# 2

# Literature Review

## 2.1 Introduction to the chapter

This chapter will provide a short summary of the existing mathematical models that are utilised to define pest control ecological interactions. This chapter will discuss the significance of mathematical models in agriculture, as well as the main ecological phenomena in the agricultural pest control framework. The main objective of the eco-epidemiological model beneath the pesticide application is to reduce the harm produced by the infected plant populations and the application of pesticides as a control measure.

Existing works in the literature have analysed eco-epidemic models, emphasising either predatory or prey populations, where they are segregated into infected or susceptible categories. As a result, this chapter focuses on the scholarly works that deal with plant populations enduring a disease epidemic and the role of pesticides. Moreover, this chapter explains the investigations that explore strategies for mitigating epidemics and promoting the recovery of plant populations.

Plant epidemics have been recognised in various scholarly works. Since the eco-epidemiological model permits:

- To analyse the associations prevailing among biotic and abiotic disease system constituents impacting the development of disease in a host plant [62].
- To enumerate the spatial and temporal spread of the disease [66].
- To evaluate the ability of management possibilities for vector and disease mechanism [60].
- To improve scenario examination for the assessment of various disease management programme [22].

Moreover, epidemiological model improvement and utilisation can recognise data or knowledge gaps and, therefore, prioritise further investigation work. This research subject has attracted a surge of attention, and the number of papers involving various mathematical models and issues is discussed in this chapter. This chapter provides an exploration of the academic works that concentrate on the mathematical modelling, application of pesticides, eco-epidemic model for plant diseases, and integrated pest management.

## 2.2 Employment of Mathematical modelling in agriculture

The benefits of agricultural production for society and its collaboration with the environment are not precise or subjective, as it is appropriate to implement mathematical models of fuzzy logic for such correlations. It is needed to define the promising application of these techniques to Cuba’s agriculture decision-making. Accordingly, the work done by Rodríguez and López [151] denotes the method of organising an agro-systems group based on soil overuse syndrome symptoms in order to rearrange biodiversity to reduce the central scheme. The academic work deals with the application of Big6 as a mathematical instrument constructed on fuzzy logic that is easy to implement to address environmental management issues in agricultural soils, thus satisfying the aim of the investigation. The outcomes attained permit us to recognise that two of the plots in this agro-system require a huge amount of attention to reduce the impacts of the main mechanism of soil overuse syndrome. Generally, managers can employ the improvement of these mathematical models on the basis of fuzzy logic in place of an instrument to encourage environmental management’s decision-making.

Currently, several agricultural products have started to execute a specific farming method. In this context, it is necessary to develop strategies for assessing and sustaining the growth of agricultural crops. Hence, in order to evaluate the administration of the plant-soil-air system, the growth of the plant's prospective is introduced, which is the proportion of the power disbursed on the unit of a plant's mass formation. The study contemplates the theoretical requisites for measuring the potential of growth. Hence, keeping in mind the need for a precise scheme of agricultural science on the one hand and, on the other hand, the consistency of acquiring the intended crop yield, Maksimov et al. [124] intend to develop the functioning of the plant-soil-air system using mathematical modelling for producing choices related to operational management.

### 2.2.1 Significance

In recent decades, an investigation into biological issues has been heavily reliant on mathematical modelling. The main motive of the investigation carried out by Li et al. [116] is to analyse innovative solutions to the predator-prey model with two nonlocal and local fractional operators. Through the incorporation of two effective analytical techniques, a few categories of analytical solutions were established for the system through the beta-time fractional operator. Furthermore, a numerical technical basis for the model was established. The article displays the finite difference schemes, which are the basis for the numerical approach. The concurrent investigation of numerical techniques for this problem category makes it conceivable to deliver an appropriate numerical description of the problem solution in the instance of analytical method failure. Moreover, the article embraces numerical simulations of the outcomes.

### 2.2.2 Applications

Rudoy [156] devotes his classic study to the modern methods of mathematical modelling in agriculture, specifically focusing on the applied aspects of its development. The main goal of this work is to improve and perfect one of the most important techniques that can move the science of mathematical modelling and its application for the needs of agriculture to a new level of development. The study shows that mathematical models based on advanced scientific knowledge play a huge role in optimising agricultural processes and ensuring high efficiency in crop production.

The study demonstrates that mathematical models are a vital part of agricultural processes, especially in economic calculation and in modelling the processes of different sections of farms. By using mathematical models, farmers or agricultural managers can predict the results more accurately, improve the decision-making process, and optimise resource allocation for higher yields. Strategic use of the models could lead to tangible improvements in farm management, including higher yields and better resource allocation, according to the statistical data. Overall, Rudoy's study demonstrates that advanced mathematical models are vital for improving agricultural processes and ensuring a more sustainable agricultural industry.

### 2.2.3 Mathematical modelling and integrated management of Eco-Epidemiology model

Jeger and Bragard [92] carried out the conventional study, which intends to reveal that insect-transmitted diseases have several features in common, irrespective of whether the casual pathogen is a virus, phytoplasma, or virus. The introduced framework can be established, which condenses the fundamental interactions among the insect vector, plant, and pathogen with respect to key epidemiological parameters from which a principle can be formed that measures whether the pathogen will establish, persist, and lead to an epidemic in a host population.

Likewise, another inquiry conducted by Greenhalgh et al. [70] analysed a model comprising a predatory population and both the infected and susceptible populations. The predator can forage on either prey species, but instead of selecting individuals at random, the predator forages specifically on the most abundant prey species. Precisely, it is assumed that the predator's likelihood of grasping an infected or susceptible prey is proportional to the numbers of these two distinct kinds of prey species. This phenomenon, concerning changing preference from susceptible to infected prey, is termed switching. According to the proposed model, the researcher assumes that the predator will ultimately die as a result of eating infected prey.

One of the significant needs for the utilisation of advanced numerical approaches to resolving fractional issues is their application in real-world problems. The primary part of the improvement made in this region has been the improvement of effective numerical approaches and methods. Similarly, Ghanbari [64] investigates an advanced technique for evaluating the eco-epidemiological dynamics of a nonlinear fractional system using differential equations. One of the primary topics of computational biology is eco-epidemiology, which relates epidemiology to ecology. The existence of a disease in one of the populations in the environment provides major alterations in the necessary system components in those models. The system's equilibrium points are assessed for this model. Finally, the study displays the uniqueness and convergence theorems of the solution acquired from the employment of fractional derivative operators.

Another inquiry performed by Ayembillah et al. [17] established and evaluated the mathematical model of the maize steak virus disease dynamics. The outcomes displayed that the system was positive and uniformly bounded. The disease-free and the points of endemic equilibrium and its local stability analysis are performed. The sensitivity analysis of every parameter is performed in the model analysis to provide a detailed picture of the effect of every factor on spread. Figure 2.1 below depicts the maize steak virus disease spread dynamics model.

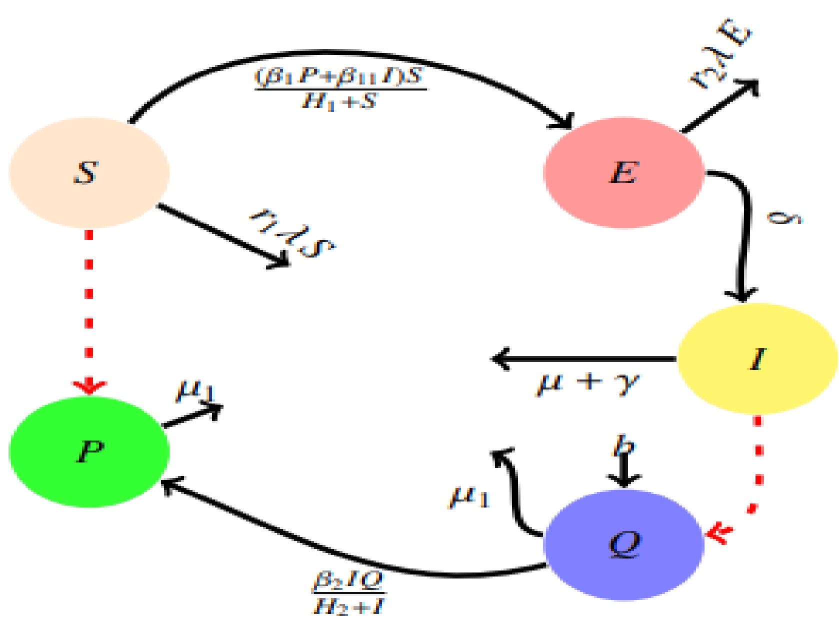


**Figure 2.1. Schematic illustration of the Maize steak virus disease model**

**(Source: [41])**

The analytical outcomes were established by the numerical simulations with realistic parameter values. The outcomes attained are given as follows:

- Altering the predation and infection rates of leafhoppers and maize, $\beta_1, \beta_{11}$ and $\beta_2$ correspondingly and the susceptible leafhopper's birth rate $b$ have a direct association with the standard basic reproduction number $R_0$ .
- Meanwhile, fluctuating the infected maize's death rate, $\mu$ ,death rate of infected and susceptible leafhopper, $\mu_1$, and the infected maize plants removal rate have an inverse association with the standard basic reproduction number $R_0$.

