Chapter 3

An Eco-epidemic model with disease in Plant populations and Pesticides as control measure

This article has been published under the title "***An Eco-epidemic Model with Disease in Plant Populations and Pesticides as Control Measure***" in the IOSR Journal of Mathematics (IOSR-JM), **Volume 18, Issue 3 Series II (May – June 2022), pages 48-54.**

3

An Eco-epidemic model with disease in Plant populations and Pesticides as control measure

3.1 Introduction

The first section of the paper aims to highlight an explanation of the significance of ecology and ecosystems in terms of understanding how nature functions. Acquiring this knowledge is critical for understanding ecological processes and the methods by which diseases spread throughout their respective plant populations. Also, throughout the course of this chapter, we discussed the significance of mathematical eco-epidemiology in terms of acquiring an understanding of the functioning of ecosystems that are substantial in scale. By implementing mathematical models, researchers are able to acquire a more thorough understanding of the development of diseases as well as the management of their particular agricultural ecosystems. The use of mathematical models makes this possible. Applying this approach, which necessitates examining everything simultaneously, we can gain a comprehensive understanding of ecological phenomena. The information it provides enables us to devise strategies to prevent the spread of disease and ensure the proper maintenance of agricultural supplies [163].

3.2 Eco-Epidemic Model Formulation and Assumptions

3.2.1 Proposed mathematical model

The mathematical model displayed changes in plant populations when they are infected with a disease and when pesticides are used to get rid of the disease. Differential equations determine the rates of change in the populations of susceptible (x) and infective (y) plant species over time. According to the model, plant populations are divided into classes of susceptible and infective plants, and both groups are subjected to diseases. The complex relationship between disease spread and pesticide effects on plant populations can be studied using this formulation. It is also easy to find ways to stop the spread of disease in agricultural ecosystems [3].

3.2.2 Assumptions underlying the model formulation

The formulation of the model is based on important aspects of understanding plant populations, disease spread, and pesticide activity. First, it is assumed that there are healthy and susceptible plants. This decline allows for the for the study of plant disease dissemination. The model also assumes a direct correlation between susceptible and infected populations. Let $x(t)$ and $y(t)$ be the susceptible population and infective population of the total plant populations, respectively. Let $z(t)$ be the amount of pesticide used to control infectious diseases in plant populations. The following assumptions are taken into account while formulating the mathematical eco-epidemiological model:

i. In the presence of disease, the total plant population is composed of two classes: the susceptible population $x(t)$ and the infected population $y(t)$. Therefore, the total plant population at any point in time t is given by $x(t) + y(t) = N(t)$.

ii. In the absence of the disease, the plant populations grow at a growth rate of r.

iii. The pesticide $z(t)$ is applied to both the susceptible and the infective populations.

iv. A linear response for βxy between susceptible and infective is considered.

v. A linear response for dyz between the infectives and pesticides is considered.

vi. All the model parameters are assumed to be positive.

This study examines how pesticides influence susceptible and infective plants. These enable us to study how chemicals like pesticides combat diseases. Using these assumptions, the model illustrates the interconnection between plant populations, disease transmission, and pesticide usage. This will aid in understanding and preventing ecological diseases in agricultural ecosystems [9].

The model equations below are based on the assumptions taken above:

$$\frac{dx}{dt} = \mathrm{r} - \beta xy - \mu x - cxz$$

$$\frac{dy}{dt} = \beta xy - \mathrm{dyz} - \mu y$$

$$\frac{dz}{dt} = \alpha z(1 - z) \tag{3.1}$$

With initial conditions $x(0) \equiv x_0 > 0, y(0) \equiv y_0 > 0 \; and \; z(0) \equiv z_0 > 0.$ (3.2)

Where $\frac{dx}{dt}, \frac{dy}{dt}$ and $\frac{dz}{dt}$ represent the rate of change of the quantities $x(t), y(t)$ and $z(t)$ respectively.

Here, r is the constant growth rate of the plant populations, β is the susceptible and infective contact rate, μ is the natural death rate of plant populations, c is the susceptible and pesticides contact rate, d is the infective and pesticides contact rate and α is the amount of pesticides used.

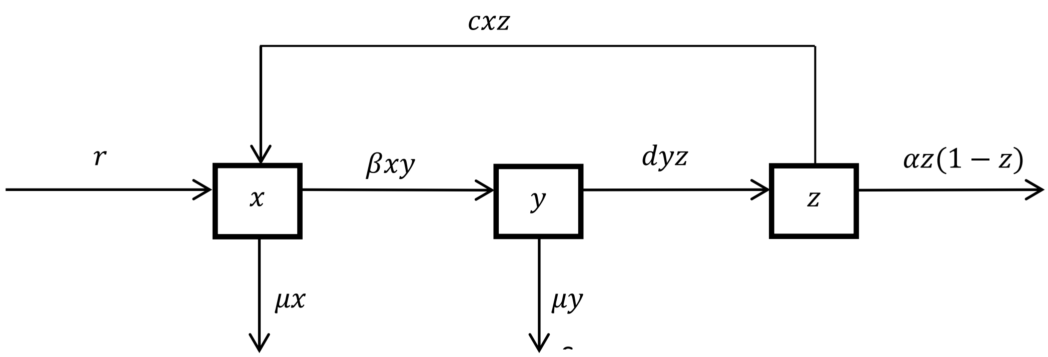

Figure 3.1: Transfer diagram of model (3.1)

