\gamma,\lambda)}P \cap {}^{(\gamma,\lambda)}T) + ({}^{(\gamma,\lambda)}M \cap {}^{(\gamma,\lambda)}T)$[using **proposition 5.2.1**]

$\Rightarrow (X+I) \cap L = (X \cap L) + (I \cap L)$.

Thus, ${}^{(\gamma,\lambda)}E$ is a DN-group.

Conversely, suppose ${}^{(\gamma,\lambda)}E$ is a DN-group.

Let P, M, T are IF N-subgroups of E.

Then by **proposition 5.2.2**, ${}^{(\gamma,\lambda)}P, {}^{(\gamma,\lambda)}M, {}^{(\gamma,\lambda)}T$ are N-subgroups of ${}^{(\gamma,\lambda)}E$.

Since ${}^{(\gamma,\lambda)}E$ is a DN-group,

$({}^{(\gamma,\lambda)}P + {}^{(\gamma,\lambda)}M) \cap {}^{(\gamma,\lambda)}T = ({}^{(\gamma,\lambda)}P \cap {}^{(\gamma,\lambda)}T) + ({}^{(\gamma,\lambda)}M \cap {}^{(\gamma,\lambda)}T)$

$\Rightarrow {}^{(\gamma,\lambda)}[(P+M) \cap T] = {}^{(\gamma,\lambda)}[(P \cap T) + (M \cap T)]$ [using **proposition 5.2.1**]

$\Rightarrow (P+M) \cap T = (P \cap T) + (M \cap T)$[using **lemma 5.2.2**] .

Thus E is an IF DN-group.

Definition 5.2.5. *E is called an IF uniserial N-group if any two of its IF N-subgroups are comparable to each other.*

Theorem 5.2.2. *An IF uniserial N-group is an IF DN-group.*

Proof : Let E be an IF uniserial N-group.

Let $P = < \phi_P, \psi_P >, B = < \phi_B, \psi_B >, T = < \phi_T, \psi_T >$ be IF N-subgroups of E.

Since E is an IF uniserial N-group, any two of its IF N-subgroups are comparable to each other.

So, it may assume $P \subseteq B \subseteq T$.

Thus, $\phi_P \leq \phi_B \leq \phi_T$ and $\psi_P \geq \psi_B \geq \psi_T$.

Therefore, $P+B =< \phi_{P+B}, \psi_{P+B} >$, where

$\phi_{P+B}(s) = \vee\{\phi_P(a) \wedge \phi_B(q) : a,q \in E, s = a+q\}$ and $\psi_{P+B}(s) = \wedge\{\psi_P(a) \vee \psi_B(q) : a,q \in E, s = a+q\}, \forall\, s \in E$.

So, $(P+B) \cap T =< \phi_{P+B} \wedge \phi_T, \psi_{P+B} \vee \psi_T >$.

Now, $P \cap T =< \phi_P \wedge \phi_T, \psi_P \vee \psi_T >=< \phi_P, \psi_P >= P$ and $B \cap T =< \phi_B \wedge \phi_T, \psi_B \vee \psi_T >=< \phi_B, \psi_B >= B$.

Therefore, $(P \cap T) + (B \cap T) = P+B$.

Again, for any $s,a,q \in E$,

$\phi_{P+B}(s)$

$= \vee\{\phi_P(a) \wedge \phi_B(q) : s = a+q\}$

$\leq \vee\{\phi_T(a) \wedge \phi_T(q) : s = a+q\}$

$\leq \vee\{\phi_T(a+q) : s = a+q\}$[Since T is an IF N-subgroup]

$= \phi_T(s)$.

Therefore, $\phi_{P+B} \wedge \phi_T = \phi_{P+B}$.

Similarly, $\psi_{P+B} \vee \psi_T = \psi_{P+B}$.

Thus, $(P+B) \cap T = P+B$.

Hence the result.

Definition 5.2.6. $Ann(E) = \{n \in N : ns = 0, \forall s \in E\}$.

Lemma 5.2.3. *If $E = E_1 + E_2$ is commutative with E_1, E_2 being unitary DN-groups such that $Ann(E_1) + Ann(E_2) = N$, then $E = E_1 + E_2$ is a DN-group.*

Proof : Let P, M, T be N-subgroups of $E = E_1 + E_2$.

Then $P = P_1 + P_2, M = M_1 + M_2, T = T_1 + T_2$, for some subsets P_1, M_1, T_1 of E_1 and P_2, M_2, T_2 of E_2.