The introduced model reveals that the disease spread mostly relies on how these parameters are deployed [41].

Similarly, another paper, which was carried out by Alemneh et al. [6], evaluated an ideal deterministic eco-epidemiological model for the dynamics of the dynamics of the maize steak virus and investigated the optimal strategy to combat the maize population from the maize steak virus. The model of optimal control is improved with three control intermediations, such as:

- $u_1$ - $Prevention$
- $u_2$ - $quarantine$
- $u_3$ – $chemical\ control$

In order to attain an optimum control approach, the researcher utilised Pontryagin's maximal principle to acquire the Hamiltonian, the control characterization, adjoint variables, and the system's optimality. Numerical simulations are executed by means of the forward-backward sweep iterative technique. The outcomes of the study reveal that every integrated approach has the capability to reduce the disease at a particular time. On the other hand, because of the inadequate resources, it is vital to identify a profitable tactic. The utilisation of an *incremental cost-effectiveness* ratio and a cost-effectiveness analysis method is investigated, and it is inferred that the collaboration of quarantine and prevention is an excellent and profitable tactic from various integrated approaches. Hence, stakeholders and policymakers must smear the integrated association to prevent the maize steak virus from spreading in the maize populace.

Kebede and Muchie [101] have presented another inquiry that establishes a mathematical model for the transmission dynamics of the cotton leaf curl virus disease. This model considers both the vector and cotton populations. The model takes into account the following primary factors:

- The vector populations are categorized as infected ($Y$) and susceptible ($X$)
- Cotton populations are categorized as infected ($B$) and Susceptible ($A$)
- The solutions of the model are positive and confined to the initial circumstances of the particular meaningful set.

The existence of distinct cotton leaf curl viruses and the endemic equilibrium points are discovered, and the basic reproduction number is estimated utilising the subsequent generation matrix technique. The conditions for this equilibrium's global and local asymptotic stability are then introduced. While the standard reproduction is below one, the system has the globally and locally asymptotically stable cotton leaf curl virus-free equilibrium point, and if the standard reproduction number is greater than 1, the system has the globally and locally asymptotically stable endemic equilibrium point. The outcomes of the simulation agree with the analytical outcomes.

Another article studied by Ibrahim [86] intends to learn about the eco-epidemiological model's dynamic behaviour. The paper presents a prey-predator model that incorporates infectious diseases in prey species as well as stage structure in predator species. The Lotka-Volterra functional response led to the structural growth of the prey species, even in the absence of predators and ferocity procedures. The boundedness, uniqueness, and existence of the entire equilibrium point are measured. The restrictions on the models' persistence are established. The local bifurcation towards all equilibrium points is evaluated. The global dynamics of the model are examined in a numerical manner and contrasted with the acquired results.

Likewise, Ma et al. [123] conducted a study that presented a stochastic eco-epidemiological system with a patchy structure and infection associated with transportation. Moreover, the assessment of stochastic dynamical behaviours is conducted. Initially, through the construction of appropriate Lyapunov functions, the study demonstrates the existence of a distinct globally positive solution, commencing from the initial positive value. Subsequently, it is revealed that the displayed system is stochastically ultimately bounded, and the average in the second moment of solution time is confined. Finally, it is demonstrated that stochastic perturbations can manage the predator population, and the diseases in the predator can be non-existent while being constant in the deterministic system. Consequently, few numerical simulations are provided to examine the theoretical outcomes.

Savadogo et al. [160] conducted another paper in which they introduced and assessed a mathematical model defining the dynamics of phytoplankton, toxin production, and fish population using an ordinary differential equations system. They segregate the phytoplankton into two groups: the susceptible phytoplankton and the infected phytoplankton. The study intends to evaluate the impact of the toxic substance on the fish population. The contributions of the study are as follows:

- The model's equilibria stability has been learned globally and locally towards the standard basic reproduction number $R_0$.
- The model's mathematical analysis reveals that the equilibrium in the absence of diseases is globally asymptotically stable when $R_0 \leq 1$ and the endemic equilibrium is globally asymptotically stable if $R_0 \geq 1$.
- Numerical simulations are performed to demonstrate the theoretical results feasibility.

Gaber et al. [61] conducted an inquiry to construct an eco-epidemiological model in addition to the nonlinear incidence rate that Gumel and Moghadas [72] recommended. The introduced model delivers a realistic and reasonable approach to ecological systems across the globe since we track Holling type II for the considered predator-susceptible prey interaction. Also, the model follows the simple mass action low for the predator for the predator-infected prey interaction since the infected prey will be weak. The inquiry reveals that:

- The time for judgement is suggestively longer compared to the time required to hold the healthy prey.
- The solutions for positivity, existence, and boundlessness are evidenced.

Also, the equilibrium points are measured according to the feasibility conditions of the study. Local stability has been evaluated by Routh Hurwitz criteria, and the Lyapunov function has been created to learn global stability as per the La Salle theorem. Hopf and Sotomayor's theorems provide insight into various types of bifurcation. The solutions to the numerical analysis were implemented with the aid of fourth-order Runge-Kutta. The simulations executed in the study encouraged the theoretical findings.

### 2.2.4 Role of Pesticides in evolving plant diseases

In an eco-epidemic model, the impact of pesticides on plant diseases can be seen through interactions among plants, pathogens, and the environment. Pesticides, meant to control pathogens and pests, can unintentionally affect disease transmission and evolution in ecosystems. For example, pesticide use can influence pathogen populations, leading to the emergence of pesticide-resistant strains. This can be represented in the model by including a parameter for the rate of pesticide resistance development among pathogens.

Additionally, the indirect effects of pesticides on plant health and ecosystem dynamics are important in eco-epidemic models. Pesticides can disrupt ecological balances by harming natural enemies of pathogens or changing soil microbial communities, which can impact disease dynamics. These indirect effects can be included in the model by adding equations or parameters that show changes in ecosystem structure and function due to pesticide application. By considering these factors, researchers can gain a better understanding of how pesticides, pathogens, and plants interact and develop more effective strategies for managing plant diseases while minimising environmental harm.

An academic study carried out by Vavre et al. [185] gives insights into the interactions between pesticides and plant disease-causing pathogens from an ecological and epidemiological perspective. It discusses the ecological and evolutionary effects of using pesticides in farming systems. The paper looks at how pesticides impact disease spread and the development of pesticide resistance in pathogens. It emphasises the need to combine ecological and epidemiological methods when designing pest control approaches. The overall meaning is that pesticides interact with pathogens in complex ways, and sustainable pest management requires understanding these interactions from multiple scientific viewpoints.

### 2.2.5 Modelling of Eco-epidemiology Model for effective Integrated Pest management

The extensive use of chemical pesticides in agricultural pest management systems can pose risks to the multifaceted ecosystems, which consist of ecological, social, and economic subsystems. Wan et al. [188] introduced an ecological two-sidedness technique that has been implemented in pest-control strategies for pesticide pollution management. On the other hand, the catastrophe theory has not been initially implemented with this technique. Hence, the researcher utilises the technique of incorporating ecological two-sidedness with a multi-criterion assessment method of catastrophe theory to evaluate the agro-ecosystem complexity concerned by insecticides and display the optimal insect pest-control tactic in the production of cabbage. The outcomes of the research reveal that applying environment-friendly insecticides and frequency vibration lamps was regarded as the optimal insect pest-controlling approach in the production of cabbage in China.