3.2.3 Representation of plant populations, disease dynamics, and pesticide intervention

Variables that illustrate the spread of diseases, plant populations, and pesticide use are combined in the eco-epidemic model. This helps us understand how different parts of agricultural ecosystems work together. Both $x(t)$ and $y(t)$, which are variables, show plant populations that are susceptible and infective. The total plant population is shown by $N(t) = x(t) + y(t)$. By adding the same number of pesticides to both susceptible and infective plants, the variable $z(t)$ illustrates how crucial it is to control diseases. The idea behind this model is that there are straight lines that connect pesticides and susceptible plants (cxz), infective plants and pesticides (dyz), and susceptible plants and infective plants (βxy). These assumptions allow for the formulation of a set of ordinary differential equations (ODEs) that show the rate of change of $x(t)$, $y(t)$, and $z(t)$ over time. How diseases spread and how well pesticides work in agricultural ecosystems to keep plants from getting infected are shown by the number of links in this model.

Plant development, the spread of disease, and the application of pesticides are all examples of ecological processes that can be simulated. Using this model, one can conduct a comprehensive investigation into the progression of disease. Through the use of a variety of variables, the model explains how everything works and how they link together using a variety of variables. The results demonstrate that there is a connection between the level of sensitivity, the number of diseases, and the effectiveness of pesticides.

3.2.4 Linear responses between populations and pesticides

Through the use of this model, we are able to gain a better understanding of how epidemics occur and how to effectively manage diseases in plant ecosystems. It has been hypothesised that chemical substances have a direct influence on plant populations that are sensitive to and afflicted with diseases. The model displays more specific information. There are straight lines connecting these populations, which indicate which ones are susceptible to the disease and which ones are affected by it. There is a connection between the population that is diseased and the population that is sensitive to the use of pesticides. It is possible to evaluate the effectiveness of pesticides as control measures by studying how they influence disease-prone plant types as well as plant types that are sensitive to disease. The model provides a framework for assessing how different pesticide treatments affect disease dissemination and pesticide population dynamics within plant groups. This is accomplished by including these linear responses in the model. The development of the most effective strategies for preserving the health of ecological systems and reducing the incidence of diseases that are prevalent in agricultural regions is facilitated by the information here [202].

3.3 Positivity and Boundedness Analysis

3.3.1 Theorem 3.1: Positivity of solutions in the system

Theorem 3.1: All solutions of the system represented by (3.1) that start in R^3 remain positive at R_+^3 for all $t \geq 0$.

Proof: Considering (3.1) in a matrix form, $\dot{X} = F(X)$,

where, $X = [x, y, z]^T$ and $\bar{X}(0) = [x_0, y_0, z_0]^T \in {R_+}^3$ and

$$F(X) = \begin{bmatrix} \mathrm{r} - \beta xy - \mu x - cxz \\ \beta xy - dyz - \mu y \\ \alpha z(1 - z) \end{bmatrix}.$$

It is observed that, for $\bar{X}(0) \in {R_+}^3$, whenever and $\bar{X}(0) = [0,0,0]^T, F(X) \geq 0$, So the solution of (3.1) will always lie in ${R_+}^3$.

Hence, the theorem with the help of this derivation above is proved.

The results of the eco-epidemic model are investigated through the use of a positive analysis. In accordance with Theorem 3.1, which deals with the positivity of solutions, it is imperative that all variables that characterise plant populations and disease processes remain positive as the model expands. An explanation is provided by the model, which places an emphasis on the positive aspects of responses. According to these concepts, the rate that is predicted by the ecological population model is biologically acceptable. This is because diseases and low population numbers are not expected to occur in nature. It is stated in Theorem 3.1 that the model is able to reliably predict how plant populations will evolve and how diseases will spread, provided that its findings are satisfactory [14]. This concept provides an explanation for the results of the model in agricultural contexts, which are characterised by low disease populations that would defy biological rules. Performing the eco-epidemiological positivity analysis, which is the final phase, is necessary in order to guarantee that the model provides us with informative information regarding the ways in which eco-epidemiological processes evolve over time.

3.3.2 Theorem 3.2: Boundedness of solutions in the system

Theorem 3.2: All solutions of the system (3.1) that start in R_+^3 are uniformly bounded in the solution set $\Omega = \left\{(x, y, z) : 0 \leq x \leq \frac{r}{\mu}, 0 \leq y \leq \frac{r}{\mu}, 0 \leq z \leq 1, 0 \leq x + y \leq \frac{r}{\mu}\right\}$.

Proof: Let $x(t), y(t), z(t))$ be the solution of the system (3.1).

Let $w = x + y$.

Then,

$$\frac{dw}{dt} = \frac{dx}{dt} + \frac{dy}{dt}$$

$$\frac{dw}{dt} = \mathrm{r} - \beta xy - \mu x - cxz + \beta xy - \mathrm{dyz} - \mu y$$

$$\frac{dw}{dt} + \mu w = r - (cx + dy)z$$

Therefore, $\frac{dw}{dt} + \mu w \leq r \Rightarrow w \leq \frac{r}{\mu} + \mathrm{C}e^{-\mu t}$ and $w \leq \frac{r}{\mu} + \left(w_0 - \frac{r}{\mu}\right) e^{-\mu t}$

As t tends to infinity, $e^{-\mu t}$ tends to 0 implies W tends to $\frac{r}{\mu}$

Thus $w(t) \leq \frac{r}{\mu}$ and hence w is bounded.

Now $\frac{dz}{dt} = \alpha z(1 - z) \Rightarrow z = \frac{1}{1-Ce^{-\alpha t}}$

As t tends to infinity, z tends to 1.