Since $Ann(E_1) + Ann(E_2) = N$ and $1 \in N$, then $\exists\, n_1 \in Ann(E_1), n_2 \in Ann(E_2)$ such that $n_1 + n_2 = 1$.

Now, for any $a_1 \in P_1$,

$1.a_1 = (n_1 + n_2)a_1$

$\Rightarrow a_1 = n_1a_1 + n_2a_1$[since E_1 is unitary] .

But, $a_1 \in P_1 \subseteq E_1$

$\Rightarrow a_1 \in E_1$

$\Rightarrow n_1a_1 = 0$[since $n_1 \in Ann(E_1)$] .

Therefore, $n_2a_1 = a_1 \in P_1$.

But $n_2a_1 \in NP_1$.

Therefore, $NP_1 \subseteq P_1$.

This shows that P_1 is an N-subgroup of E_1.

Similarly, it can be shown that M_1, T_1 are N-subgroups of E_1 and P_2, M_2, T_2 are N-subgroups of E_2.

Now, $(P+M) \cap T$

$= [(P_1 + P_2) + (M_1 + M_2)] \cap (T_1 + T_2)$

$= [(P_1 + M_1) \cap T_1] + [(P_2 + M_2)] \cap T_2]$

$= [(P_1 \cap T_1) + (M_1 \cap T_1)] + [(P_2 \cap T_2) + (M_2 \cap T_2)]$[since E_1, E_2 are DN-groups]

$= [(P_1 + P_2) \cap (T_1 + T_2)] + [(M_1 + M_2) \cap (T_1 + T_2)]$

$= (P \cap T) + (M \cap T)$.

This shows that $E = E_1 + E_2$ is a DN-group.

Theorem 5.2.3. *Let E, K be unitary IF DN-subgroups and $Ann(E) + Ann(K) = N$ with ${}^{(\gamma,\lambda)}E = E, {}^{(\gamma,\lambda)}K = K$, then $E + K$ is an DN-group with $\gamma, \lambda \in (0, 1], \gamma + \lambda = 1$.*

Proof : Since E, K are unitary IF DN-groups and $Ann(E) + Ann(K) = N$, by **theorem 5.2.1**, ${}^{(\gamma,\lambda)}E, {}^{(\gamma,\lambda)}K$ are also DN-groups with $\gamma + \lambda = 1$ and $\gamma, \lambda \in (0, 1]$.

Since E, K are unitary, ${}^{(\gamma,\lambda)}E, {}^{(\gamma,\lambda)}K$ are also unitary.

Also, by hypothesis $Ann({}^{(\gamma,\lambda)}E + Ann({}^{(\gamma,\lambda)}K = N$.

Therefore, By **lemma 5.2.3**, ${}^{(\gamma,\lambda)}E + {}^{(\gamma,\lambda)}K$ is a DN-group with $\gamma, \lambda \in (0, 1], \gamma + \lambda = 1$.

So, $E + K$ is an DN-group with $\gamma, \lambda \in (0, 1], \gamma + \lambda = 1$.

5.3 Intuitionistic fuzzy weak DN-groups

Definition 5.3.1. *E is said to be a weak DN-group if and only if $(P+M)\cap T=(P\cap T)+(M\cap T)$, for all ideals P,M,T of E.*

Definition 5.3.2. *E is referred to as an IF weak DN-group if and only if $(P+M)\cap T=(P\cap T)+(M\cap T)$, for all IF ideals P,M,T of E.*

Definition 5.3.3. *${}^{(\gamma,\lambda)}E$ is said to be a weak DN-group if and only if $(P+M)\cap T=(P\cap T)+(M\cap T)$, for all ideals P,M,T of ${}^{(\gamma,\lambda)}E$.*

Theorem 5.3.1. *If E is an IF weak DN-group, then ${}^{(\gamma,\lambda)}E$ is also a weak DN-group with $\gamma,\lambda\in(0,1],\gamma+\lambda\leq 1$.*