Within the framework of integrated pest management programmes (IPM), the study conducted by Anguelov et al. [13] introduced a generic model to learn the effect of mating disruption control utilising an artificial female pheromone to obscure males and adversely impact one's mating prospects. Subsequently, the reproduction rate is reduced, resulting in a fall in the size of the population. Trapping is executed to seize the males fascinated by the artificial pheromone for more effective control. A model's theoretical analysis in the absence of control initially takes place to execute the endemic equilibrium's properties. Next, control is included, and the model's theoretical analysis permits the pheromone's threshold values, which is practically fascinating for field applications. Lastly, the numerical experiments are executed to illustrate the theoretical outcomes.

The conventional study that was carried out by Bazarra et al. [21] describes the numerical and mathematical insight of a problem emerging in vector-borne plant infections. The introduced model is described as a non-linear structure that consists of a parabolic partial differential equation for the function of vector abundance and a 1st-order normal differential equation for the function of plant well-being. A uniqueness and existence outcome are demonstrated utilising uniform estimates, finite differences, and short-lived distances to the boundary. The solutions regulatory are also acquired. Subsequently, utilising the implicit Euler scheme and the finite element method, fully discrete approximations are established. Lastly, a few numerical outcomes in 1 or 2 dimensions are displayed to validate the approximation's accuracy and the solution's behaviour.

A main threat to livestock and agricultural resources is the pest outbreak. Integrated pest management is extensively being employed now-a-days to manage the pest population. In the process of managing the pest's outbreak, the natural enemies are extremely valuable. As a result, Kalra et al. [96] employed biological and microbial pest control approaches that involve simultaneously using infected pests and spontaneously discharging natural enemies. Hence, a susceptible-infected-recovered model involving prey's infection with two classes and the predator's stage structure is analysed for the purpose of integrated pest management. Predators acts as the natural enemy, while prey behaves as the pest. Initially, global and local stability of pest extinction periodic solutions is performed, followed by the system's performance, utilising a stroboscopic map, the floquet theory of impulsive differential equations, and a comparison analysis approach. Moreover, it is revealed that an impulsive period's threshold value plays a significant role in the system's dynamics. Furthermore, to endorse the suggested outcomes, numerical simulations are performed.

Al-Jubouri et al. [8] present another paper that aims to study and evaluate the impact of optimal harvesting on the dynamic behaviour of the eco-epidemiological model, which deals with an assailant species (locust insect) and a wheat field (host species). The research provides an interpretation and analysis of a mathematical model that incorporates an epidemiological environment containing various infectious diseases in the presence of optimal harvesting. The introduced system is segregated into two species such as predator individuals – $Y(t)$ – Locust insect and Prey individuals – $X(t)$ – wheat (host species).

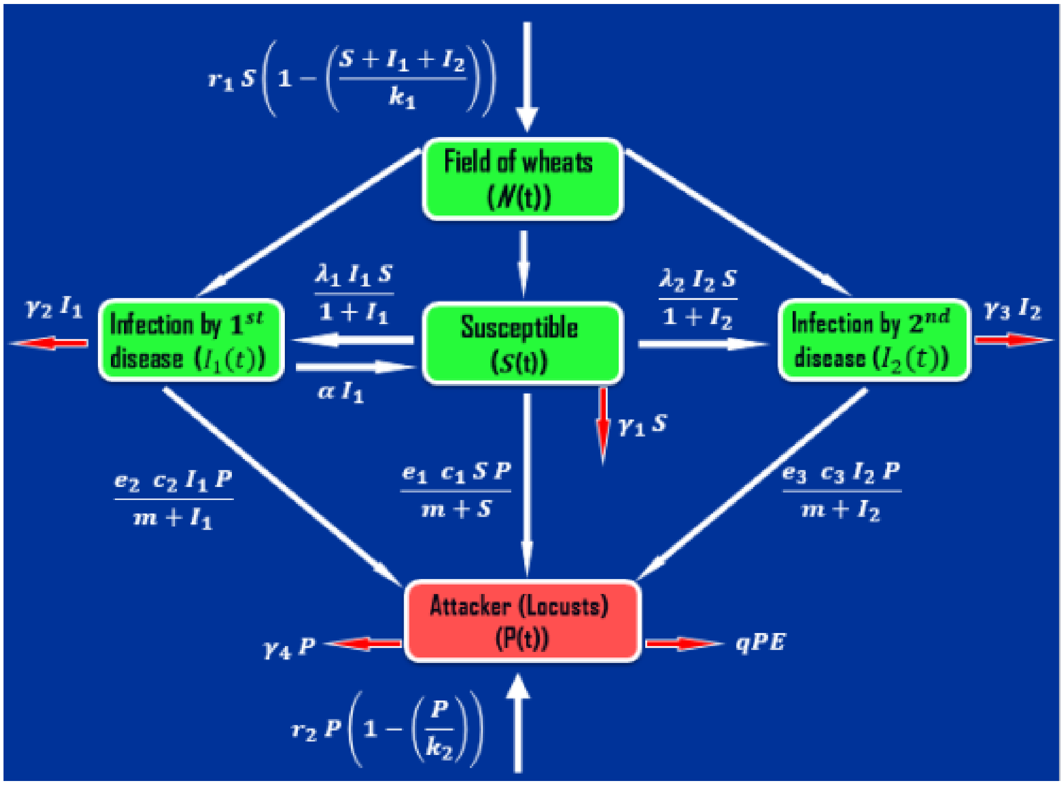


**Figure 2.2. The introduced eco-epidemiological system's chart**

**(Source: [8])**

Figure 2.2 illustrates the chart of eco-epidemiological system with two various diseases. The primary work contributions are:

- Understanding of the mathematical analysis for the eco-epidemiological model's dynamic behaviour and discoursing the proportional harvesting effects.
- Processing and safeguarding the wheat crops from the danger of being visible to locus pest infest.
- Forming a steady, balanced eco-epidemiological system that can creates high-income.

Similarly, Bakhtiar et al. [18] examine the complex interaction among the pest insects and plants beneath the intervention of natural enemies discharges coupled with sterile insect approaches. A nonlinear ordinary differential equations set is created with respect to an optimal control model involving the population's characteristics. Also, the optimal control measures are pursued in such a manner that they reduce the density of pests concurrently with the control energies. The study investigated the three distinct strategies associating with the sterile insect release rate and predators in place of natural enemies, correspondingly proportional, constant, and engulfing proportional release rates for the control objective attainability. The main findings of the study are:

- The essential optimality conditions of the control problem are attained by utilising the Pontryagin maximum principle. Following this, the forward-backward sweep technique is then executed to estimate the optimal solution numerically.
- It is also revealed that the environment comprises of brown plant hoppers as pests and rice plants and lady beetles as natural enemies, and the releases of sterile plant hoppers can reduce the density of pests, thus increasing the biomass of plants.
- The natural enemies release with constant rate and sterile insects release with proportional rate are considered to be the most cost-efficient approach in managing pest insects.
- The introduced approach reduces the pest population by about thirty-five percent and, hence, increases the plant density by up to thirteen percent at the time of control implementation.

Maintenance of aphids using insecticides is ineffective to control the potato virus. Meanwhile, a mixed-cropping system that contains potatoes and non-host crop plants is the main strategy. Investigations regarding the significance of those practices will benefit smallholder framers. Accordingly, Degefa et al. [40] introduce a mathematical model that intends to analyse the impacts of a mixed-cropping system on the potato virus Y disease dynamics. The evaluation results show that the non-virus host plant has a significant impact on decreasing the basic reproduction number, 1. Subsequently, an endemic equilibrium is formed, in which one is locally asymptotically stable. The study implements a geometric technique and carefully considers conditions to ensure the global stability of the endemic equilibrium. The sensitivity analysis and bifurcation of the embedded parameters are performed. Lastly, a few numerical simulations are performed to encourage the analytic outcomes.

Panja [143] conducted another inquiry that presents a 3-species model of the interactions between the crop, predator, and pest. In the absence of crop harvesting and pests, researchers regard the crop growth rate as logistical. The study considers that pests consume crops, while in the interim, predators consume pests. The research focuses on the process of harvesting crops. Holling type II functional response has been regarded as important for pest and crop depletion. It is also presumed that the predators will consume pests. The main contributions of this study are:

- Boundedness and positivity of the model's solution have been analysed.
- Several equilibrium model points are assessed and global and local stability at that time are examined.
- Investigations on the Hopf bifurcation existence principle in association with the central model parameter have been performed.
- Through the utilisation of a defined objective function, the optimum crop harvesting has been measured.
- The best crop harvesting rate is estimated by Pontryagin's maximum theory.