Hence, z is bounded for all t and any initial value.

Therefore, it is proved that x, y, z are bounded.

The boundedness analysis of Theorem 3.2 demonstrates that the eco-epidemic model's results remain within biologically significant time ranges. This result of boundedness gives the model more weight because it ensures that disease or population numbers don't continue to rise or exceed ecological limits [65]. The model adheres to ecological truth by utilising only a limited number of options. This makes it better for understanding how diseases spread and working out how to run agricultural ecosystems. The positivity and boundedness tests [15] clearly demonstrate that we can use the eco-epidemic model to study how plant populations change over time and how diseases spread.

The boundedness of solutions in the eco-epidemic system can provide insights on the long-term behaviour of the epidemic inside the ecological model. It asserts that the system's responses remain constant. This halts ecological and epidemiological processes from either speeding up or decelerating. In the field of mathematical operations, this refers to the condition where all variables that describe the susceptible and affected plant populations, as well as the amounts of pesticides, stay within certain boundaries throughout the system's growth. This must be supplied to ensure the model is ecologically accurate. If pesticide quantities or levels were able to increase or decrease indefinitely, it would result in the breakdown of several systems. An effective approach to illustrating boundedness is by analysing the behaviour of the system and proving that solutions cannot tend towards infinity regardless of the initial conditions. Evaluate the stability and resilience of ecosystems damaged by disease and pesticides by assessing the boundedness of solutions. Furthermore, it has the advantage of protecting agricultural ecosystems and effectively managing plant diseases [16].

Hence, based on Theorems 3.1 and 3.2, we can conclude that the model is well-defined and biologically valid.

3.3.3 Proof and implications of positivity and boundedness for the model's biological validity

The eco-epidemic model is subjected to a positivity and boundedness analysis in order to see how well it can demonstrate how things function in the real world and how well it can explain how things work in the biological world. The purpose of this investigation is to demonstrate that the answers provided by the model continue to be biologically true and accurate throughout time. The proof of positivity demonstrates that all of the state variables in the system, such as disease rates and plant populations, remain positive even as the model takes on more characteristics. The model will be able to avoid receiving negative numbers that are devoid of any biological meaning, provided that this is done. In addition to this, the boundedness analysis demonstrates that the outcomes of the model remain contained within particular regions.

3.4 Equilibria

3.4.1 Identification of equilibrium points in the eco-epidemic model

Equilibria are stable states in the eco-epidemic model where all of the system's variables remain constant. By putting the derivatives of all state variables to zero and finding these points, a set of equations that describe the conditions under which the system reaches stability is created. When all populations have died out or the disease has spread so far that there are no longer any susceptible hosts, trivial answers are found. In non-trivial solutions, populations have a certain amount of disease and live in peace [20].

For finding the equilibrium points, we set the right-hand side of the system (3.1) equals to zero. The system (3.1) has been identified to be having the following equilibrium points:

I. The trivial equilibrium point $E_0(0,0,0)$.

II. The equilibrium point $E_1(0,0,1)$.

III. The equilibrium point $E_2(x_1,0,0)$ where $x_1 = \frac{r}{\mu}$ which always exist.

IV. The equilibrium point $E_3(x_2, 0, z_2)$where $x_2 = \frac{r}{\mu+c}$ and $z_2 = 1$.

V. The equilibrium point $E_4(x_3, y_3, z_3)$ where $x_3 = \frac{\mu}{\beta}$, $y_3 = \frac{1}{\mu}\left(r - \frac{\mu^2}{\beta}\right)$ and $z_3 = 0$.

VI. The endemic equilibrium point $E_5(x_4, y_4, z_4)$ where $x_4 = \frac{d+\mu}{\beta}$, $y_4 = \frac{r}{d+\mu} - \frac{(\mu+c)}{\beta}$ and $z_4 = 1$.

3.4.2 Analysis of trivial and non-trivial equilibrium points

When analysing eco-epidemiological models, it is crucial to comprehend the equilibrium points. In these stages, the system remains constant. Trivial and non-trivial equilibria can be identified. Trivial equilibria occur when all state variables are zero, leaving no plant populations or disease rates. Non-trivial equilibria, on the other hand, are stable conditions where populations and disease behaviour have achieved equilibrium. They coincide with biologically significant events, such as community-wide chronic disease transmission or the steady coexistence of healthy and diseased plants.

3.4.3. Conditions for existence and stability of equilibrium points

Theorem 3.3: The following equilibrium points

(i) $E_0(0,0,0)$, $E_2(x_1, 0,0)$ and $E_4(x_3, y_3, z_3)$ are saddle.

(ii) $E_5(x_4, y_4, z_4)$ is locally asymptotically stable in R^3_+ under the conditions:

A. $\beta r > (c+\mu)(d+\mu)$,

B. $\beta r + \alpha(d+\mu) > 0$,

C. $\beta r\left(1 + \frac{\alpha}{d+\mu}\right) > (c+\mu)(d+\mu)$,

D. $\left(\frac{\beta r}{d+\mu} + \alpha\right)\left[\beta r\left(1 + \frac{\alpha}{d+\mu}\right) - (c+\mu)(d+\mu)\right] > \alpha[\beta r - (c+\mu)(d+\mu)]$.