Proof : Let X,I,L be ideals of ${}^{(\gamma,\lambda)}E$ and defined as in **theorem 5.2.1.**
Then as above, ${}^{(\gamma,\lambda)}P=X,{}^{(\gamma,\lambda)}M=I,{}^{(\gamma,\lambda)}T=L$ with $\gamma,\lambda\in(0,1],\gamma+\lambda=1$.
Claim P,M,T are IF ideals of E.
Since X is an ideal of ${}^{(\gamma,\lambda)}E$, therefore X is normal subgroup of $(E,+)$ and $n(s+e)-ne\in X,\forall\, n\in N,s\in X,e\in E$
$\Rightarrow \phi_P\{n(s+e)-ne\}=1=\phi_P(s)$[since $s\in X$] .
Also, since X is normal subgroup of $(E,+)$, $s-a,s+a\in X,\forall\, s,a\in X$.
$\Rightarrow \phi_P(s-a)=1,\phi_P(s+a)=1$
$\Rightarrow \phi_P(s-a)=1\wedge 1=\phi_P(s)\wedge\phi_P(a)$ and $\phi_P(s+a)=1=\phi_P(a+s)$[since $s,a\in X$].
Again, $n\in N,s\in X$,
$ns\in X$
$\Rightarrow \phi_P(ns)=1=\phi_P(s)$.
Now, $s,a\in X$
$\Rightarrow a+s-a\in X$
$\Rightarrow \phi_P(a+s-a)=1=\phi_P(s)$.
Similarly, it can be shown that
$\psi_P(s-a)=\psi_P(s)\vee\psi(a),\psi_P(ns)=\psi_P(s),\psi_P(a+s-a)=\psi_P(s),\psi_P(n(s+a)-ns)=\psi_P(a)$.

This shows that P is an IF ideal of E.

Similarly, M, T are IF ideals of E.

Since E is an IF weak DN-group,

$(P+M)\cap T$

$=(P\cap T)+(M\cap T)$

$\Rightarrow {}^{(\gamma,\lambda)}[(P+M)\cap T]={}^{(\gamma,\lambda)}[(P\cap T)+(M\cap T)]$ [using **lemma 5.2.2**]

$\Rightarrow {}^{(\gamma,\lambda)}(P+M)\cap {}^{(\gamma,\lambda)}T={}^{(\gamma,\lambda)}(P\cap T)+{}^{(\gamma,\lambda)}(M\cap T)$[using **proposition 5.2.1**]

$\Rightarrow ({}^{(\gamma,\lambda)}P+{}^{(\gamma,\lambda)}M)\cap {}^{(\gamma,\lambda)}T=({}^{(\gamma,\lambda)}P\cap {}^{(\gamma,\lambda)}T)+({}^{(\gamma,\lambda)}M\cap {}^{(\gamma,\lambda)}T)$[using **proposition 5.2.1**]

$\Rightarrow (X+I)\cap L=(X\cap L)+(I\cap L)$.

Thus, ${}^{(\gamma,\lambda)}E$ is a weak DN-group with $\gamma,\lambda\in(0,1],\gamma+\lambda\leq 1$.

Lemma 5.3.1. *If $P=<\phi_P,\psi_P>$ is an IF ideal of E , then ${}^{(\gamma,\lambda)}P$ is also an ideal of ${}^{(\gamma,\lambda)}E$.*

Proof : Let $P=<\phi_P,\psi_P>$ be an IF ideal of E.

So, ${}^{(\gamma,\lambda)}P\subseteq E$.

Let $s,a\in {}^{(\gamma,\lambda)}P$.

Then, $\phi_P(s)\geq\gamma,\psi_P(s)\leq\lambda,\phi_P(a)\geq\gamma,\psi_P(a)\leq\lambda$.

Since P is an IF ideal of E and $s,a\in E$, therefore

$\phi_P(s-a)\geq\phi_P(s)\wedge\phi_P(a)\geq\gamma\wedge\gamma=\gamma$ and $\psi_P(s-a)\leq\psi_P(s)\vee\psi_P(a)\leq\lambda\vee\lambda=\lambda$.

Therefore, $s-a\in {}^{(\gamma,\lambda)}P$.

Also, if $a\in E$ and $s\in {}^{(\gamma,\lambda)}P$, then $s,a\in E$.

Therefore, $\phi_P(a+s-a)\geq\phi_P(s)\geq\gamma$ and $\psi_P(a+s-a)\leq\psi_P(s)\leq\lambda$.

So, $a+s-a\in {}^{(\gamma,\lambda)}P$.

Again, if $a\in E$, $s\in {}^{(\gamma,\lambda)}P$ and $n\in N$, then $s,a\in E$.

Therefore, $\phi_P(n(s+a)-na)\geq\phi_P(s)\geq\gamma$ and $\psi_P(n(s+a)-na)\leq\psi_P(s)\leq\lambda$.

Thus, $n(s+a)-na\in {}^{(\gamma,\lambda)}P$.