According to the study's findings, an increase in optimal harvesting yields contributes to the model's stability, indicating that strategic harvesting practices can enhance the consistency and predictability of agricultural outputs. Furthermore, the study demonstrates that a higher rate of pest consumption significantly reduces agricultural yields, emphasizing pests' detrimental impact on crop productivity. These findings highlight the intricate and delicate balance between harvesting practices and pest management. Effective optimisation of one aspect, such as maximising harvest yields, can positively influence the overall productivity of the agricultural system. However, the increased crop losses due to pests can undermine the benefits of optimised harvesting without adequate pest control measures. Thus, the study underscores the importance of integrated approaches that simultaneously address both harvesting and pest management to ensure the sustainability and efficiency of agricultural systems.

Jana and Kar [90] conducted another investigation with the purpose of establishing and analysing the hybrid impulsive eco-epidemic model, which incorporates disease in the population of pests. The primary purpose of this study is to investigate the existence of pest-eradication periodic solutions, both in terms of their stability and their existence. It is as a result of this that the research presents an updated eco-epidemic model that incorporates alternating frequencies of pesticide sprays and the release of both natural enemies and infected bugs for the purpose of pest control. As a result of the findings of the study, the threshold values for the susceptible pest eradication periodic solution were displayed below the various scenarios, which demonstrated the solution's global attractiveness.

### 2.3 Research gaps

Some of the limitations or future works found in discussed existing studies are as follows,

A. Existing literature on eco-epidemic models focuses primarily on the dynamics of animal populations involving prey and predators. Researchers have devoted significant attention to studying epidemics within either the prey or predator populations, and the subsequent spread of diseases within these animal populations. Nevertheless, these studies usually do not investigate the dynamics of epidemics in plant populations.

B. Given that eco-epidemic modelling primarily focuses on prey-predator animal populations, measures for disease control tend to vary correspondingly. There is a lack of research in the literature on using pesticides to prevent and manage plant diseases within the eco-epidemic modelling framework. This gap emphasizes the need for further research to clarify the potential impact of pesticides on reducing disease transmission among plant populations.

C. There is a significant lack of research, especially focused on the creation of plant epidemic models, despite an abundance of literature on plant diseases and control measures. Even though there are studies on plant disease management available, the current state of research mostly ignores the complex dynamics of epidemic spread within plant populations and the development of successful control techniques. This omission emphasizes the need for future research efforts to address this gap and improve our understanding of plant epidemic dynamics and corresponding control methods.

D. The paper [64] suggests to prolong the numerical method utilised by the researcher to variable-order fractional systems in upcoming future. These systems can produce intense alterations in the potential ways of modelling of eco-epidemiological issues.

E. The application of pesticides does not provide quick recovery of the infected plants. There is a possible interruption in the recovery procedure. The introduced model can be prolonged to a time-delay model with the aid of delay differential equations. The suggested work can be protracted utilizing fractional order derivatives. Moreover, the investigators can extend the model by investigating contact rates among pesticides and plants, which are totally reliant on the measure of pesticides smeared. It can be attained by means of utilization of functions which depend on the variable $\alpha$.

F. The theoretical findings of the study [148] present various pest management control strategies. On the other hand, the wide-ranging eradication of the pest populace is often unachievable in reality. Hence, the main goal is to stop the pest populace from attaining harmful levels and to manage its density beneath the economic threshold. Hence, the study introduces the improvement of state-dependent impulsive differential equations to resolve this issue and this strategy will serve as the futuristic recommendation for the investigators.

## 2.4 Research Questions

I. What are the main dynamics and factors that affect the spread of plant epidemics in an ecosystem, taking into account the interactions between plant populations and disease agents?

II. What are the various ecological components that contribute to the occurrence and spread of plant epidemics in ecosystems, and how can we integrate these factors into modelling frameworks?

III. Within eco-epidemic modelling frameworks, how can we measure and examine the possible impacts of plant epidemics on the stability of ecosystems?

IV. What are the possible approaches to intervene in order to control plant epidemics in ecosystems, specifically focusing on the use of pesticides, and how successful are these approaches in reducing the spread of diseases while minimising their impact on the environment?

V. How can we adapt current models and frameworks to account for the dynamics of prey-predator animal populations, the transmission of diseases to plant populations, and plant epidemics?

### 2.5 Objectives of the study

Based on the literature reviewed above, the primary objective of this research is to develop a Mathematical Model for an Eco-epidemic Model. The focus is on studying the impact of a specific disease on an age-structured plant ecosystem. Through the model, we aim to simulate the growth of plants and the transmission of the disease within an interacting eco-epidemic framework. The susceptible plant population will be represented using an age-specific Lotka-Volterra model, while disease-carrying vectors will be considered as infectives. To accurately capture the complexity of this interactive system, a comprehensive mathematical model will be formulated. The main objectives of this research are as follows:

1. The aim is to develop a basic eco-epidemic model that analyses the spread of diseases between susceptible and infected plants, utilising pesticides as a control measure. This model will look at how well pesticides control the spread of disease, which will help experts come up with long-term plans for managing plant epidemics.
2. The goal is to establish the stability and control of a plant epidemic model using pesticide intervention and the Holling Type II functional response. This involves developing a mathematical framework to understand how pesticides can effectively control disease spread while maintaining ecological balance.
3. The aim of this study is to analyse the stability, sensitivity index, and Hopf bifurcation of the Eco-Epidemiological SIR model under pesticide application. This involves investigating the conditions for maintaining system stability, evaluating the sensitivity of model outcomes to changes in key parameters, and examining the potential for Hopf bifurcation, which could lead to periodic disease outbreaks. Through these analyses, the study seeks to understand how pesticides impact plant disease dynamics, ultimately contributing to more effective and sustainable disease management strategies.
4. To find an optimal control policy for the plant epidemic model, integrate susceptible infective plants and herbivores with pesticide intervention as a control measure. This study aims to determine how pesticides affect populations of plants and herbivores infected with disease. By developing mathematical models that account for pesticide impact, the study aims to understand the dynamics of disease transmission within plant communities and its consequences for herbivore populations. Through analysis and

validation of these models, the research seeks to provide insights into optimizing pesticide strategies for controlling disease while maintaining ecological balance, thus informing sustainable management practices and conservation efforts.

5. To examine the existence and uniqueness of solutions to the model equations developed in **Chapter 4**, with a particular focus on plant population dynamics under pesticide application. It seeks to identify the conditions that ensure the existence and uniqueness of these solutions through mathematical analysis. Additionally, the chapter explores optimal control strategies for regulating plant population dynamics, aiming to balance the benefits of pesticide use with environmental and economic concerns. These findings enhance the understanding of plant population control in agricultural systems and provide crucial insights into environmentally responsible methods of pesticide administration.

## 2.6 Mathematics preliminaries and general methodology

Differential equations serve as a means to describe and formulate scientific problems, necessitating the use of various mathematical tools for their analysis. This section offers a thorough overview of these tools, accompanied by pertinent definitions, fundamental concepts, and theorems that consistently apply throughout this thesis.

**2.6.1 State variables:** A state variable is a group of variables that describe the mathematical "state" of a dynamic system. These variables explain the behaviour of the factors that impact the system. Models are comprised of first-order differential equations that are expressed in the form of state variables.

**2.6.2 Equilibrium points:** Equilibrium points, also referred to as steady states or fixed points, are the solutions of a dynamical system. These systems are typically described by ordinary differential equations, and at equilibrium points, the rate of change of the system's state variables is zero. Mathematically, an equilibrium point $x^*$ satisfies the condition $\dot{x} = f(x^*) = 0$, where $\dot{x}$ represents the time derivative of the state variables and $f(x)$ represents the vector field defining the dynamics of the system. Equilibrium points play a crucial role in understanding the behavior and stability of dynamical systems. They serve as reference states where the system remains unchanged over time [145].