It has "equilibrium points," which are conditions in which all variables in the eco-epidemic model remain constant across time. Conditions for the existence and stability of equilibrium points are critical to understanding how the model acts and forecasting its long-term behaviour. The system may have equilibrium points if there are existence conditions, typically represented as a set of mathematical rules. The stability conditions, on the other hand, assess how the system reacts to changes in equilibrium and determine whether a point of equilibrium is stable, unstable, or partially stable. When things are in a state of safe balance, they attempt to revert to their former state following minor alterations. However, it behaves differently when it is at an undetermined equilibrium point. Semi-stable equilibria are stable over a large range of state space dimensions, but not across a wide range of dimensions [31]. It helps us figure out how to get rid of pests in agricultural ecosystems by revealing which plant populations are resistant to disease and which are susceptible to poisons.

3.4.4. Biological significance of different equilibria

The equilibria of the eco-epidemic model are positions that are safe from a biological perspective. A situation is said to be in trivial equilibrium when one or more populations do not undergo any changes. It is possible that there is no disease and no one is making any efforts to change the situation. Simple ecological processes, such as population dynamics and disease rates, are subject to alteration when the ecological system is in a null state. These reactions highlight the result of striking a balance between disease control, population growth, and management considerations. The outcome could be a long-term coexistence or a period of transition [34]. If legislators and ecologists are aware of the biological effects that each stability has, they will be able to devise individualized solutions that will enhance ecosystem health and vitality while also reducing the chance of disease outbreaks and population declines [36].

3.5. Stability Analysis

3.5.1 Application of Jacobian matrix for stability analysis

A Jacobian matrix is required in order to do an analysis of the stability properties that correspond to the model. The Jacobian matrix J of the system (3.1) can be expressed in the following manner:

$$J = \begin{bmatrix} -(\beta y + \mu + cz) & -\beta x & -cx \\ \beta y & \beta x - dz - \mu & -dy \\ 0 & 0 & \alpha - 2\alpha z \end{bmatrix} \tag{3.3}$$

A discussion of the stability of six points of equilibrium for the system (3.1) is presented in the following theorems.

Theorem 3.4: The trivial equilibrium point $E_0(0,0,0)$ is unstable.

Proof: The Jacobian matrix of E_0 is given by $J_{E_0} = \begin{bmatrix} -\mu & 0 & 0 \\ 0 & -\mu & 0 \\ 0 & 0 & \alpha \end{bmatrix}$.

Eigenvalues of the above matrix are $\lambda_1 = -\mu, \lambda_2 = -\mu, \lambda_3 = \alpha$.

Eigenvalues λ_1, λ_2 are always negative and λ_3 is positive,

Hence, E_0 can be identified to be unstable.

Theorem 3.5: The equilibrium point $E_1(0,0,1)$ is locally asymptotically stable.

Proof: The Jacobian matrix of E_1 is given by $J_{E_1} = \begin{bmatrix} -\mu - c & 0 & 0 \\ 0 & -d - \mu & 0 \\ 0 & 0 & -\alpha \end{bmatrix}$.

Eigenvalues of the above matrix are $\lambda_1 = -(c + \mu), \lambda_2 = -(d + \mu), \lambda_3 = -\alpha$.

Clearly, the eigenvalues $\lambda_1 < 0, \lambda_2 < 0, \lambda_3 < 0$ and therefore $\lambda_1, \lambda_2, \lambda_3$ have negative real parts.

Hence, the equilibrium E_1 is locally asymptotically stable.

Theorem 3.6: The equilibrium point $E_2(x_1, 0,0)$ is unstable.

Proof: The Jacobian matrix of E_2 is given by $J_{E_2} = \begin{bmatrix} -\mu & -\frac{\beta r}{\mu} & -\frac{cr}{\mu} \\ 0 & \frac{\beta r}{\mu} - \mu & 0 \\ 0 & 0 & \alpha \end{bmatrix}$.

Eigenvalues of the above matrix are $\lambda_1 = -\mu, \lambda_2 = \frac{\beta r}{\mu} - \mu, \lambda_3 = \alpha$.

Clearly, eigenvalue λ_3 is positive.

Hence E_2 is unstable.

Theorem 3.7: The equilibrium point $E_3(x_2, 0,1)$ is locally asymptotically stable if $A_0 < 1$, where $A_0 = \frac{\beta r}{(c+\mu)(d+\mu)}$.

Proof: The Jacobian matrix of E_3 is given by $J_{E_3} = \begin{bmatrix} -\mu - c & -\frac{\beta r}{c+\mu} & -\frac{cr}{c+\mu} \\ 0 & \frac{\beta r}{c+\mu} - d - \mu & 0 \\ 0 & 0 & -\alpha \end{bmatrix}$.

Eigenvalues of the above matrix are $\lambda_1 = -(c+\mu), \lambda_2 = \frac{\beta r}{\mu+c} - d - \mu, \lambda_3 = -\alpha$.

Here, the eigenvalues $\lambda_1 < 0, \lambda_3 < 0 \Rightarrow \lambda_1, \lambda_3$ have negative real parts.

Now for stability, we must have: $\lambda_2 < 0$ i.e. $\frac{\beta r}{\mu+c} - d - \mu < 0 \Rightarrow \frac{\beta r}{(c+\mu)(d+\mu)} < 1 \Rightarrow A_0 < 1$.

Hence, the equilibrium E_3 is locally asymptotically stable if $A_0 < 1$, where $A_0 = \frac{\beta r}{(c+\mu)(d+\mu)}$.

Theorem 3.8: The equilibrium point $E_4(x_3, y_3, z_3)$ is unstable.

