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The Assam Royal Global University, Guwahati

Royal School of Applied & Pure Sciences

B.Sc. Mathematics, 6th Semester

Semester End Examination, June 2023

Course Title: Abstract Algebra

Course Code: MAT012C602

Time: 3 Hours **Maximum Marks: 70**

Note: Attempt all questions as per instructions given.

The figures in the right-hand margin indicate marks.

Section – A

1. Attempt **all questions.** (Maximum word limit 50) **2 x 8**

a. Find the identity element of the binary operation $*$ on $\mathbb{N}$ defined by $a * b = \max\{a, b\}$, if it exists.

b. Is the relation $R = \{(x, y): x^2 - y^2 \leq 5\}$ an equivalence relation on the set of integers $\mathbb{Z}$? Justify.

c. "Every abelian group is cyclic." Is the statement true? Justify.

d. Define order of an element of a group. If $f = (123)(45) \in S_5$ then find the order of f.

e. Show that every subgroup of an abelian group is normal.

f. Is the function $f: (\mathbb{R}, +) \to (\mathbb{R}, +)$ defined by $f(x) = x + 1$ a homomorphism? Justify.

g. If S is an ideal of a ring R and $1 \in S$ then show that $S = R$.

h. Define idempotent elements. Find the idempotent elements in the ring $\mathbb{Z}_4$ of integers modulo 4.

Section – B

2. Attempt **any two** of the following: **6x 2**

a. Show that the set $\mathbb{N}$ with respect to the relation 'division' is a poset. Check if the poset $(D_{220}, \div)$ is a lattice or not, where D_{220} is the set all positive divisors of 220.

b. Prove that if $f : A \to B$ is a bijection then there exists a bijection $g : B \to A$ such that $g_o f : A \to A$ and $f_o g : B \to B$ are identity functions.

c. Show that the relation R on $\mathbb{N} \times \mathbb{N}$ defined by $(a, b)R(c, d) \Leftrightarrow ad = bc \quad \forall\, a, b, c, d \in \mathbb{N}$ is an equivalence relation. Also, find the equivalence class of $(1,2)$.

3. Attempt **any two** of the following: **7 x 2**

a. Show that an alternating group A_n is a subgroup of S_n. Also, find $[S_n : A_n]$.

b. Define order of an element of a group with an example. Show that order of every element of a group divides the order of the group.

c. (i) If G is a group of order 35 then show that it cannot have two subgroups of order 7.
(ii) Let x be an element of a group G such that $x^2 \neq e$ and $x^6 = e$ then what can you say about the order of x ? (where e is the identity element)

P.T.O.

7 x 2

4. Attempt **any two** of the following:

a. If H and K are two finite normal subgroups of co-prime orders of a group G then show that $H \cap K = \{e\}$ and $hk = kh \;\forall\; h \in H, k \in K$.

b. State and prove Cayley's theorem.

c. If N is a normal subgroup of a group G then show that the mapping $f : G \to \frac{G}{N}$ defined by $f(x) = Nx \;\forall\; x \in G$ is an epimorphism. Also, find the kernel of f.

7 x 2

5. Attempt **any two** of the following:

a. Prove that a non-empty subset S of a ring R is a subring iff $x - y, xy \in S \;\forall\; x, y \in S$.

b. Prove that an ideal P of a commutative ring R is a prime ideal if and only if $\frac{R}{P}$ is an integral domain.

c. Define prime ideal. In the ring of integers $\mathbb{Z}$, prove that the set $S = \{3r : r \in \mathbb{Z}\}$ is a prime ideal of $\mathbb{Z}$ generated by 3.

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