**2.6.3 Time dependent equilibrium point:** A time-dependent equilibrium point refers to a solution of a dynamical system where the state variables of the system reach a steady state, but this steady state varies with time. Mathematically, consider a dynamical system described by ordinary differential equations $\dot{x} = f(t, x)$, where $x$ represents the state variables of the system and $f(t, x)$ is a vector field that defines the evolution of the system over time [145]. A time-dependent equilibrium point $x^*(t)$ satisfies the condition $\dot{x} = f(t, x^*(t)) = 0$.

**2.6.4 Types of Equilibrium points:** There are three types of equilibrium points and they are (1) Nodes, (2) Sink and (3) Sources [145].

I. **Nodes:** At a node, the dynamics of the system with respect to the state variable evaluated at the equilibrium point has a derivative that equals zero, while the dynamics at nearby points have a derivative that is negative. Mathematically, it is characterized by $f(x_e) = 0$ and $\frac{df}{dx} < 0$ for $x$ close to $x_e$.

II. **Sink:** A sink is a special case of a node where the derivative of the system's dynamics with respect to the state variable evaluated at the equilibrium point equals zero, and the derivative of the dynamics with respect to the state variable at nearby points is negative, and there are no zero eigenvalues of the Jacobian matrix. Mathematically, it is characterized by $f(x_e) = 0$, $\frac{df}{dx} < 0$ for $x$ close to $x_e$, and all eigenvalues of the Jacobian matrix have negative real parts.

III. **Sources:** At a source, the dynamics of the system with respect to the state variable evaluated at the equilibrium point has a derivative that equals zero, while the dynamics at nearby points have a derivative that is positive. Mathematically, it is characterized by $f(x_e) = 0$ and $\frac{df}{dx} > 0$ for $x$ close to $x_e$.

### 2.6.5 Malthusian and Logistic Growth Model

In the early stages of modelling biological processes, the growth rate of a population (birth-death) was calculated using the following equation:

$$\frac{dx}{dt} = kx \tag{2.1}$$

Here, $k$ is a constant. The population experiences a constant growth rate '$k$' time the current numbers, without any restrictions on resources [125]. However, it is important to note that this model has limitations, as populations in ecology cannot sustain indefinite growth. Instead, this model may hold true for a certain period of time. In response to the shortcomings of the Malthusian model, the growth of the population is reevaluated using the following Logistic growth model equation:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) \tag{2.2}$$

Where '$r$' is the intrinsic growth rate and '$K$' is the maximum carrying capacity of the environment. In this logistic growth model, the population increases similarly to the Malthusian growth model when '$x$' is small. However, when '$x$' is large, the species start competing for limited resources.

### 2.6.6 Prey-predator Model: Lotka-Volterra System

Lotka suggested a simple model for predator-prey relationships to explain fluctuations in fish catches [120]. The Volterra model is given by the equations below, where $n(t)$ represents the prey population and $p(t)$ represents the predator at time $t$,

$$\frac{dn}{dt} = n(\delta - \vartheta p)$$

$$\frac{dp}{dt} = p(\sigma n - \tau) \tag{2.3}$$

Where $\delta, \vartheta, \sigma, \tau$ are non-negative constants.

The above-mentioned model is built upon certain assumptions:

a) The prey population experiences unrestricted growth in the absence of predators.

b) The term $\vartheta np$ signifies the impact of predation on prey, leading to a decrease in its per capita growth rate.

c) In the event of prey scarcity, the predator mortality rate leads to exponential decline, which is directly proportional to the term $\tau p$.

d) The growth rate of the predator is influenced by the contribution of the prey, which is denoted as $\sigma np$. This means that the prey's contribution is directly proportional to both the abundance of available prey and the size of the predator population.

### 2.6.7 Prey dependent functional response

The ecological concept of functional response refers to the rate at which a consumer consumes food in relation to its density. This concept is closely linked to the numerical response, which measures the reproductive rate of a consumer in relation to food density. In the following discussion, we will explore various types of functional responses.

1. **Prey dependent functional response:** The consumption rate of predators is solely determined by the prey in the prey-dependent function type, denoted as $F(x,y) = F(x)$. Prey-dependent functional responses can be classified in various ways [82]. Here are some of the most commonly recognized types.

- **Lotka-Volterra Type:**

  In this type, the rate at which prey is consumed by each predator steadily increases in a linear manner. This functional response can be mathematically represented as: $F(x) = ax$, where $a > 0$ denotes the rate at which a predator consumes its prey.

- **Holling type-I:**

$$F(x,y) = F(x) = ax \quad \text{for } 0 < x < a$$
$$= b \quad \text{for } x \geq a. \tag{2.4}$$

Here, $x$ represents the density of the prey population, while $y$ represents the density of the predator population. The value of $a$ signifies the resource level at which the predator reaches satiation at $b$. Consequently, when $a$ takes on significantly large values, the equation simplifies to the Lotka Volterra form.

- **Holling type-II:**

$$F(x,y) = F(x) = \frac{Mx}{1+Mnx} = \frac{\alpha x}{\beta + x} \tag{2.5}$$

Here, $M$ represents the search rate, while $n$ denotes the handling time. The variable '$\alpha$' stands for the maximum harvest rate and '$\beta$' represents the half saturation level.

- **Holling type-III:**

$$F(X) = \frac{Mx^2}{1+Mnx^2} = \frac{\alpha x^2}{\beta + x^2} \tag{2.6}$$

This behaviour is characteristic of a generalist predator that transitions between different food species or predators that focus their feeding activities in areas with plentiful resources.

**2. Ratio dependent functional response:**

The model is referred to as ratio-dependent when

$$f_1(x_1, x_2) = f_1\left(\frac{x_1}{x_2}\right) \tag{2.7}$$

Leslie introduced the earliest ratio dependent model, which was later examined by Leslie-Gower and Pielou as follows:

$$\frac{dx_1}{dt} = \alpha_1 x_1 - \alpha_2 x_1 x_2$$

$$\frac{dx_2}{dt} = \alpha_1 x_2 \left(1 - \frac{x_2}{\sigma_1 x_1}\right) \tag{2.8}$$

In this particular model, the carrying capacity $K$ is determined by the presence of resources, denoted as $x_1$. As a result, $K$ is directly proportional to the abundance of prey population $x_1$, with $\sigma_1$ representing the constant of proportionality.

It is becoming increasingly evident that in certain scenarios, the per-capita growth rate of predators should be determined by the ratio of prey-predator abundance. The concept of a ratio-dependent functional response was given as:

$$F(X,Y) = \frac{\alpha\left(\frac{X}{Y}\right)}{1+Mn\left(\frac{X}{Y}\right)} = \frac{\alpha X}{\beta Y + X} \tag{2.9}$$

The consumption of prey in these models is not directly linked to prey density, but rather to the ratio of prey and predator densities. This allows for the consideration of predator abundance or scarcity in the ecosystem.

### 2.6.8 Stability and Instability

This process enables the validation of a dynamical system to determine if its trajectories converge towards the equilibrium point [145]. Validation can be accomplished through numerical methods, Lyapunov stability analysis, or structural stability assessment.

I. **Stability:** Let $D$ represent the spherical region centered at the equilibrium point $x^*$ defined by $|x - x^*| < R$ in the state space, and let the sphere $|x - x^*| = R$ be denoted by $S$. The equilibrium point $x^*$ is considered stable if, for any $R > 0$, there exists $r > 0$ such that $|x(0) - x^*| < r$ implies $|x(t) - x^*| < R$ for all $t > 0$. Conversely, if this condition does not hold, then $x^*$ is termed an unstable equilibrium point.

II. **Asymptotic stability:** An equilibrium point $x^*$ is considered asymptotically stable if it is stable and there exists a positive $r > 0$ such that $|x - x^*| < r$ which implies $x(t) \to x^*$ as $t \to \infty$. Additionally, $x^*$ is termed locally asymptotically stable if it is locally stable and all solutions starting near $x^*$ converge to $x^*$ as $t \to \infty$.

III. **Local stability:** One way of finding out the local stability of an equilibrium point in a dynamical system is through linearization of the dynamics by means of the Jacobian matrix $J(x^*)$, which expresses the variable $x^*$ as the equilibrium point. If all eigenvalues for $J(x^*)$ have negative real parts $(Re(\lambda_i) < 0 \; for \; all \; i)$, then the equilibrium point can be considered locally stable. However, if any eigenvalue for $J(x^*)$ has a positive real part, then it would be regarded as locally unstable [145]. This analysis allows us to know what kinds of behaviours trajectories can have in a neighbourhood around an equilibrium point, and this knowledge provides valuable information about system behaviour without considering its global dynamic features.