Proof: The Jacobian matrix of E_4 is given by $J_{E_4} = \begin{bmatrix} -\frac{\beta r}{\mu} & -\mu & -\frac{c\mu}{\beta} \\ \frac{\beta r}{\mu} - \mu & 0 & \frac{-d}{\mu}\left(r - \frac{\mu^2}{\beta}\right) \\ 0 & 0 & \alpha \end{bmatrix}$.

Eigenvalues of the above matrix are $\lambda_1 = \frac{\frac{-\beta r}{\mu} - \sqrt{\left(\frac{\beta r}{\mu}\right)^2 - 4\mu\left(\frac{\beta r}{\mu} - \mu\right)}}{2}, \lambda_2 = \frac{\frac{-\beta r}{\mu} + \sqrt{\left(\frac{\beta r}{\mu}\right)^2 - 4\mu\left(\frac{\beta r}{\mu} - \mu\right)}}{2}$ and $\lambda_3 = \alpha$. Clearly, the eigenvalue λ_3 is positive.

Hence E_4 is unstable.

Theorem 3.9: The endemic equilibrium point $E_5(x_4, y_4, z_4)$ is locally asymptotically stable if and only if the following condition holds:

I. $\beta r > (c+\mu)(d+\mu)$,

II. $\beta r + \alpha(d+\mu) > 0$,

III. $\beta r\left(1 + \frac{\alpha}{d+\mu}\right) > (c+\mu)(d+\mu)$,

IV. $\left(\frac{\beta r}{d+\mu}+\alpha\right)\left[\beta r\left(1+\frac{\alpha}{d+\mu}\right)-(c+\mu)(d+\mu)\right]>\alpha[\beta r-(c+\mu)(d+\mu)]$.

Proof: We will use the Routh-Hurwitz Criterion to analysed the stability of the equilibrium point E_5.

The Jacobian matrix of E_5 is given by $J_{E_5}=\begin{bmatrix} -\frac{\beta r}{d+\mu} & -(d+\mu) & -\frac{c(d+\mu)}{\beta} \\ \frac{\beta r}{d+\mu}-(c+\mu) & 0 & -d\left[\frac{r}{d+\mu}-\frac{(c+\mu)}{\beta}\right] \\ 0 & 0 & -\alpha \end{bmatrix}$.

Here,

$w_1=-\det\left(J_{E_5}\right)=\alpha[\beta r-(c+\mu)(d+\mu)]$,

$w_2=-\mathrm{tr}\left(J_{E_5}\right)=\frac{\beta r}{d+\mu}+\alpha$,

$$w_3=\det\begin{bmatrix} -\frac{\beta r}{d+\mu} & -(d+\mu) \\ \frac{\beta r}{d+\mu}-(c+\mu) & 0 \end{bmatrix}+\det\begin{bmatrix} -\frac{\beta r}{d+\mu} & -\frac{c(d+\mu)}{\beta} \\ 0 & -\alpha \end{bmatrix}+\det\begin{bmatrix} 0 & -d\left[\frac{r}{d+\mu}-\frac{(c+\mu)}{\beta}\right] \\ 0 & -\alpha \end{bmatrix},$$

$$=\beta r\left(1+\frac{\alpha}{d+\mu}\right)-(c+\mu)(d+\mu).$$

According to the Routh-Hurwitz Criterion, the real parts of all eigenvalues of J_{E_5} are negative if and only if $w_i>0$ for $i=1,2,3$ (3.4) and $w_2w_3>w_1$.

Now, from equation (3.4), we have;

I. $w_1>0\Rightarrow\alpha[\beta r-(c+\mu)(d+\mu)]>0\Rightarrow\beta r>(c+\mu)(d+\mu)$,

II. $w_2>0\Rightarrow\frac{\beta r}{d+\mu}+\alpha>0$,

III. $w_3>0\Rightarrow\beta r\left(1+\frac{\alpha}{d+\mu}\right)-(c+\mu)(d+\mu)>0\Rightarrow\beta r\left(1+\frac{\alpha}{d+\mu}\right)>(c+\mu)(d+\mu)$,

IV. $w_2w_3>w_1\Rightarrow\left(\frac{\beta r}{d+\mu}+\alpha\right)\left[\beta r\left(1+\frac{\alpha}{d+\mu}\right)-(c+\mu)(d+\mu)\right]>\alpha[\beta r-(c+\mu)(d+\mu)]$.

Hence, the equilibrium point E_5 is considered to be locally asymptotically stable if and only if the criterion described above is satisfied.

The Jacobian matrix is utilised in the stability study of the eco-epidemic model in order to investigate the functioning of the system in close proximity to its equilibrium points. Through the process of linearizing the model equations around each equilibrium point, the Jacobian matrix is able to provide information regarding the stability of the system. One can determine if the equilibrium points of a term are stable or unstable by examining its eigenvalues. In the event that the points remain consistent over time, even negligible shifts will become less noticeable. It is necessary for the points to be less steady in order to observe changes more clearly, the points must be less stable. As a result of this, their behaviour is going to shift [37].

The Routh-Hurwitz Criterion is a useful tool for stability analysis. It does this by analysing the coefficient values in characteristic equations that are generated from the term. When determining whether or not a system is stable, several researchers use approaches other than eigenvalues. It is also possible to utilise this measurement.