Thus, ${}^{(\gamma,\lambda)}P$ is an ideal of ${}^{(\gamma,\lambda)}E$.

Theorem 5.3.2. *If $^{(\gamma,\lambda)}E$ is a weak DN-group , then $^{(\gamma,\lambda)}(D\cap M)+{}^{(\gamma,\lambda)}T=({}^{(\gamma,\lambda)}D+{}^{(\gamma,\lambda)}T)\cap({}^{(\gamma,\lambda)}M+{}^{(\gamma,\lambda)}T)$, for all ideals $^{(\gamma,\lambda)}D,{}^{(\gamma,\lambda)}M,{}^{(\gamma,\lambda)}T$ of $^{(\gamma,\lambda)}E$ with $T\subseteq M$.*

Proof : Let $s\in{}^{(\gamma,\lambda)}(D\cap M)+{}^{(\gamma,\lambda)}T$.

Then $s=y+m$, where $y\in{}^{(\gamma,\lambda)}(D\cap M),m\in{}^{(\gamma,\lambda)}T$

$\Rightarrow y\in{}^{(\gamma,\lambda)}D$ and $^{(\gamma,\lambda)}M,m\in{}^{(\gamma,\lambda)}T$

$\Rightarrow s=y+m\in({}^{(\gamma,\lambda)}D+{}^{(\gamma,\lambda)}T)\cap({}^{(\gamma,\lambda)}M+{}^{(\gamma,\lambda)}T)$.

Again, let $y\in({}^{(\gamma,\lambda)}D+{}^{(\gamma,\lambda)}T)\cap({}^{(\gamma,\lambda)}M+{}^{(\gamma,\lambda)}T)$.

Since Sum of two ideals of $^{(\gamma,\lambda)}E$ is also an ideal and $^{(\gamma,\lambda)}E$ is a weak DN-group, therefore

$y\in[{}^{(\gamma,\lambda)}D\cap({}^{(\gamma,\lambda)}M+{}^{(\gamma,\lambda)}T)]+[{}^{(\gamma,\lambda)}T\cap({}^{(\gamma,\lambda)}M+{}^{(\gamma,\lambda)}T)]$

$\Rightarrow y\in[{}^{(\gamma,\lambda)}D\cap{}^{(\gamma,\lambda)}M]+{}^{(\gamma,\lambda)}T)]$[since $T\subseteq M$] .

Thus, $^{(\gamma,\lambda)}(D\cap M)+{}^{(\gamma,\lambda)}T=({}^{(\gamma,\lambda)}D+{}^{(\gamma,\lambda)}T)\cap({}^{(\gamma,\lambda)}M+{}^{(\gamma,\lambda)}T)$.

Theorem 5.3.3. *If $^{(\gamma,\lambda)}E$ is a weak DN-group with $\gamma+\lambda=1$, then E is an IF weak DN-group.*

Proof : Let D,M,T be IF ideals of E, then by **lemma 5.3.1**, $^{(\gamma,\lambda)}D,{}^{(\gamma,\lambda)}M,{}^{(\gamma,\lambda)}T$ are ideals of $^{(\gamma,\lambda)}E$.

Since $^{(\gamma,\lambda)}E$ is a weak DN-group,

$({}^{(\gamma,\lambda)}D+{}^{(\gamma,\lambda)}M)\cap{}^{(\gamma,\lambda)}T=({}^{(\gamma,\lambda)}D\cap{}^{(\gamma,\lambda)}T)+({}^{(\gamma,\lambda)}M\cap{}^{(\gamma,\lambda)}T)$

$\Rightarrow{}^{(\gamma,\lambda)}[(D+M)\cap T]={}^{(\gamma,\lambda)}[(D\cap T)+(M\cap T)]$[using **proposition 5.2.1**]

$\Rightarrow(D+M)\cap T=(D\cap T)+(M\cap T)$][using **lemma 5.2.2**]

$\Rightarrow E$ is an IF weak DN-group.

5.4 Conclusion

The main goal of this chapter is to extend the notion of distributive near-ring groups to intuitionistic fuzzy distributive near-ring groups as well as intuitionistic fuzzy weak distributive near-ring groups. Some lemmas and theorems related to IF N-subgroups, IF DN-groups and IF weak DN-groups are studied. The **theorem** 5.2.2 describes the relationships between IF *DN*-groups and IF uniserial *DN*-groups. **Theorems** 5.3.1 to 5.3.2 illustrate the association between IF weak *DN*-group and weak *DN*-group.