**IV.** **Global stability:** Global stability involves examining the behaviour of a dynamical system within its entire state space, taking into account all the possible initial conditions. In contrast with local stability, which concentrates only on the behaviour near given equilibrium points, global stability implies the study of the whole dynamics of the system and how its trajectories evolve through time independently of their initial states. A dynamical system is said to be globally stable if, regardless of starting positions in the state space, all paths eventually converge to some stable configurations such as equilibria, limit cycles, or invariant sets [145]. Analysis of global stability often focuses on the behaviour of trajectories in the long run and the recognition of attractors with their basins. A fundamental result is that trajectories from different initial conditions must tend to one and only one attractor. This analysis helps to understand the nature of the system as a whole, ensuring that it behaves predictably and reliably throughout its phase space.

### 2.6.9 Linear stability analysis:

In order to determine the stability of a dynamic system, it is important to observe how it behaves near its equilibrium states. These equilibrium states are achieved by setting all time-dependent variables' derivative values to zero. Even though the system's input and output remain constant at the equilibrium state, it is crucial to understand whether any deviations from this state will cause significant changes in the entire system or if they will be dampened out. Local stability analysis focuses on determining the survival and coexistence of a species when there are small deviations from the equilibrium state [145]. On the other hand, the global stability approach is used to discuss the long-term behaviour of a system. The following methodology is employed to find the stability of a dynamic system.

Consider a dynamical system described by the equation $\dot{x} = f(x)$, where $x$ represents the state vector and $f(x)$ represents the dynamics of the system. Suppose $x^*$ s an equilibrium point, such that $f(x^*) = 0$.

We can expand the dynamics $f(x)$ around $x^*$ using a Taylor series:

$$f(x) = f(x^*) + \frac{\partial f}{\partial x}\bigg|_{x^*} (x - x^*) + \Phi((x - x^*)^2).$$

Since, $f(x^*) = 0$, the linear term becomes:

$$f(x) \approx \frac{\partial f}{\partial x}\bigg|_{x^*} (x - x^*).$$

This expression resembles the linear approximation of a function around a point, where $\frac{\partial f}{\partial x}\Big|_{x^*}$ is the Jacobian matrix $J(x^*)$. Therefore, the linearized dynamics around $x^*$ can be written as:

$$\dot{x} \approx J(x^*)(x - x^*).$$

The stability of $x^*$ can then be analysed by examining the eigenvalues of $J(x^*)$. If all eigenvalues have negative real parts, $x^*$ is locally stable. If any eigenvalue has a positive real part, $x^*$ is locally unstable. This linear stability analysis provides insights into the behavior of the system near $x^*$ based on its linearized dynamics.

**2.6.10 Eigen Values and Eigen Vectors**

The historical origins of the concept of eigenvalues and eigenvectors can be traced back to the 18th century, where mathematicians like Leonhard Euler and Joseph Louis Lagrange explored the idea of characteristic roots and vectors. However, it was not until the 19th century that the formal study of eigenvalues and eigenvectors began. It was during this time that mathematicians such as Carl Friedrich Gauss, Augustin-Louis Cauchy, and Hermann von Helmholtz made significant contributions to the theory. The term "eigenvalue" was later coined by David Hilbert in 1904. The early 20th century witnessed the growth of linear algebra as a distinct field, with luminaries like David Hilbert, Emmy Noether, and Hermann Weyl leading the way in advancing the topic. Since then, eigenvalues and eigenvectors have become fundamental concepts in many areas of mathematics, including linear algebra, differential equations, and functional analysis, and they play an important role in a wide range of applications across scientific fields.

Eigenvalues and eigenvectors are mathematical properties of square matrices $A$. An eigenvalue, symbolized as $\lambda$, is a scalar that possesses the property of having a nonzero vector $v$ (known as the eigenvector) that satisfies the equation $Av = \lambda v$ [145]. This relationship can be mathematically expressed as the matrix equation $(A-\lambda I)v=0$, where $I$ represents the identity matrix. The process of determining the eigenvalues involves solving the characteristic equation $|A-\lambda I|=0$, which provides the values of $\lambda$ for which the matrix $A-\lambda I$ becomes singular. Once the eigenvalues have been obtained, the corresponding eigenvectors can be found by solving the system of equations $(A-\lambda I)v=0$ for each eigenvalue $\lambda$. Eigenvectors, being nonzero vectors, define directions within the vector space that remain unchanged or are scaled by the linear transformation represented by the matrix $A$. On the other hand, eigenvalues represent the scaling factors associated with these eigenvectors.

### 2.6.11 Jacobian matrix:

The Jacobian matrix represents the derivatives of a vector-valued function in terms of its input variables [145]. It is a matrix of partial derivatives, with each entry representing the rate of change of one component of the output vector in relation to one input variable.

Consider a vector-valued function $f:\mathbb{R}^n \rightarrow \mathbb{R}^m$, which maps an $n$-dimensional input vector $x$ to an $m$-dimensional output vector $y=f(x)$.

The Jacobian matrix of $f$, denoted by $J(f)$, is defined as follows:

$$J(f)=\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \tag{2.10}$$

Here:

$f_i$ represents the $i$-th component function of $f$.

$\frac{\partial f_i}{\partial x_j}$ denotes the partial derivative of the $i$-th component function with respect to the $j$-th input variable $x_j$.

The Jacobian matrix $J(f)$ shows how the $i$-th output variable changes based on the $j$-th input variable. For example, in the Jacobian matrix, $\frac{\partial f_i}{\partial x_j}$ represents the rate of change of the $i$-th output variable with respect to the $j$-th input variable in the $i$-th row and $j$-th column.

The Jacobian matrix is a fundamental tool in calculus, optimization, and various fields of applied mathematics and physics. It plays a vital role in capturing significant information about the local behaviour of a function, specifically regarding the impact of slight variations in the input variables on the corresponding changes in the output variables.

### 2.6.12 Routh Hurwitz criteria

The Routh-Hurwitz criterion is a technique utilized in control theory to determine the stability of a linear time-invariant (LTI) system by examining the coefficients of its characteristic polynomial. This method offers a structured approach to stability analysis without the need to directly find the roots of the polynomial. The criterion involves constructing a tabular array called the Routh array, where the polynomial coefficients are organized in a specific manner. By observing the sign changes in the first column of the array, conclusions can be drawn regarding the system's stability. A stable system is indicated when there are no sign changes in the first column. Conversely, the number of sign changes corresponds to the number of unstable roots in the characteristic polynomial. The Routh-Hurwitz criterion plays a crucial role in the analysis and design of control systems [155].

Consider the constants $a_1, a_2, \ldots\ldots, a_n$ to be real numbers.

The equation $\mathcal{L}(\kappa) = \kappa^n + a_1\kappa^{n-1} + \ldots\ldots + a_n = 0$ possesses roots with negative real parts if and only if the determinant values of the subsequent matrices:

$$Q_1 = (a_1),\ Q_2 = \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix},\ Q_3 = \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{pmatrix} \ldots\ldots\ldots Q_n = \begin{pmatrix} a_1 & 1 & 0 \ldots\ldots & 0 \\ a_3 & a_2 & a_1 \ldots\ldots & 0 \\ \ldots & \ldots & \ldots\ldots & 0 \\ \ldots & \ldots & \ldots\ldots & a_n \end{pmatrix}$$

are all positives.

Here $(i,k)^{th}$ entry in the matrix $Q_j = \begin{cases} a_{2i-k}, & 0 < 2i-k < n \\ 1 & 2i-k=0 \\ 0 & 2i-k<0 \; or \;\; 2i-k>n \end{cases}$ (2.11)

In particular, for cubic polynomials, these criteria simplify to $a_1 > 0$, $a_3 > 0$, $a_1 a_2 > a_3$.