3.5.2 Theorem 3.4-3.9: Stability analysis of equilibrium points using Eigenvalue theorem and Routh-Hurwitz Criterion

According to Theorems 3.4 to 3.9, it is clear that the analysis of stability at points of equilibrium in the eco-epidemic model is either stable or unstable. Both the Routh-Hurwitz criterion and the Eigenvalue theory are utilised in these applications. By providing us with insights into the system's behaviour over the long run, these theories, which are key instruments for analysing the stability of equilibrium points, also provide us with information. With the use of the Eigenvalue theorem, one may determine if the dynamics of the system lead to stable, unstable, or barely stable behaviour by examining the eigenvalues of the Jacobian matrix at each equilibrium point. Applying the Routh-Hurwitz criterion to analyse system stability is a truly fascinating experience. This mathematical approach provides a clear and methodical way to determine whether a system will remain stable, which is both intellectually rewarding and practically valuable. It is exciting to see how theoretical principles translate into real-world applications, enhancing our understanding and control of dynamic systems. Specifically, it examines the formulas for the characteristic polynomial of the system in order to determine the values that they contain. Researchers are able to assess whether equilibrium points are stable, in which case minor changes will have less of an impact over time, or unstable, in which case modifications will have a larger impact and could lead to significant changes in population and disease patterns. It is only the plant stability test that can give solid evidence of how well steps taken to stop diseases are working and how strong plant populations would be in agricultural ecosystems [51].

3.5.3 Interpretation of stability results of disease control and population dynamics

The stability analysis of the eco-epidemic model reveals the effectiveness of disease prevention measures and the evolution of plant populations over time. Researchers may find the mathematical term for the stability of equilibrium points in the model using the Jacobian matrix, the Eigenvalue theorem, and the Routh-Hurwitz criterion. Systems that aren't stable usually balance out once stable answers are found [52]. This shows that keeping diseases away may help keep plant populations stable. It is important for researchers to exercise caution since unstable solutions demonstrate that even minor adjustments can have significant consequences for populations.

To gain an understanding of the long-term stability of disease management, it is necessary to consider the ways in which various control strategies may alter the system's functioning over time. Based on the findings of fixed equilibria, it is possible that certain methods of plant management maintain healthy plant populations and reduce the number of infected plants. In order to prevent disease outbreaks or population losses, it may be necessary to take tougher or more targeted efforts to stabilise a fragile equilibrium. When taken as a whole, eco-analysis provides an explanation for the change in eco-epidemiological systems [53]. This makes it easier for people to think of methods to maintain the long-term health of agricultural ecosystems.

3.6 Numerical Analysis

For the purpose of gaining an understanding of the behaviour of the disease transmission dynamics model and the influence of control measures on the prevalence of disease, numerical analysis was utilised. Using MATLAB 2015a, this numerical analysis enabled the execution of a variety of simulations based on a wide range of scenarios and parameter values. This shed light on how varying levels of pesticide application and other interventions influence the dynamics of the SIR model in diverse settings. The accuracy of the predictions was ensured by comparing the results of the model with real-world data obtained from agricultural field observations. Furthermore, this data typically included recommendations for specific actions that farmers should take to manage diseases on their farms.

3.6.1 Parameters and initial conditions for numerical simulations

Set factors and starting conditions for the eco-epidemic model so that the computer analysis can give useful results. Factors like transfer rates, recovery rates, and pesticide efficiency are examples of parameters that influence ecological and epidemiological trends. These factors are often based on real-life facts or ideas. The basic factors display the state of the machine at the start. Some of these factors include the number of people who lived there at the beginning, the disease rate, and the number of chemicals used. To make sure they have the right starting conditions, models always begin in places that are like real ecological situations. Changes in parameters can also be tested using parameter sensitivity analysis to see how the model's results change. As a result, the analysis becomes more accurate [57]. Tools that work with numbers, like MATLAB, are used to run the eco-epidemic model over time, given the factors and starting conditions. Visualisation tools make it easier to understand simulation results by providing more information about how diseases spread, how populations change, and how well control measures work. Ecological analysis aids decision-making for disease control methods by carefully examining a range of parameter values and starting conditions. The parameters and initial conditions for numerical simulations are given as follows:

$S(0) = 100$ (100% of plant population), $I(0) = 0.1$(10% of plant population infected), $P(0) = 10$(proportion of pesticide used). $r = 0.1$, $k = 0.001$, $\beta_1 = 0.001$, $\gamma = 0.001$, $\mu = 0.01$, $\beta_2 = 0.02$, $\alpha = 0.02$, τ =0.01.

3.6.2 Numerical simulation of the eco-epidemic model using MATLAB

For the purpose of analysing eco-epidemic models on a computer, MATLAB simulations are utilised. This allows one to observe how the system develops in response to either positive or negative outcomes. For the purpose of obtaining precise results over time, MATLAB makes use of the model's components and initial conditions. Researchers are able to gain a better understanding of the mechanisms that cause diseases to spread and the effectiveness of the preventative measures that are taken by conducting these simulations. They were able to comprehend by demonstrating to them what the model predicted would take place. Due to the fact that MATLAB is capable of generating equations, researchers are able to investigate the ways in which a variety of elements, including as the weather or the usage of pesticides, influence the spread of disease and the behaviour of populations. MATLAB's drawing tools make it simple to discuss term models and determine whether or not the assertions made by the models about the real world are consistent with the real situation of reality [68]. The model will be compared to data from the real world in order to determine how accurately it captures the complex nature of ecological systems. This will allow them to determine how accurate the model is over the long term. As a results, numerical analysis is a useful instrument for comprehending eco-epidemiological processes, determining farming decisions, and exercising control over agricultural administration systems. The model (3.1) is simulated for time $t = 700$ using the parameters and initial values mentioned above. Figure 3.2-Figure 3.4 show the results of the simulation with and without any control measure, as well as with pesticide as a control measure for the proposed eco-epidemic model.