### 2.6.13 Phase portrait or phase plane diagram

A phase portrait, also called a phase plane diagram, is a visual representation used in dynamical systems theory to analyse how a system of differential equations behaves. It shows the paths that the system's state variables follow over time. Each axis in the phase portrait represents one of the system's state variables, and the paths show how these variables change over time. These paths can provide important information about the system's behaviour, such as stability, periodicity, and the presence of attractors or repellers. By studying the shape and direction of the paths, we can understand the long-term behaviour of the system without explicitly solving the differential equations.

Phase portraits are very helpful for understanding nonlinear systems when it is hard or impossible to find analytical solutions. They give us a qualitative understanding of how the system behaves and can predict its behaviour with different starting conditions. To make a phase portrait, we usually graph the state variables against each other. For instance, if we have a system with two state variables $x$ and $y$, we graph $x$ on the $x$-axis and $y$ on the $y$-axis. Every point on the phase plane shows a distinct combination of the state variables at a specific moment in time.

### 2.6.14 Slope Field/ Direction Field:

A slope field, also known as a direction field, for a first-order ordinary differential equation (ODE) $\frac{dy}{dx} = f(x,y)$ is a visual depiction of a function $f(x,y)$ on a grid of points in the $xy$-plane. Every point $(x,y)$on the grid has a little line segment or arrow with a slope equal to $f(x,y)$. This generates a field of slope vectors indicating the direction a solution curve is going to take at each point. This representation allows us to better understand the behaviour of differential equation solutions and identify zones of increasing decreasing, or constant behaviour dependent on vector direction. We can also identify nullclines, where $f(x, y) = 0$, which provide further information on the behaviour of solutions.

### 2.6.15 Trajectory:

In differential equations and dynamical systems, a trajectory is the path that a system solution takes over time. It illustrates how the state variables of the system change over time, going from one initial condition to several successive ones. A trajectory is defined mathematically by integrating a set of differential equations with specified initial conditions. Every point on the trajectory represents a distinct system state at a given point in time. Depending on the kind of differential equations, trajectories may show a variety of behaviours, such as periodic oscillations around limit cycles, chaotic motion characterised by sensitivity to initial conditions, or convergence to stable equilibrium points. Trajectory analysis offers important information about the stability characteristics and long-term behaviour of dynamical systems.

### 2.6.16 Limit Cycles:

In a dynamical system, a periodic orbit is termed a limit cycle if all paths that begin near the orbit converge towards it as $t$ approaches infinity. Another way to define it is as an isolated periodic solution in which any neighbouring trajectory is dragged towards or away from the limit cycle. If the neighbouring trajectories are pulled to the limit cycle for every "$x$" in a neighbourhood, the limit cycle is stable; if the neighbouring trajectories are repulsed from the limit cycle for every "$x$" in a neighbourhood, the limit cycle is unstable. Henri Poincaré (1854-1912) pioneered the study of limit cycles [145].

### 2.6.17 Basic Theorems of Lyapunov on Stability:

There are several variations of Lyapunov's theorems, but the two most basic ones are often referred to as the Direct Method and the Indirect Method [145].

I. **Direct Method (First Theorem):**

Consider an autonomous system of ordinary differential equations represented as $\dot{x} = f(x)$, where $\dot{x}$ denotes the state vector of the system and $f(x)$ represents the vector field that defines the dynamics of the system.

The first theorem states that there exists a continuously differentiable function $\Upsilon(x)$, termed the Lyapunov function, such that:

- $\Upsilon(x)$ is positive definite (i.e., $\Upsilon(x) > 0$ $\forall x \neq 0$ and $\Upsilon(0) = 0$).
- The derivative of function $\Upsilon(x)$ along the system's trajectories, represented as $\dot{\Upsilon}(x)$, is negative semidefinite (i.e., $\dot{\Upsilon}(x) \leq 0$ $\forall$ $x$).
- If $\dot{\Upsilon}(x) < 0$, then $x = 0$ is asymptotically stable.

II. **Indirect Method (Second Theorem):**

Similarly, for the Indirect Method, there exists a continuously differentiable function $\Upsilon(x)$, termed the Lyapunov function, such that:

- $\Upsilon(x)$ is positive definite.
- The derivative of function $\Upsilon(x)$ along the system's trajectories, represented as $\dot{\Upsilon}(x)$, negative semidefinite.
- If $\dot{\Upsilon}(x) < 0$, then $x = 0$ is asymptotically stable.

These theorems outline the conditions for the stability of the equilibrium point $x = 0$ within the system. The direct method involves a direct evaluation of the derivative of the Lyapunov function, while the indirect method assesses the derivative indirectly through the system dynamics. Both approaches provide effective tools for analysing stability in nonlinear systems.

### 2.6.18 Poincare–Lyapunov Theorem:

The Poincare–Lyapunov theorem, also called the Poincare–Bendixson theorem, deals with the way solutions of two-dimensional autonomous differential equations behave. It gives conditions for the trajectories of these systems to show specific types of behaviour, like the presence of limit cycles [145].

The theorem states:

If the trajectory of a planar autonomous system remains within a closed, bounded subset of the phase plane without converging to a stable equilibrium point or diverging to infinity over time, it must either: (1) As time approaches infinity, the system tends to converge towards a periodic orbit, also known as a limit cycle, or (2) Remain in a finite region without oscillating indefinitely.

In other words, the theorem states that under specific conditions, trajectories of planar autonomous systems either display periodic behaviour by converging to a closed orbit (limit cycle) or remain restricted within a finite region of the phase plane without expanding to infinity.

The theorem has important consequences in studying dynamical systems, especially in understanding the presence and behaviour of limit cycles. These cycles are repeated paths seen in various real-life systems. The theorem offers valuable understanding of the qualitative behaviour of solutions without the need to explicitly solve the differential equations.

### 2.6.19 Basic Reproduction Number:

The Basic Reproduction Number ($R_0$) in an eco-epidemic model measures the potential for an infectious disease to spread throughout a coupled ecological and epidemiological system. It offers insights into how diseases spread and population dynamics are simultaneously influenced by interactions between hosts, pathogens, and ecological factors [129].

In the context of an eco-epidemic model, $R_0$ can be expressed as the product of several components:

$$R_0 = \Psi \times \Gamma \times \Phi, \tag{2.12}$$

Where:

$\Psi$: represents the transmission rate of the pathogen which signifies the likelihood of transmission occurring during each contact between an infected host and a susceptible host.

Γ: stand for the average amount of time an infected host is contagious, or the duration of infectiousness.

Φ: represents the average number of susceptible hosts in contact with an infected host during its infectious phase.

Mathematically, $R_0$ represents how ecological and epidemiological processes interact to drive the spread of disease within a population. Changes in host population density, predator-prey interactions, or environmental conditions, for instance, can have an impact on the contact rate Φ and the transmission rate Γ, which in turn can have an impact on $R_0$ and the dynamics of disease spread.

The $R_0$ estimation of an eco-epidemic model is critical for understanding how infectious diseases may affect host populations and ecosystems. By calculating $R_0$, researchers can identify important factors driving disease transmission, assess the effectiveness of interventions, and develop conservation and disease management methods that account for the interconnectedness of ecological and epidemiological processes.

### 2.6.20 Sensitivity analysis:

Sensitivity analysis is the process of measuring how modifications to a mathematical model's parameters or input variables impact the model's solution or output [144]. In Eco-epidemiology, understanding the relative significance of the different factors involved in its transmission is necessary to identify the most sensitive parameters. We calculate the basic reproduction number $R_0$ sensitivity index for different model parameters. These indices show the relative importance of each parameter for the spread of disease.

1. **Local Sensitivity Analysis:** Through local sensitivity analysis, we investigate how slight modifications to model parameters surrounding a particular scenario impact $R_0$ in an eco-epidemic model. To do this, one must compute the partial derivatives of $R_0$ for each individual parameter at a given set of parameter values. Examples of these include disease transmission rates, host population growth rates, predator-prey interaction coefficients, and environmental carrying capacities. These partial derivatives measure

how sensitive $R_0$ is to variations in each parameter, which aids in determining which ecological and epidemiological elements influence the disease's potential to spread within that particular ecological context.