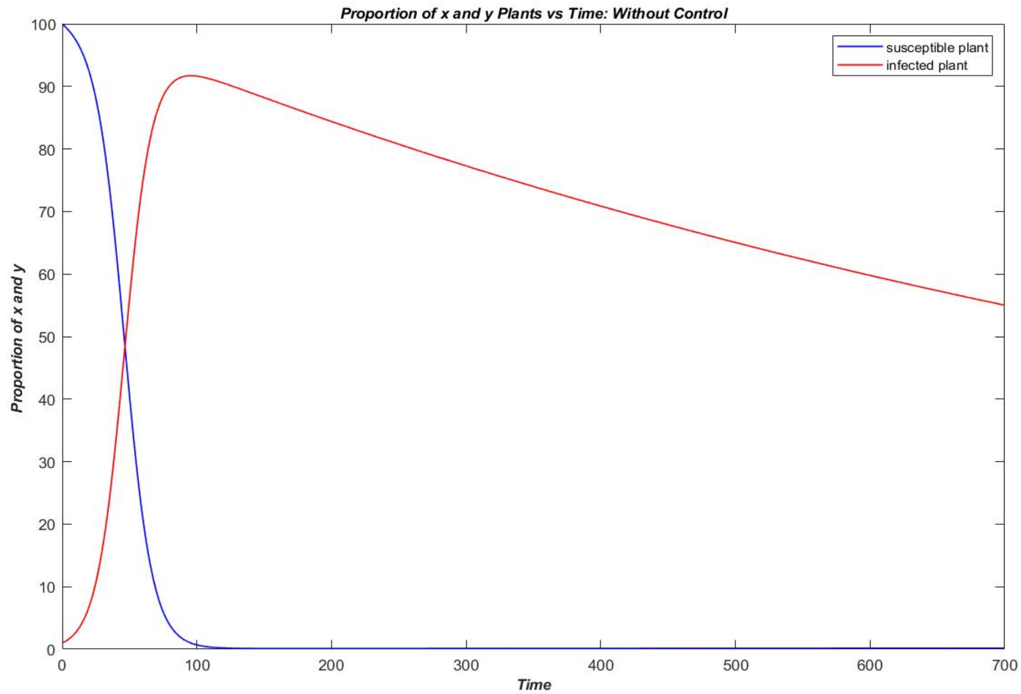

Figure 3.2: x-y model without control measure

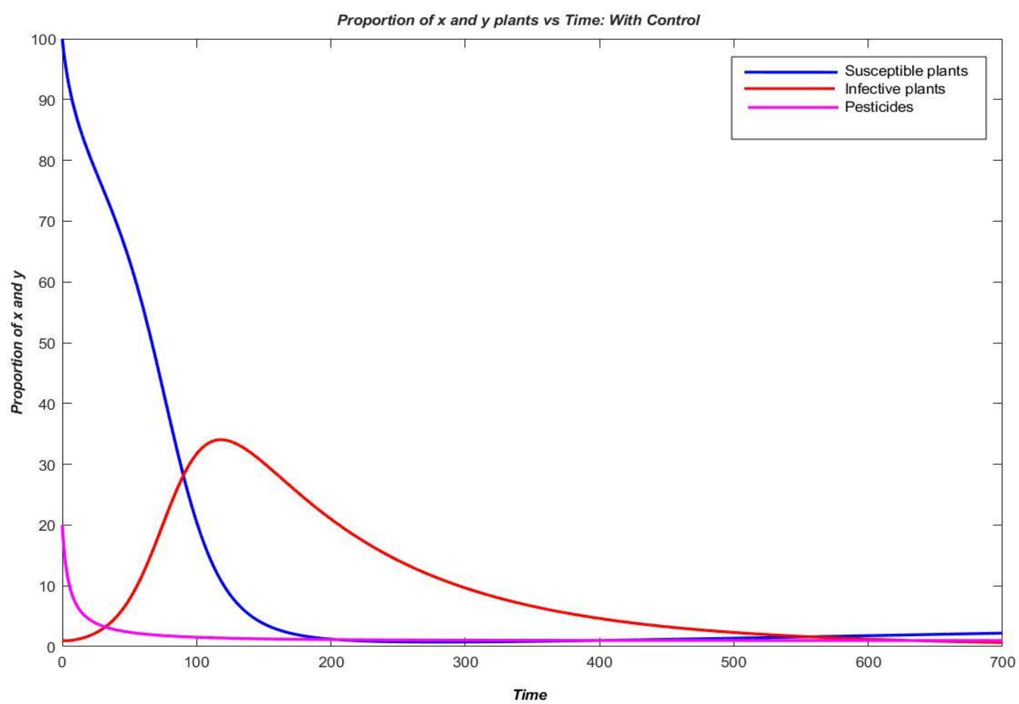

Figure 3.3: x-y model with pesticide as control measure

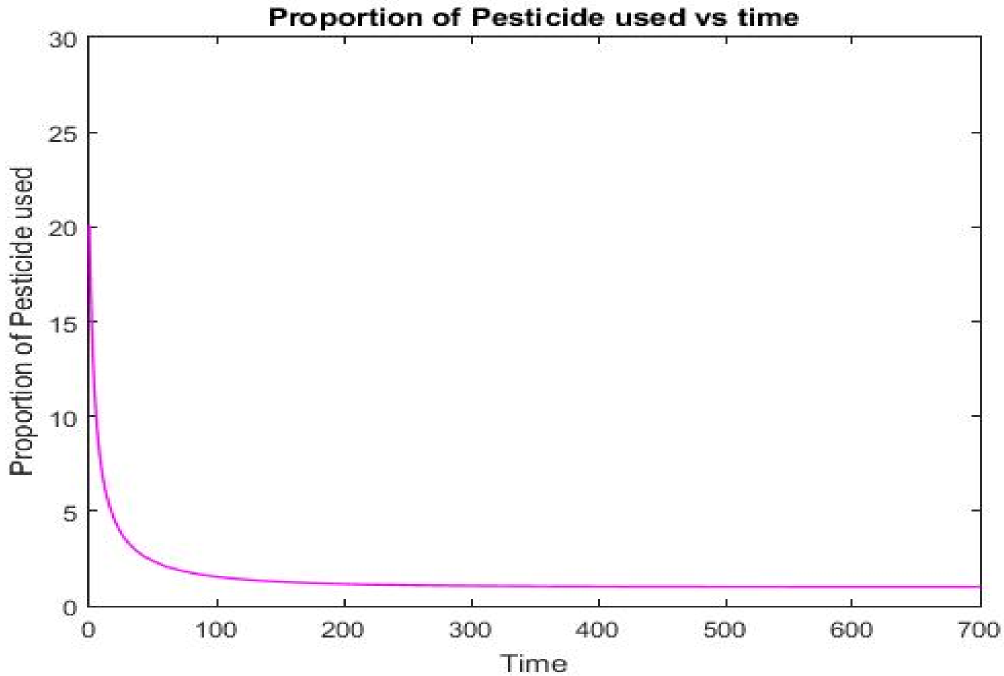

Figure 3.4: Amount of pesticides used

3.6.3 Visualisation and interpretation of simulation results

Through computer interpretation of the development of plant ecosystems and accounts of plant ecosystems, the emergence of disease populations and the spread of diseases in agricultural environments may be better understood. One can make reasonable assumptions about how the model will behave in various situations by entering a number and the starting conditions into a computer tool like MATLAB [71]. When the simulation results are shown as graphs, charts, or heatmaps, the data is easier to understand. There is a chance that this information could help us comprehend the ecological mechanisms and factors that lead to the spread of diseases.

Figuring out what the simulation results mean involves looking for patterns, links, and trends in the data. Researchers look at the effects of shifting factors, such as pesticide application rate or disease incidence starting point, on population changes and disease incidence over time. There are several methods to evaluate the eco-epidemic rate model. One way to do this is to observe how it works in real life. One advantage of the simulation results is that they help find effective ways to treat and prevent diseases in agricultural settings [73]. Computer-based data analysis aids scientists in understanding the interactions between disease, viruses, and plants. This will help them stop the spread of disease and keep long-term control over ecological processes.

3.7 Discussion

3.7.1 Synthesis of analytical and numerical findings

To understand the behaviour and effects of the eco-epidemic model, the discussion blends analytical and numerical findings. Analytical models can help to fully comprehend the model and its application. Some people think they can guess how the system will behave and stay stable in the long term by doing a stability analysis and finding equilibrium. It is possible to see that these results are true because computer simulations show how the model changes when events and factors change [75].

In this discussion, the advantages and disadvantages of the model are illustrated for the purpose of understanding ecological changes and the spread of disease within plant populations by combining all of these different techniques. By illustrating the areas in which the model is a good fit with known phenomena and the areas in which it is not, the graph demonstrates how the model could be improved. In addition, the findings of the model and the consequences such discoveries have for agricultural practices and the management of ecosystems are investigated in great detail. This study investigates the potential of the model to assist in the management of diseases, the application of pesticides, and the enhancement of plant populations within agricultural environments.