2. **Global Sensitivity Analysis:** Analysing variations in model parameters over their whole range or distribution allows us to determine how these variations affect the overall value of $R_0$ and the dynamics of the eco-epidemic system. This is known as global sensitivity analysis. We can measure how important various parameters and interactions are in determining how diseases spread and how ecological dynamics work by methodically changing the values of the model's parameters and tracking changes in $R_0$ and related model outcomes, such as disease prevalence, host and predator populations, or ecosystem stability. Strategies including variance-based approaches (e.g., Sobol' indices) or Monte Carlo simulation, we can break down the variability in $R_0$ and pinpoint important variables that influence the unpredictability and uncertainty of disease transmission and ecosystem dynamics.

A model of eco-epidemic sensitivity analysis of $R_0$ helps inform management decisions and intervention tactics for infectious disease control in natural ecosystems by offering important insights into the intricate relationships between ecological and epidemiological processes. Researchers can determine important ecological and epidemiological factors influencing disease transmission, evaluate the possible effects of interventions on disease spread and ecosystem health, and create more efficient plans for disease control and conservation management by measuring the sensitivity of $R_0$ to changes in parameters.

### 2.6.21 Hopf bifurcation:

The concept of Hopf bifurcation holds significant importance in the analysis of dynamic systems, especially in nonlinear systems theory. It characterizes the way a system behaves differently when a parameter is changed. In particular, Hopf bifurcation is the result of a change in a stable equilibrium point that produces stable limit cycles or periodic orbits [145]. This can be explained mathematically as follows:

Consider a system described by a set of ordinary differential equations:

$$\frac{dx}{dt} = f(x, u) \quad (2.13)$$

where $u$ is a parameter, $x$ stands for the system's state variables, and $f(x, u)$ is a vector field that describes the dynamics of the system.

When the Jacobian matrix of the system evaluated at the equilibrium point has two purely imaginary eigenvalues, there exist a Hopf bifurcation at an equilibrium point $x^* = 0$ of the system, where $f(x, u) = 0$.

Mathematically, let $J$ be the Jacobian matrix of the system evaluated at the equilibrium point $x^*$, defined as:

$$J = \left.\frac{\partial f}{\partial x}\right|_{x=0}$$

A Hopf bifurcation takes place if the eigenvalues of $J$ are of the form $\lambda = \pm i\omega$, where $\omega$ is a non-zero real number and $i$ is the imaginary unit.

Moreover, the stability of the periodic orbits resulting from the Hopf bifurcation depends on the sign of the real part of the eigenvalues. Periodic orbits exhibit stability when the real part is negative (subcritical Hopf bifurcation), whereas instability occurs when the real part is positive (supercritical Hopf bifurcation).

Alternatively, it is not necessary to explicitly determine eigenvalues when using the equivalent condition for simple Hopf bifurcation, as suggested by Wei-Min Liu [119]. This criterion, also known as the Liu criterion, offers a useful method for locating Hopf bifurcations in dynamical systems without the need to compute eigenvalues.

The following two conditions must be satisfied, according to the Liu criterion, for a simple Hopf bifurcation to occur at an equilibrium point $x^*$:

1. At the equilibrium point $x^*$, the trace of the Jacobian matrix must be zero, i.e., $tr(J) = 0$.
2. The determinant of the Jacobian matrix at $x^*$ must be positive, i.e., $det(J) > 0$.

The first condition, $tr(J) = 0$, ensures that the equilibrium point is neither a stable nor an unstable node or spiral. It indicates that the linearized dynamics at the equilibrium point $x^*$ are purely rotating, which is a key feature of Hopf bifurcations and in order for oscillatory behaviour to occur, the second criterion, $det(J) > 0$, must guarantee that the linearized dynamics rotate anticlockwise around the equilibrium point $x^*$.

By fulfilling both conditions of the Liu criterion, it is possible to clearly detect the presence of a simple Hopf bifurcation in the dynamical system without explicitly identifying eigenvalues. This criterion provides a useful and computationally efficient way for identifying Hopf bifurcations in a variety of dynamical systems theory applications. Hopf bifurcation is also known as a Poincare–Andronov–Hopf bifurcation and is named after Henri Poincare, Aleksandr Andronov and Eberhard Hopf.

### 2.6.22 Optimal Control

The study of optimal control focuses on identifying the most effective strategies or control policies to enhance the behaviour or performance of a dynamic system over a period of time [154]. This area of research involves determining how to manipulate the inputs or controls of a system in order to achieve specific goals, while taking into account constraints and optimizing performance criteria.

Optimal control problems are typically presented as optimization tasks, where the objective is to discover a control function that either minimizes or maximizes an objective function, while considering system dynamics and constraints. The objective function represents the performance criterion that needs to be optimized, such as maximizing profit, minimizing cost, or attaining a desired state of the system. The control function outlines how inputs or controls should vary over time to accomplish the desired objectives.

From a mathematical perspective, consider a system described by a set of ordinary differential equations:

$$\frac{dx}{dt} = f(x, u, t) \tag{2.14}$$

In the above system, $x$ represents the state variables, $u$ represents the control inputs and $t$ represents time. The function $f$ explains the dynamics of the system.

The goal is to discover the control inputs $u(t)$ that minimize or maximize an objective function while considering constraints on the system dynamics, control inputs, and potentially the state variables. Mathematically, this can be expressed as an optimization problem:

Minimize $J^*(x, u, t)$

subject to:

$$\frac{dx}{dt} = f(x, u, t)$$

$$u_{min} \leq u(t) \leq u_{max}$$

$$x_{min} \leq x(t) \leq x_{max}$$

$$x(t_0) = x_0$$

$$x(t_f) = x_f$$

Here, the function $J^*(x, u, t)$ is the objective function, which is to be optimised, with $u_{min}$ and $u_{max}$ representing the lower and upper bounds on the control inputs, and $x_{min}$ and $x_{max}$ denoting the lower and upper bounds on the state variables. The initial and final conditions of the state variables are represented by $x_0$ and $x_f$, while the initial and final times are denoted by $t_0$ and $t_f$.

The objective of optimal control analysis is to identify the optimal control inputs $u(t)$ that either minimize or maximize the objective function $J^*(x, u, t)$, while adhering to the specified constraints. This process often requires the application of mathematical optimization methods like calculus of variations, Pontryagin's maximum principle, dynamic programming, or numerical optimization algorithms to identify the optimal solution.

### 2.6.23 Pontryagin's Maximum Principle

Pontryagin's Maximum Principle (PMP) is an important concept in optimum control theory, and it serves as an effective tool for analysing and solving optimal control problems. It specifies the required conditions that optimal control trajectories need to fulfil [146].

Mathematically, consider a system of an optimal control problem described by a set of the following dynamics:

$$\frac{dx}{dt} = f(x, u, t) \tag{2.15}$$

In the above system, $x$ represents the state variables, $u$ represents the control inputs and $t$ represents time. The function $f$ explains the dynamics of the system. The objective is to minimise or maximise the cost function $J^*(x, u, t)$ based on system dynamics and constraints.

The existence of an adjoint vector $p(t)$ that satisfies the following differential equation is stated by Pontryagin's Maximum Principle in the context of an optimal control problem without any state constraints:

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x}$$

Here, $H(x, p, u, t)$ represents the Hamiltonian function and is defined as follows:

$$H(x, p, u, t) = p^T . f(x, u, t) + J^*(x, u, t).$$

Additionally, the optimal control $u^*(t)$ that minimizes or maximizes the cost function must satisfy:

$$u^*(t) = argmin/max\ H(x, p, u, t)$$

Furthermore, the terminal condition for the adjoint vector $p(t_f)$ is given by:

$$p(t_f) = \frac{\partial \chi}{\partial x(t_f)}$$

Where $\chi\big(x(t_f), t_f\big)$ represents a terminal cost function, if it exists.

Pontryagin's Maximum Principle offers valuable insights into the framework of optimal control problems and presents a methodical way to discover optimal control strategies. It serves as the foundation for numerous optimization algorithms and methodologies applied in the examination and resolution of optimal control problems in diverse areas such as engineering, economics, and biology.

### 2.6.24 Conclusion / Summary

This chapter begins with an overview of the study, followed by a discussion of various existing scholarly works related to the research topic. Some of these are the importance of mathematical modelling in agriculture, eco-epidemiology models, the role of pesticides in new plant diseases, and how eco-epidemiology models can be used for integrated pest management, along with the associated mathematical methodologies. The chapter also identifies research gaps, outlines research questions, and states the objectives of the study in comparison with the previously existing studies. Finally, the chapter concludes with a summary section.