3.7.2 Implications of model assumptions and limitations

Details of the model's assumptions and restrictions demonstrate the eco-epidemic framework's importance. Everyone believes that mathematical models aid comprehension. They're flawed and based on assumptions that may understate how complex ecosystems really are. Thanks to these restrictions, ecologists can use model results and understand the hazards of making predictions. The seminar also discusses ways to make models more accurate and dependable by testing assumptions using real-world data and field reports [78].

It can be concluded that the interpretation of pesticides and the design of control strategies could be influenced by model assumptions. Some of these assumptions include the predictability of interactions between populations and pesticides. This is due to the fact that statistical models serve as the foundation for model assumptions. Upon completion of a sensitivity analysis, one can ascertain the consistency of the model's outcomes with changes to important assumptions and inputs. This is possible because sensitivity analysis is a type of statistical study. Creating more realistic eco-epidemic models is a complex undertaking because it necessitates the inclusion of stochastic elements and spatial variations. As a result, developing such models is a very difficult endeavour.

Despite these issues, the discussion emphasises the eco-agricultural decision-making model for safe growth and decision-making [81]. The model helps understand disease spread across ecological systems. It gives us ways to prevent diseases without harming the environment. Better eco-epidemiological models preserve ecosystems. It needs to be remembered that to make modifications, the study requires that model [1].

3.8 Conclusion

This study has provided significant insights into the functioning of eco-epidemiological systems and the management of diseases in agricultural ecosystems. The mathematical model's analysis and development provided crucial findings on plant population treatments, disease patterns, and pesticide application. The study and description of equilibrium points have contributed to the development of efficient disease control measures by making the system's stability and behaviour evident. This model is ecologically important because of its boundedness rate and positivity, which are similar to the results seen in the real world. The coupling of mathematical models with ecological and epidemiological data in this study demonstrated the significance of employing a range of approaches to complex environmental challenges.